Homotopietheorie Seminar (S2D4)
Topological K-theory

Wednesdays, 14:15-15:45
Organisiational meeting on Tuesday, July 16, 2023, at 16:15 in N0.008.

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Topological K-theory, just called K-theory from now on, is a cohomology theory built from vector bundles. The goal for this seminar is to study vector bundles, define the cohomology of K-theory, and then prove statements in algebra and topology using this theory. The main references are the books of Atiyah [Ati67], Hatcher [Hat], and Karoubi [Kar78]. We will mostly focus on complex K-theory. Each talk should be 90 minutes long, accounting for questions and comments, and so it is up to each presenter to chose exactly what should be presented from topic, although the main theorems and definitions should always be given.

(16.10.2024) Vector bundles I (TBD) Define vector bundles, both complex and real, give some examples, make the basic constructions including the pullback, direct sum, tensor product, exterior algebra, etc. Prove that over a compact Hausdorff space, all vector bundles are stably trivial. See [Ati67, §1.1-1.2], [Hat, §1.1], and [Kar78, §I].

(23.10.2024) Vector bundles II (TBD) Prove the homotopy invariance of vector bundles and their classification over compact Hausdorff spaces in terms of homotopy classes of maps into a Grassmannian. Discuss clutching functions and the classification of vector bundles over spheres. See [Ati67, §1.3], [Hat, §1.2], and [Kar78, §I].

(30.10.2024) The ring $K(X)$ (TBD) Define the K-theory group $K(X)$ for a compact Hausdorff space X and show it is a ring. Define the reduced group $\widetilde{K}(X)$. Prove that $K(-)$ satisfies the axioms of a cohomology theory defined only in degree zero. See [Ati67, §2.1 & 2.4] and [Hat, §2.1-2.2], and [Kar78, §II].

(06.11.2024) The fundamental product theorem (TBD) State and prove the theorem that the exterior product map $K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff spaces X. See [Ati67, §2.2], [Hat, §2.1], and [Kar78, §III].

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(13.11.2024) The cohomology theory $K^*(X)$ (TBD) Use the fundamental product theory to define the groups $K^n(X)$ for all $n \in \mathbf{Z}$, prove Bott periodicity, and show the resulting functor $K^*(-)$ defines a cohomology theory. Show that $K^n(X)$ is represented by a sequence of spaces K_n, meaning that there are isomorphisms $K^n(X) \cong [X, K_n]$ for compact Hausdorff spaces X, together with homotopy equivalences $\Omega K_{n+1} \simeq K_n$. See [Ati67, §2.2 & 2.4], [Hat, §2.2], and [Kar78, §III].

(20.11.2024) The splitting principle and the Leray–Hisch theorem (TBD) Prove the splitting principle and the Leray–Hirsch theorem for K-theory. Use this to compute the complex K-theory of complex projective spaces and Grassmannians. See [Hat, §2.3] and [Kar78, §IV.2].

(27.11.2024) Adams operations and λ-ring structures (TBD) Define λ-rings, construct the Adams operations on $K(X)$, and combine these results to show that $K(X)$ has a natural structure of a λ-ring. Calculate the λ-ring structure on spheres and projective spaces. See [Ati67, §3.1-3.2], [Hat, §2.3], and [Yau10].

(04.12.2024) Hopf invariant one theorem (TBD) Show that the Hopf invariant one theorem implies the nonexistence of H-space structure of spheres, and hence the parallelisability of spheres and division algebra structure of Euclidean space. Prove the Hopf invariant one theorem using Adams operations and complex K-theory. See [Hat, §2.3] and [Kar78, §V.1].

(11.12.2024) K-theory and reality (TBD) Following Atiyah [Ati66], construct a C_2-equivariant cohomology theory KR, prove equivariant Bott periodicity, and use this to conclude Bott periodicity for real K-theory. See [Ati66].

(18.12.2024) Clifford algebras (TBD) Describe the relationship between Clifford algebras, Fredholm operators, and topological K-theory, in particular with regard to real topological K-theory. Discuss how these can be used to give a reproof of Bott periodicity. See [ABS64] and [Kar78, §II.1 & III.3].

(08.01.2025) Thom isomorphism (TBD) Discuss the Thom isomorphism for complex K-theory (and singular cohomology, if you have seen this). Use this result to calculate the complex K-theory of complex projective spaces again. If there is time, discuss the Thom isomorphism for spinor bundles. See [Ati67, §2.7] and [Kar78, §IV.5].

(15.01.2025) Vector fields on spheres (TBD) Discuss the result of Adams [Ada62] which calculates the precise maximal number of linearly independent trangent vector fields on spheres. See [Kar78, §V.2] for an overview.

References

[ABS64] Michael F. Atiyah, Raoul Bott, and A. Shapiro. Clifford modules. *Topology*, 3:3–38, 1964.

[Ada62] J. F. Adams. Vector fields on spheres. *Topology*, 1:63–65, 1962.

[Ati66] Michael F. Atiyah. K-theory and reality. *Q. J. Math., Oxf. II. Ser.*, 17:367–386, 1966.

[Ati67] Michael F. Atiyah. K-theory. Lecture notes by D. W. Anderson. Fall 1964. With reprints of M. F. Atiyah: Power operations in K-theory; K-theory and reality. New York-Amsterdam: W.A. Benjamin, Inc. 166 p. (1967)., 1967.

[Hat] Allen Hatcher. Vector bundles and K-theory. Available at `https://pi.math.cornell.edu/~hatcher/VBKT/VBpage.html`.

[Kar78] Max Karoubi. *K-theory. An introduction*, volume 226 of *Grundlehren Math. Wiss.* Springer, Cham, 1978.

[Yau10] Donald Yau. *Lambda-rings.* Hackensack, NJ: World Scientific, 2010.