Fundamental Notions in Algebra – Exercise. 13

1. Let G be a finite group.

   (a) For a representation $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ denote by $\check{\rho} : G \to \mathrm{Aut}(\check{V})$ the contragredient representation $\check{\rho}(g)(f)(v) := f(\rho(g^{-1})(v))$. Describe character $\chi_{\check{\rho}}$ in terms of χ_ρ.

   (b) Let ρ be the representation of $G \times G$ on the space $\mathbb{C}[G]$ given by the rule $(g_1, g_2)(\sum_{h\in G} a_h h) := \sum_{h\in G} a_h g_1 h (g_2)^{-1}$. Show that ρ decomposes as a direct sum $\rho \cong \oplus_{\rho_i \in Irr(G)}[\rho_i \boxtimes \check{\rho_i}]$.

   (c) Let ρ be the representation of G on the space $\mathbb{C}[G]$ given by the rule $g_1(\sum_{h\in G} a_h h) := \sum_{h\in G} a_h g h g^{-1}$. Show that $\chi_\rho(g) = |Cent_G(g)|$ for each $g \in G$.

   (d) Show that for each $g \in G$, we have $|Cent_G(g)| = \sum_{\rho_i \in Irr(G)} |\chi_{\rho_i}(g)|^2$.

2. Let $A = (\chi_i(g_j))_{i,j=1}^r$ be the character table of a finite group G, whose rows are indexed by characters $\chi_i = \chi_{\rho_i}$ of irreducible representations of G, and columns are indexed by representatives g_j of conjugacy classes of G. The goal of this question is to show that many properties of G can be obtained from its character table.

   (a) Determine which column of A corresponds to the conjugacy class of 1. (Hint: Show that $|\chi_\rho(g)| \leq \chi_\rho(1)$ for all g and ρ and that $\chi_\rho(g) = \chi_\rho(1)$ if and only if $g \in Ker\rho$.)

   (b) Determine from A dimensions of each ρ_i and the cardinality of G.

   (c) Determine from A the cardinality of the conjugacy class of each g_j. (Hint: use question 1).

   (d) Determine from A all normal subgroups of G. (Here and below to describe a normal subgroup means to present it as a union of conjugacy classes of G).

   (e) For each normal subgroup N of G, give an algorithm how to construct from A the character table of the quotient group G/N. (Hint: Show that $Irr(G/N)$ can be identified with the set of $\rho \in Irr(G)$ such that $N \subset Ker\rho$).

   (f) Determine from A the commutator subgroup (G, G) of G.

   (g) Determine from A the center $Z(G)$ of G (together the group structure!).

   (h) Determine from A whether G is abelian, simple, nilpotent.

3. Let $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$ be an irredicible representation. Show that $dimV$ divides $[G : Z(G)]$. Hint:

   (a) Consider the representation $\rho^{\boxtimes n} : G^n \to \mathrm{Aut}_{\mathbb{C}}(V^{\boxtimes n})$ (the exterior n-th power of ρ). Show that $\rho^{\boxtimes n}$ is irredicible and that the subgroup $H = \{(z_1, \ldots, z_n) \in Z(G)^n \mid z_1 \cdot \ldots z_n = 1\}$ acts trivially on $V^{\boxtimes n}$.

   (b) Deduce that $(dimV)^n$ divides $|G|^n/|Z(G)|^{n-1}$.

   (c) Conclude from this that $dimV$ divides $[G : Z(G)]$.