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# The $p$-order of topological triangulated categories

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## Abstract

The $p$-order of a triangulated category is an invariant that measures 'how strongly' $p$ annihilates objects of the form $Y/p$. In this paper, we show that the $p$-order of a topological triangulated category is at least $p-1$; here we call a triangulated category *topological* if it admits a model as a stable cofibration category. Our main new tools are enrichments of cofibration categories by $\Delta$-sets; in particular, we generalize the theory of 'framings' (or 'cosimplicial resolutions') from model categories to cofibration categories.

## *Introduction*

Many triangulated categories arise from chain complexes in an additive or abelian category by passing to chain homotopy classes or inverting quasi-isomorphisms. Such examples are called 'algebraic' because they have underlying additive categories. Stable homotopy theory produces examples of triangulated categories by quite different means, and in this context the underlying categories are usually very 'non-additive' before passing to homotopy classes of morphisms. We call such triangulated categories *topological*, and formalize this in Definition 1.4 via homotopy categories of stable cofibration categories.

The purpose of this paper is to explain some basic properties of topological triangulated categories and provide a lower bound on their $p$-order for a prime $p$. The *p-order* of a triangulated category is a non-negative integer (or infinity), and it measures, informally speaking, 'how strongly' the relation $p \cdot Y/p = 0$ holds, where $Y/p$ denotes a cone of multiplication by $p$ on $Y$. Our main result is Theorem 5.3, saying that the $p$-order of a topological triangulated category is at least $p-1$. In the companion paper [**30**], we prove some complementary results: the $p$-order of every algebraic triangulated category is infinite [**30**, Theorem 2.2] and the $p$-order of the $p$-local stable homotopy category is at most $p-1$ [**30**, Theorem 3.1]. Hence the $p$-order of the $p$-local stable homotopy category is exactly $p-1$, and the $p$-local stable homotopy category is not algebraic for any prime $p$.

Our main technical innovation is the use of $\Delta$-sets for studying cofibration categories. We develop certain foundations about enrichments of cofibration categories by $\Delta$-sets that we hope to be of independent interest. In particular, we generalize the theory of 'framings' (or 'cosimplicial resolutions') from model categories to cofibration categories. Theorem 3.10 shows that the category of *frames* (certain homotopically constant co-$\Delta$-objects) in any cofibration category is again a cofibration category, and that the homotopy category does not change when passing to frames. Theorem 3.17 shows that the category of frames in a saturated cofibration category is always a $\Delta$-cofibration category (cf. Definition 3.15), the analog of simplicial model category. So one could say that $\Delta$-sets are to cofibration categories what simplicial sets are to model categories.

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Here is a summary of the contents of this paper. In Section 1, we introduce and discuss topological triangulated categories as the homotopy categories of stable cofibration categories. In Section 2, we review basic properties of $\Delta$-sets. Section 3 develops the theory of *framings* in cofibration categories; this is needed later to make sense of an action of the category of finite $\Delta$-sets on a cofibration category. Section 4 is devoted to coherent actions of mod-$n$ Moore spaces (that is, $\Delta$-sets) on objects in pointed cofibration categories. In Section 5, we prove our main result, that the $p$-order of every topological triangulated category is at least $p-1$. Appendix A recalls the basic facts about the homotopy category of a cofibration category and gives a self-contained proof that the homotopy category of a stable cofibration category is triangulated.

Some results of this paper were announced in the survey article [**29**], where we based the definition of topological triangulated categories on Quillen model categories. Here, however, we use the more general concept of cofibration categories, so our results are somewhat more general. Also, the contents of this paper and its companion [**30**] were originally combined as a single paper (`arXiv:1201.0899`); the referee convinced the author to divide up the results into two separate papers.

## 1. *Topological triangulated categories*

We implement the notion of 'topological enhancement' of triangulated categories via *cofibration categories*. This notion was first introduced and studied (in the dual formulation) by Brown [**4**] under the name 'categories of fibrant objects'. Closely related sets of axioms have been explored by various authors; cf. Remark 1.3.

Definition 1.1. A *cofibration category* is a category $\mathcal{C}$ equipped with two classes of morphisms, called *cofibrations* respectively *weak equivalences*, that satisfy the following axioms (C1)–(C4).

(C1) All isomorphisms are cofibrations and weak equivalences. Cofibrations are stable under composition. The category $\mathcal{C}$ has an initial object and every morphism from an initial object is a cofibration.

(C2) Given two composable morphisms $f$ and $g$ in $\mathcal{C}$, if two of the three morphisms $f, g$ and $gf$ are weak equivalences, then so is the third.

(C3) Given a cofibration $i : A \to B$ and any morphism $f : A \to C$, there exists a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \longrightarrow & P \end{array} \tag{1.2}$$

in $\mathcal{C}$ and the morphism $j$ is a cofibration. If additionally $i$ is a weak equivalence, then so is $j$.

(C4) Every morphism in $\mathcal{C}$ can be factored as the composite of a cofibration followed by a weak equivalence.

An *acyclic cofibration* is a morphism that is both a cofibration and a weak equivalence. We note that in a cofibration category a coproduct $B \vee C$ of any two objects in $\mathcal{C}$ exists by (C3) with $A$ an initial object, and the canonical morphisms from $B$ and $C$ to $B \vee C$ are cofibrations.

A property that we will frequently use is the *gluing lemma*. This starts with a commutative diagram

$$\begin{tikzcd} A \arrow[d, "\sim"'] & B \arrow[l, "i"'] \arrow[r] \arrow[d, "\sim"'] & C \arrow[d, "\sim"] \\ A' & B' \arrow[l, "i'"] \arrow[r] & C' \end{tikzcd}$$

in a cofibration category $\mathcal{C}$ such that $i$ and $i'$ are cofibrations and all three vertical morphisms are weak equivalences. The gluing lemma says that then the induced morphism on pushouts $A \cup_B C \to A' \cup_{B'} C'$ is a weak equivalence. A proof of the gluing lemma can be found in [**26**, Lemma 1.4.1(1)].

The *homotopy category* of a cofibration category is a localization at the class of weak equivalences, that is, a functor $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ that takes all weak equivalences to isomorphisms and is initial among such functors. The homotopy category always exists if one is willing to pass to a larger universe. To get a locally small homotopy category (that is, have 'small hom-sets'), additional assumptions are necessary; one possibility is to assume that $\mathcal{C}$ has 'enough fibrant objects'; cf. Remark A.2. We recall some basic facts about the homotopy category of a cofibration category in Theorem A.1.

Remark 1.3. The above notion of cofibration category is due to Brown [**4**]. More precisely, Brown introduced 'categories of fibrant objects', and the axioms (C1)–(C4) are equivalent to the dual of the axioms (A)–(E) of Part I.1 in [**4**]. The concept of a cofibration category is a substantial generalization of Quillen's notion of a 'closed model category' [**25**]: from a Quillen model category one obtains a cofibration category by restricting to the full subcategory of cofibrant objects and forgetting the class of fibrations.

Cofibration categories are closely related to 'categories with cofibrations and weak equivalences' in the sense of Waldhausen [**35**]. In fact, a category with cofibrations and weak equivalences that also satisfies the *saturation axiom* [**35**, 1.2] and the *cylinder axiom* [**35**, 1.6] is, in particular, a cofibration category as in Definition 1.1. Further relevant references on closely related axiomatic frameworks are Baues' monograph [**1**] and Cisinski's work [**6**]. Radulescu-Banu's [**26**] extensive paper is the most comprehensive source for basic results on cofibration categories and, among other things, contains a survey of the different kinds of cofibration categories and their relationships.

A cofibration category is *pointed* if every initial object is also terminal, hence a zero object, which we denote by $*$. In a pointed cofibration category, the factorization axiom (C4) provides a *cone* for every object $A$, that is, a cofibration $i_A : A \to CA$ whose target is weakly equivalent to the zero object. The *suspension* $\Sigma A$ of $A$ is the quotient of the 'cone inclusion', that is, a pushout:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d] & CA \arrow[d] \\ * \arrow[r] & \Sigma A. \end{tikzcd}$$

We recall in Proposition A.4 that there is a preferred way to make the suspension construction into functor $\Sigma : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(\mathcal{C})$ on the level homotopy categories. In other words: on the level of cofibration categories the cone, and hence the suspension, constitute a choice, but any set of choices becomes functorial upon passage to the homotopy category. Moreover, different choices of cones lead to canonically isomorphic suspension functors; cf. Remark A.15.

Every cofibration $j : A \to B$ in a pointed cofibration category $\mathcal{C}$ gives rise to a preferred and natural *connecting morphism* $\delta(j) : B/A \to \Sigma A$ in $\mathbf{Ho}(\mathcal{C})$; see (A.10). The *elementary distinguished triangle* associated to the cofibration $j$ is the triangle

$$A \xrightarrow{\gamma(j)} B \xrightarrow{\gamma(q)} B/A \xrightarrow{\delta(j)} \Sigma A;$$

here $q : B \to B/A$ is a quotient morphism. A *distinguished triangle* is any triangle in the homotopy category that is isomorphic to the elementary distinguished triangle of a cofibration.

A pointed cofibration category is *stable* if the suspension functor $\Sigma : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(\mathcal{C})$ is an auto-equivalence. We recall in Theorem A.12 that the suspension functor and the class of distinguished triangles make the homotopy category $\mathbf{Ho}(\mathcal{C})$ of a stable cofibration category into a triangulated category. We call the class of triangulated categories arising in this way the 'topological triangulated categories'. The adjective 'topological' does *not* mean that the category or its hom-sets are topologized; rather, 'topological' is supposed to indicate that these examples are constructed by topological methods, or that they have models in the spirit of abstract homotopy theory.

Definition 1.4. A triangulated category is *topological* if it is equivalent, as a triangulated category, to the homotopy category of a stable cofibration category.

Now we prove a basic closure property of topological triangulated categories.

Proposition 1.5. *Every full triangulated subcategory of a topological triangulated category is topological.*

*Proof.* Let $\mathcal{C}$ be a stable cofibration category and $\mathcal{T}$ be a full triangulated subcategory of $\mathbf{Ho}(\mathcal{C})$. We let $\bar{\mathcal{C}}$ denote the full subcategory of $\mathcal{C}$ consisting of those objects that are isomorphic in $\mathbf{Ho}(\mathcal{C})$ to an object in $\mathcal{T}$. We claim that $\bar{\mathcal{C}}$ becomes a stable cofibration category when we restrict the classes of cofibrations and weak equivalences from $\mathcal{C}$ to $\bar{\mathcal{C}}$.

Indeed, axioms (C1) and (C2) are directly inherited from $\mathcal{C}$. Concerning axiom (C3) we observe that a pushout square (1.2) in $\mathcal{C}$ in which $i : A \to B$ is a cofibration gives rise to two elementary distinguished triangles

$$A \xrightarrow{\gamma(i)} B \longrightarrow B/A \xrightarrow{\delta(i)} \Sigma A \quad \text{and} \quad C \xrightarrow{\gamma(j)} P \longrightarrow P/C \xrightarrow{\delta(j)} \Sigma C$$

in $\mathbf{Ho}(\mathcal{C})$. So if $A$ and $B$ belong to $\bar{\mathcal{C}}$, then so does the quotient $B/A$. If, moreover, $C$ belongs to $\bar{\mathcal{C}}$, then so does the pushout $P$ (since $B/A$ and $P/C$ are isomorphic). Hence the square (1.2) is in fact a pushout in the subcategory $\bar{\mathcal{C}}$, so (C3) holds in $\bar{\mathcal{C}}$. Since $\bar{\mathcal{C}}$ is closed under weak equivalences, a factorization as in axiom (C4) in the ambient category $\mathcal{C}$ is also a factorization in $\bar{\mathcal{C}}$, so $\bar{\mathcal{C}}$ inherits property (C4).

As we just saw, the inclusion $\bar{\mathcal{C}} \to \mathcal{C}$ preserves the particular pushouts required by (C3), so it is an exact functor. Moreover, the functor $\mathbf{Ho}(\bar{\mathcal{C}}) \to \mathbf{Ho}(\mathcal{C})$ induced by the inclusion is fully faithful (use parts (i) and (ii) of Theorem A.1). Since the inclusion is exact, this induced functor is also exact (by Proposition A.14) and hence an embedding of triangulated categories whose image is the original triangulated subcategory $\mathcal{T}$ of $\mathbf{Ho}(\mathcal{C})$. □

Examples of topological triangulated categories abound. An important example is the *stable homotopy category* of algebraic topology which, to my knowledge, was first published in the form it is used today by Kan [**16**]. When Kan's paper appeared in 1963, neither model categories nor cofibration categories had been formalized, but it is straightforward to deduce from the results therein that Kan's 'semisimplicial spectra' form a stable cofibration category. Later Brown showed in [**4**, II, Theorem 5] that the semisimplicial spectra even support a model category structure. By now there is an abundance of models for the stable homotopy category; see, for example, [**3**, **9**, **15**, **22**]. The *Spanier–Whitehead category* [**33**], which predates the stable homotopy category, can be obtained from finite based CW-complexes by formally inverting the suspension functor; it is equivalent to the full subcategory of compact objects in the stable homotopy category, so it is a topological triangulated category in our sense.

Further examples of topological triangulated categories are the homotopy categories of stable model categories, including the 'derived' (that is, homotopy) categories of structured ring spectra or spectral categories, equivariant and motivic stable homotopy categories, sheaves of spectra on a Grothendieck site or (Bousfield-)localizations of the above; a more detailed list of specific references can be found in [**31**, Section 2.3].

Triangulated categories that are not topological do not come up as frequently; in fact, the author's point of view is precisely that topological triangulated categories are the ones that come up 'in nature', and other examples have to be 'manufactured by hand'. The simplest example of a triangulated category that is not topological is the following, due to Muro. The category $\mathcal{F}(\mathbb{Z}/4)$ of finitely generated free modules over the ring $\mathbb{Z}/4$ has a unique triangulation with the identity shift functor and such that the triangle

$$\mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4$$

is distinguished. For details we refer the reader to [**23**], where Muro, Strickland and the author discuss an entire family of 'exotic' (that is, non-topological) triangulated categories that includes $\mathcal{F}(\mathbb{Z}/4)$ as the simplest case. I do not know any non-topological triangulated category in which 2 is invertible.

Example 1.6 (Algebraic triangulated categories). Another important class of triangulated categories are the *algebraic* triangulated categories. The earliest formalization of this class of triangulated categories seems to be the notion of *enhanced triangulated category* of Bondal and Kapranov [**2**, Section 3]. Algebraic triangulated categories can be introduced in at least two other, equivalent, ways. One way is as the full triangulated subcategories of homotopy categories of additive categories. Another way is as the stable categories of exact Frobenius categories. For the equivalence of these three approaches, and for more details, background and references, we refer the reader to [**2**, **19**, **20**].

For the purposes of this paper, the most convenient definition of algebraic triangulated categories is via chain complexes. We let $\mathcal{A}$ be an additive category and denote by $C(\mathcal{A})$ the category of $\mathbb{Z}$-graded chain complexes of objects in $\mathcal{A}$, with morphisms the chain maps of homogeneous degree 0. The *homotopy category* $\mathbf{K}(\mathcal{A})$ has the same objects as $C(\mathcal{A})$, but morphisms are chain homotopy classes of chain maps. The category $C(\mathcal{A})$ has a natural structure of a stable cofibration category: The weak equivalences are the chain homotopy equivalences, and the cofibrations are those chain maps that are dimensionwise split monomorphisms. This cofibration structure is special in that every object is fibrant (see Remark A.2). The abstract homotopy category $\mathbf{Ho}(C(\mathcal{A}))$ of this cofibration structure coincides with the concrete homotopy category $\mathbf{K}(\mathcal{A})$.

The homotopy category $\mathbf{K}(\mathcal{A})$ has a well-known triangulation, with shift functor given by shift of complexes and distinguished triangles arising from algebraic mapping cone sequences. The triangulated category axioms can be checked directly, but they also follow from Theorem A.12 applied to the stable cofibration structure. A triangulated category is *algebraic* if it is equivalent to a full triangulated subcategory of the homotopy category $\mathbf{K}(\mathcal{A})$ of an additive category. Since topological triangulated categories are closed under passage to subcategories (Proposition 1.5), every algebraic triangulated category is also a topological triangulated category.

A well known example of a topological triangulated category that is not algebraic is the stable homotopy category. To see this, we exploit that for every object $X$ of an algebraic triangulated category the object $X/n$ (a cone of multiplication by $n$ on $X$) is annihilated by $n$; see, for example, [**29**, Proposition 1]. On the other hand, the mod-2 Moore spectrum in the stable homotopy category is of the form $\mathbb{S}/2$ for $\mathbb{S}$ the sphere spectrum, and it is well known that $\mathbb{S}/2$ is *not* annihilated by 2. An account of the classical argument using Steenrod operations can be found in [**29**, Proposition 4]. It is more subtle to show that the $p$-local stable homotopy category is not algebraic when $p$ is an odd prime, and this is the main result of the companion paper; see [**30**, Theorem 3.1].

I expect that rationally there is no difference between algebraic and topological triangulated categories. In other words: every topological triangulated category whose morphism groups are uniquely divisible ought to be algebraic. There are various pieces of evidence for this claim, and all invariants I know to distinguish algebraic from topological triangulated categories vanish rationally. For example, the $n$-order (see Definition 5.1) is rationally useless since $\mathbb{Q}$-linear triangulated categories have infinite $n$-order for all natural numbers $n$. Similarly, the action of the homotopy category of finite CW-complexes on a topological triangulated category (cf. Remark 3.11) is no extra information for $\mathbb{Q}$-linear triangulated categories since the Spanier–Whitehead category is rationally equivalent to the bounded derived category of finitely generated abelian groups (both are rationally equivalent to the category of finite dimensional graded $\mathbb{Q}$-vector spaces). Moreover, under certain technical assumptions and cardinality restrictions, $\mathbb{Q}$-linear topological triangulated categories are known to be algebraic: a theorem of Shipley [**32**, Corollary 2.16] says that every $\mathbb{Q}$-linear spectral model category (a stable model category enriched over the stable model category of symmetric spectra) with a set of compact generators is Quillen equivalent to dg-modules over a certain differential graded $\mathbb{Q}$-category.

## 2. *Review of $\Delta$-sets*

In this section, we recall $\Delta$-sets and review some of their properties. In the later sections, we shall use actions of $\Delta$-sets on cofibration categories to establish lower bounds for the $p$-order in topological triangulated categories. A general reference for $\Delta$-sets is the paper by Rourke and Sanderson [**27**].

We let $\Delta$ denote the category whose objects are the totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geqslant 0$, and whose morphisms are the injective monotone maps. A $\Delta$-*set* (sometimes called a *semisimplicial set* or a *presimplicial set*) is a contravariant functor from the category $\Delta$ to the category of sets; a morphism of $\Delta$-sets is a natural transformation of functors. We write $K_n = K([n])$ for the value of a $\Delta$-set $K : \Delta^{\mathrm{op}} \to (\text{sets})$ and call the elements of this set the $n$-*simplices* of $K$. For a morphism $\alpha : [m] \to [n]$ in $\Delta$ and an $n$-simplex $x$ of $K$ we write

$$x\alpha = K(\alpha)(x) \in K_m$$

for the effect of the map induced by $\alpha$. The functor property then becomes the relation $(x\alpha)\beta = x(\alpha\beta)$. A $\Delta$-set $K$ is *finite* if the disjoint union of all the sets $K_n$ is finite. Equivalently, $K$ is finite if each $K_n$ is finite and almost all $K_n$ are empty.

For $0 \leqslant i \leqslant n$ we denote by $d_i : [n-1] \to [n]$ the unique morphism in $\Delta$ whose image does not contain $i$. A $\Delta$-set can be defined by specifying the sets of simplices and the *face maps*, that is, the effect of the morphisms $d_i$. These maps have to satisfy the relations

$$xd_jd_i = xd_id_{j-1} \quad \text{for all } i < j.$$

We will often specify the faces of an $n$-simplex $x$ in a compact way by writing

$$\partial x = (xd_0, xd_1, \ldots, xd_n).$$

The *geometric realization* of a $\Delta$-set $K$ is the topological space

$$|K| = \bigcup_{n \geqslant 0} K_n \times \nabla^n / \sim .$$

Here $\nabla^n$ is the topological $n$-simplex (the convex hull of the standard basis vectors in $\mathbb{R}^{n+1}$), and the equivalence relation is generated by

$$(x\alpha, t) \sim (x, \alpha_* t)$$

for all $x \in K_m$, $t \in \nabla^n$ and $\alpha : [n] \to [m]$, where $\alpha_*(t_0, \ldots, t_n) = (s_0, \ldots, s_m)$ with $s_i = \sum_{\alpha(j)=i} t_j$. A morphism of $\Delta$-sets is a *weak equivalence* if it becomes a homotopy equivalence after geometric realization. A $\Delta$-set is *weakly contractible* if its geometric realization is contractible. We emphasize that although the category of $\Delta$-sets has useful notions of cofibrations (the monomorphisms) and weak equivalences, $\Delta$-sets do *not* form a cofibration category because the factorization axiom (C4) fails.

Example 2.1. Important examples are the representable $\Delta$-sets $\Delta[n] = \Delta(-, [n])$, where $n \geqslant 0$. For any $\Delta$-set $K$, the Yoneda lemma says that evaluation at the unique $n$-simplex $\mathrm{Id}_{[n]}$ of $\Delta[n]$ is a natural bijection from the morphism set $\Delta\text{-set}(\Delta[n], K)$ to the set $K_n$ of $n$-simplices of $K$. The maps

$$\nabla^n \longrightarrow |\Delta[n]|, \quad t \longmapsto [\mathrm{Id}_{[n]}, t] \quad \text{and} \quad |\Delta[n]| \longrightarrow \nabla^n, \quad [\alpha, t] \longmapsto \alpha_*(t)$$

are mutually inverse homeomorphisms between the topological $n$-simplex and the geometric realization of $\Delta[n]$.

Important sub-$\Delta$-sets of $\Delta[n]$ are the *boundary* $\partial\Delta[n] = \Delta[n] - \{\mathrm{Id}_{[n]}\}$ and the *horns*

$$\Lambda^i[n] = \partial\Delta[n] - \{d_i\} = \Delta[n] - \{\mathrm{Id}_{[n]}, d_i\}$$

for $0 \leqslant i \leqslant n$. The geometric realization of $\partial\Delta[n]$ maps homeomorphically onto the boundary of the topological simplex $\nabla^n$; the geometric realization of $\Lambda^i[n]$ maps homeomorphically onto the $i$-horn of $\nabla^n$, that is, the boundary with the interior of the $i$th face removed.

Definition 2.2. Let $K$ be a sub-$\Delta$-set of $L$. The inclusion $K \to L$ is an *elementary expansion* of dimension $n$ if there is an $n$-simplex $e \in L_n - K_n$ and an $i \in \{0, \ldots, n\}$ such that $L$ is the disjoint union of $K$ and $\{e, ed_i\}$, and $ed_j \in K$ for all $j \neq i$.

Elementary expansions can be characterized as pushouts of horn inclusions: $K \to L$ is an elementary expansion of dimension $n$ if and only if there is a pushout

$$\begin{tikzcd} \Lambda^i[n] \arrow[r] \arrow[d] & \Delta[n] \arrow[d] \\ K \arrow[r] & L \end{tikzcd} \tag{2.3}$$

for some $i \in \{0, \ldots, n\}$. The simplex $e$ in the definition of elementary expansion is then the image of the $n$-simplex of $\Delta[n]$.

Geometric realization has a right adjoint, so it commutes with pushouts. So, if $K \to L$ is an elementary expansion of dimension $n$, then the realization $|L|$ is obtained from $|K|$ by attaching a topological simplex along a horn, so the inclusion $|K| \to |L|$ is a homotopy equivalence. Hence every elementary expansion of $\Delta$-sets is a weak equivalence.

EXAMPLE 2.4. For a $\Delta$-set $K$ we denote by $CK$ the *cone* of $K$, defined by $(CK)_0 = K_0 \amalg \{*\}$ and

$$(CK)_n = K_n \amalg \{\sigma x \mid x \in K_{n-1}\},$$

where $n \geqslant 1$. The face operators are determined by requiring that the inclusion of $K_n$ as the first summand of $(CK)_n$ makes $K$ a sub-$\Delta$-set of $CK$, and by the formulas

$$(\sigma x)d_i = \begin{cases} \sigma(xd_i) & \text{for } 0 \leqslant i < n \text{ and,} \\ x & \text{for } i = n, \end{cases}$$

with the interpretation $(\sigma x)d_1 = *$ for $x \in K_0$. For example, the unique morphism $\Delta[n+1] \to C\Delta[n]$ is an isomorphism. The geometric realization $|CK|$ is homeomorphic to the cone of $|K|$, hence contractible. If $K'$ is a sub-$\Delta$-set of $K$ and $K - K'$ consists of a single $n$-simplex $x$, then the inclusion $CK' \to CK$ has complement $\{x, \sigma x\}$ and is an elementary expansion of dimension $n+1$. So if $K$ is finite, then the inclusion $\{*\} \to CK$ of the cone point is a composite of elementary expansions, one for each simplex of $K$.

Categorical products of $\Delta$-sets are not homotopically well behaved; this is one of the reasons why simplicial sets are preferable for many purposes. For example, the geometric realization of the categorical product $\Delta[1] \times \Delta[1]$ is not even connected. However, there is another construction, the *geometric product* $K \otimes L$ of two $\Delta$-sets $K$ and $L$, defined as follows. An $n$-simplex of $K \otimes L$ is an equivalence class of triples $(x, y; \varphi)$, where $x \in K_i$, $y \in L_j$ and $\varphi : [n] \to [i] \times [j]$ is an injective monotone map. The equivalence relation is generated by

$$(x\alpha, y\beta; \varphi) \sim (x, y; (\alpha \times \beta)\varphi)$$

for morphisms $\alpha$ and $\beta$ in the category $\Delta$. Every equivalence class has a preferred representative, namely the unique triple $(x, y; \varphi)$ where both components $\varphi^1 : [n] \to [i]$ and $\varphi^2 : [n] \to [j]$ of $\varphi$ are surjective; however, we shall not use this. A morphism $\nu : [m] \to [n]$ of $\Delta$ acts on the equivalence class of a triple by

$$[x, y; \varphi]\nu = [x, y; \varphi\nu].$$

For example, every $n$-simplex of $\Delta[i] \otimes \Delta[j]$ has a unique representative of the form $(\mathrm{Id}_{[i]}, \mathrm{Id}_{[j]}; \varphi)$ for an injective monotone map $\varphi : [n] \to [i] \times [j]$, so $\Delta[i] \otimes \Delta[j]$ is isomorphic to the $\Delta$-set of injective monotone maps into $[i] \times [j]$.

The geometric product is symmetric monoidal with unit object $\Delta[0]$. The unit isomorphism sends an $n$-simplex $[x, \mathrm{Id}_{[0]}; \varphi]$ of $K \otimes \Delta[0]$ to $x \in K_n$. The associativity isomorphism $(K \otimes L) \otimes M \cong K \otimes (L \otimes M)$ sends an $m$-simplex $[[x, y; \varphi], z; \psi]$ to

$$[x, [y, z; \varphi']; \psi'] \in (K \otimes (L \otimes M))_m,$$

where $\varphi' : [l] \to [j] \times [k]$ and $\psi' : [m] \to [i] \times [l]$ are the unique monotone injections that make the square

$$\begin{tikzcd} {[m]} \arrow[r, "\psi"] \arrow[d, "\psi'"'] & {[n] \times [k]} \arrow[d, "\varphi \times [k]"] \\ {[i] \times [l]} \arrow[r, "{[i] \times \varphi'}"'] & {[i] \times [j] \times [k]} \end{tikzcd}$$

commute. The symmetry isomorphism $K \otimes L \cong L \otimes K$ sends $[x, y; \varphi]$ to $[y, x; \tau\varphi]$, where $\tau : [i] \times [j] \to [j] \times [i]$ interchanges the factors.

The product $\otimes$ is called 'geometric' because geometric realization takes it to cartesian product of topological spaces. More precisely, a continuous natural map $p_K : |K \otimes L| \to |K|$ is defined by sending the equivalence class of

$$([x, y;\, \varphi],\, t) \in (K \otimes L)_n \times \nabla^n$$

to the class of $(x, \varphi^1_*(t)) \in K_i \times \nabla^i$, where $\varphi^1 : [n] \to [i]$ is the first component of $\varphi : [n] \to [i] \times [j]$. One should beware that $p_K$ is not induced by a morphism of $\Delta$-sets from $K \otimes L$ to $K$. There is an analogous map for the second factor $L$, and the combined map

$$(p_K, p_L) : |K \otimes L| \longrightarrow |K| \times |L| \tag{2.5}$$

is a homeomorphism (when the product is given the compactly generated topology). It follows that the functor $K \otimes -$ preserves weak equivalences of $\Delta$-sets.

A general injective weak equivalence between finite $\Delta$-sets is *not* a sequence of elementary expansions. Nevertheless, it is well known that the localization of the category of $\Delta$-sets at the class of elementary expansions agrees with the localization of the category of $\Delta$-sets at the class of weak equivalences, and the resulting homotopy category is equivalent to the homotopy category of CW-complexes, cf. Section I.4 of [**5**]. For the convenience of the reader we recall a proof of the first of these two facts inside of the category of *finite* $\Delta$-sets.

Proposition 2.6. *Let $F$ be a functor defined on the category of finite $\Delta$-sets that takes elementary expansions to isomorphisms. Then $F$ takes all weak equivalences to isomorphisms.*

*Proof.* We let $f : K \to L$ be a weak equivalence between finite $\Delta$-sets. By ‘filling all horns’ (cf. [**27**, p. 334]) we obtain a sequence of elementary expansions

$$K = K^0 \subseteq K^1 \subseteq K^2 \subseteq \cdots \subseteq K^n \subseteq \cdots$$

such that the union $K^\infty = \bigcup_{n \geqslant 0} K^n$ is a Kan $\Delta$-set, that is, every morphism of $\Delta$-sets from a horn $\Lambda^i[n]$ to $K^\infty$ can be extended to the simplex $\Delta[n]$. Of course, $K^\infty$ is no longer finite.

We let $i_0, i_1 : K \to K \otimes \Delta[1]$ be the ‘front and back inclusion’, that is, the morphisms defined on $x \in K_n$ by

$$i_0(x) = [x, d_1;\, \psi_n] \quad \text{respectively } i_1(x) = [x, d_0;\, \psi_n],$$

where $\psi_n : [n] \to [n] \times [0]$ is the unique monotone injection. We form the mapping cylinder $Mf = K \otimes \Delta[1] \cup_f L$, that is, a pushout of the diagram:

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, "i_1"'] & L \arrow[d, "j"] \\ K \otimes \Delta[1] \arrow[r, "g"'] & Mf. \end{tikzcd}$$

The back inclusion $i_1$ is a sequence of elementary expansions, hence so is its cobase change $j : L \to Mf$. The geometric realization of $Mf$ is homeomorphic to the topological mapping cylinder of $|f| : |K| \to |L|$ and the monomorphism $gi_0 : K \to Mf$ becomes the front inclusion after realization. Hence $|gi_0| : |K| \to |Mf|$ is a homotopy equivalence since $f$ is, and $|K|$ is a retract of $|Mf|$. By the ‘generalized extension property’ of Kan $\Delta$-sets [**27**, Corollary 5.4] there is a morphism of $\Delta$-sets $\varphi : Mf \to K^\infty$ such that $\varphi g i_0 : K \to K^\infty$ is the inclusion. Since $K$ and $L$ are finite $\Delta$-sets, so is the mapping cylinder $Mf$. The image of $\varphi : Mf \to K^\infty$ is thus contained in $K^n$ for some $n \geqslant 0$.

Now let $F$ be a functor to some category $\mathcal{D}$, defined on the category of finite $\Delta$-sets, that takes elementary expansions to isomorphisms. We apply $F$ to the various morphisms of finite

$\Delta$-sets and obtain a commutative diagram in $\mathcal{D}$:

$$\begin{tikzcd}
F(K) \arrow[r, "F(f)"] \arrow[d, "F(i_1)"', "\cong"] & F(L) \arrow[d, "\cong"', "F(j)"] \\
F(K \otimes \Delta[1]) \arrow[r, "F(g)"'] & F(Mf) \arrow[d, "F(\varphi)"] \\
F(K) \arrow[u, "F(i_0)", "\cong"'] \arrow[r, "\cong"'] & F(K^n)
\end{tikzcd}$$

The morphisms decorated with the symbol '$\cong$' are isomorphisms since they arise from morphisms of $\Delta$-sets that are sequences of elementary expansions. So the diagram shows that $F(f)$ has a left inverse in $\mathcal{D}$ and $F(\varphi)$ has a right inverse in $\mathcal{D}$. Now we apply the same argument to the weak equivalence $\varphi : Mf \to K^n$ instead of $f$. We deduce that $F(\varphi)$ has a left inverse in $\mathcal{D}$. Since $F(\varphi)$ has a left inverse and a right inverse, it is an isomorphism. Hence the morphism $F(f)$ is also an isomorphism. □

## 3. *Frames in cofibration categories*

In this section, we develop the technique of 'framings', or '$\Delta$-resolutions' in cofibration categories. Framings are a way to construct homotopically meaningful pairings with $\Delta$-sets; cf. the notion of '$\Delta$-cofibration category' in Definition 3.15. In Section 4, we need such a pairing to talk about actions of Moore spaces on objects of a cofibration category.

In the context of Quillen model categories, the theory of framings is well established and goes back to Dwyer and Kan [**8**, 4.3], who use the terminology *(co-)simplicial resolutions*. In our more general set-up of cofibration categories we cannot work with cosimplicial objects, the lack of fibrations and matching objects in a cofibration category does not allow the construction of codegeneracy maps in a frame. The solution is not to ask for codegeneracy morphisms, that is, to work with co-$\Delta$-objects instead of cosimplicial objects. In other words: $\Delta$-sets are to cofibration categories what simplicial sets are to Quillen model categories.

As before we let $\Delta$ denote the category whose objects are the totally ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for $n \geqslant 0$, and whose morphisms are the injective monotone maps. A *co-$\Delta$-object* in a category $\mathcal{C}$ is a covariant functor $A : \Delta \to \mathcal{C}$. Morphisms of co-$\Delta$-objects are natural transformations of functors. In a co-$\Delta$-object $A$ we typically write $A^n$ for $A([n])$.

Now we discuss how to pair co-$\Delta$-objects with $\Delta$-sets. If $A$ is a co-$\Delta$-object in $\mathcal{C}$ and $Z$ is an object of $\mathcal{C}$, then the composite functor

$$\Delta^{\mathrm{op}} \xrightarrow{A} \mathcal{C}^{\mathrm{op}} \xrightarrow{\mathcal{C}(-,Z)} (\text{sets})$$

is a $\Delta$-set that we denote by $\mathcal{C}(A, Z)$. If $K$ is a $\Delta$-set, then we denote by $K \cap A$ a representing object, if it exists, of the functor

$$\mathcal{C} \longrightarrow (\text{sets}), \quad Z \longmapsto \Delta\text{-set}(K, \mathcal{C}(A, Z)).$$

In more detail, $K \cap A$ is an initial example of a $\mathcal{C}$-object equipped with morphisms $x_* : A^n \to K \cap A$ for every $n \geqslant 0$ and every $n$-simplex $x$ of $K$, such that, for all morphisms $\alpha : [m] \to [n]$ in the category $\Delta$, the composite

$$A^m \xrightarrow{\alpha_*} A^n \xrightarrow{x_*} K \cap A$$

is equal to $(x\alpha)_*$. The universal morphisms $x_*$ are part of the data, but we will often omit them from the notation.

The defining property of $K \cap A$ can be rephrased in several other ways. For example, $K \cap A$ is a colimit of the composite functor

$$s(K) \xrightarrow{(n,x)\mapsto[n]} \Delta \xrightarrow{\quad A \quad} \mathcal{C}.$$

Here $s(K)$ is the simplex category of $K$: objects are pairs $(n,x)$ with $x \in K_n$, and morphisms from $(n,x)$ to $(m,y)$ are those $\Delta$-morphisms $\alpha : [n] \to [m]$ that satisfy $y\alpha = x$. If $\mathcal{C}$ has coproducts, then $K \cap A$ is also a coend of the functor

$$\Delta^{\mathrm{op}} \times \Delta \longrightarrow \mathcal{C}, \quad ([n],[m]) \longmapsto K_n \times A^m,$$

where $K_n \times A^m$ denotes a coproduct of copies of $A^m$ indexed by the set $K_n$.

Example 3.1. By the Yoneda lemma, the object $A^n$ represents the functor $\Delta\text{-set}(\Delta[n], \mathcal{C}(A,-))$. So, for every co-$\Delta$-object $A$, we can, and will, take $\Delta[n] \cap A = A^n$ with respect to the structure morphisms of $A$.

The product of a co-$\Delta$-object with the boundary $\partial\Delta[n]$ of a simplex will play an important role in the following, so we use the special notation

$$\partial^n A = \partial\Delta[n] \cap A$$

and refer to this object as the $n$th *latching object* of a co-$\Delta$-object $A$. The simplex category $s(\partial\Delta[n])$ is the full subcategory of the over category $\Delta \downarrow [n]$ with objects all $\alpha : [i] \to [n]$ for $i < n$. So $\partial^n A$, if it exists, is a colimit of the functor $s(\partial\Delta[n]) \to \mathcal{C}$ that sends $\alpha : [i] \to [n]$ to $A^i$. For example, $\partial^0 A$ is an initial object and $\partial^1 A$ is a coproduct of two copies of $A^0$.

The pairing $(K, A) \mapsto K \cap A$ extends to a functor in two variables in a rather formal way, whenever the representing objects exist. Indeed, if $\lambda : K \to L$ is a morphism of $\Delta$-sets and $f : A \to B$ is a morphism of co-$\Delta$-objects, then precomposition with $\lambda$ and $f$ is a natural transformation of set-valued functors on $\mathcal{C}$

$$\Delta\text{-set}(\lambda, \mathcal{C}(f,-)) \;:\; \Delta\text{-set}(L, \mathcal{C}(B,-)) \longrightarrow \Delta\text{-set}(K, \mathcal{C}(A,-)).$$

If $L \cap B$, respectively, $K \cap A$ represent these two functors, then the Yoneda lemma provides a unique $\mathcal{C}$-morphism $\lambda \cap f : K \cap A \to L \cap B$ that represents the transformation $\Delta\text{-set}(\lambda, \mathcal{C}(f,-))$.

Definition 3.2. A co-$\Delta$-object $A$ in a cofibration category $\mathcal{C}$ is *cofibrant* if, for every $n \geqslant 0$, the latching object $\partial^n A$ exists and the canonical morphism $\nu^n : \partial^n A \to A^n$ is a cofibration. A morphism $f : A \to B$ of cofibrant co-$\Delta$-objects is a *cofibration* if, for every $n \geqslant 0$, the morphism

$$f^n \cup \nu^n \;:\; A^n \cup_{\partial^n A} \partial^n B \longrightarrow B^n$$

is a cofibration. A morphism $f : A \to B$ of co-$\Delta$-objects is a *level equivalence* if $f^n : A^n \to B^n$ is a weak equivalence for all $n \geqslant 0$.

In the definition of cofibrations, $A^n \cup_{\partial^n A} \partial^n B$ denotes a pushout of the diagram

$$A^n \xleftarrow{\nu^n} \partial^n A \xrightarrow{\partial^n f} \partial^n B,$$

which exists since $\nu^n$ is a cofibration.

Proposition 3.3. *Let $K$ be a finite $\Delta$-set and $\mathcal{C}$ be a cofibration category. Then, for every cofibrant co-$\Delta$-object $A$ in $\mathcal{C}$, the object $K \cap A$ exists in $\mathcal{C}$. Moreover, the functor $K \cap -$ takes cofibrations between cofibrant co-$\Delta$-objects to cofibrations in $\mathcal{C}$, and it takes acyclic cofibrations between cofibrant co-$\Delta$-objects to acyclic cofibrations in $\mathcal{C}$.*

*Proof.* We argue by induction over the dimension of $K$. If $K$ is empty (that is, $(-1)$-dimensional), then any initial object of $\mathcal{C}$ represents the functor $\Delta$-sets$(K, \mathcal{C}(A, -))$, and can thus be taken as $\emptyset \cap A$. Hence, for every morphism $f : A \to B$ of co-$\Delta$-objects, the morphism $\emptyset \cap f$ is an isomorphism, thus an acyclic cofibration.

Now suppose that $n \geqslant 0$ and we have established the proposition for all finite $\Delta$-sets of dimension less than $n$. We claim first that, for every morphism $f : A \to B$ between cofibrant co-$\Delta$-objects that is a cofibration and a level equivalence, the morphism $f^n \cup \nu^n : A^n \cup_{\partial^n A} \partial^n B \to B^n$ is an acyclic cofibration. Since $\partial^n A = \partial\Delta[n] \cap A$ and $\partial\Delta[n]$ has dimension $n-1$, the morphism $\partial^n f : \partial^n A \to \partial^n B$ is an acyclic cofibration by induction. So its cobase change $A^n \to A^n \cup_{\partial^n A} \partial^n B$ is an acyclic cofibration. Since $f^n : A^n \to B^n$ is a weak equivalence, the morphism $f^n \cup \nu^n : A^n \cup_{\partial^n A} \partial^n B \to B^n$ is a weak equivalence, hence an acyclic cofibration.

Now suppose that $K$ is $n$-dimensional. We can write $K$ as a pushout

$$\begin{tikzcd}
K_n \times \partial\Delta[n] \arrow[r] \arrow[d] & K_n \times \Delta[n] \arrow[d] \\
K' \arrow[r] & K
\end{tikzcd}$$

where $K'$ is the $(n-1)$-skeleton of $K$. By induction there is a representing object $K' \cap A$ for the functor $\Delta$-sets$(K', \mathcal{C}(A, -))$. The latching object $\partial^n A$ represents the functor $\Delta$-sets$(\partial\Delta[n], \mathcal{C}(A, -))$ and $A^n$ represents the functor $\Delta$-sets$(\Delta[n], \mathcal{C}(A, -))$. Moreover, the morphism $\nu^n : \partial^n A \to A^n$ is a cofibration since $A$ is cofibrant, and hence so is a finite coproduct of copies of $\nu^n$. So any pushout in $\mathcal{C}$

$$\begin{tikzcd}
K_n \times \partial^n A \arrow[r, "K_n \times \nu^n"] \arrow[d] & K_n \times A^n \arrow[d] \\
K' \cap A \arrow[r] & K \cap A
\end{tikzcd}$$

can serve as the object $K \cap A$. Here, and in the rest of the proof, we write $K_n \times X$ for a coproduct, indexed by the finite set $K_n$ of copies of an object $X$.

Now we let $f : A \to B$ be a cofibration between cofibrant co-$\Delta$-objects. The morphism $K \cap f : K \cap A \to K \cap B$ is obtained by passage to horizontal pushouts from the commutative diagram:

$$\begin{tikzcd}
K_n \times A^n \arrow[d, "K_n \times f^n"'] & K_n \times \partial^n A \arrow[l, "K_n \times \nu^n"'] \arrow[r] \arrow[d, "K_n \times \partial^n f"] & K' \cap A \arrow[d, "K' \cap f"] \\
K_n \times B^n & K_n \times \partial^n B \arrow[l, "K_n \times \nu^n"] \arrow[r] & K' \cap B.
\end{tikzcd}$$

This induced map on pushouts factors as the composite

$$K_n \times A^n \cup_{K_n \times \partial^n A} (K' \cap A) \xrightarrow{\mathrm{Id} \cup (K' \cap f)} K_n \times A^n \cup_{K_n \times \partial^n A} (K' \cap B)$$
$$\xrightarrow{(K_n \times f^n) \cup \mathrm{Id}} K_n \times B^n \cup_{K_n \times \partial^n B} (K' \cap B).$$

The first map is a cobase change of $K' \cap f$, which is a cofibration (and a weak equivalence if $f$ is also a level equivalence) by induction. The second map is a cobase change of a finite coproduct of copies of $f^n \cup \nu^n : A^n \cup_{\partial^n A} \partial^n B \to B^n$, which is a cofibration by hypothesis on $f$. If $f$ is also a level equivalence, then $f^n \cup \nu^n$, and hence the second map is an acyclic cofibration by the above claim. So the composite map $K \cap f$ is a cofibration (respectively, acyclic cofibration) as the composite of two cofibrations (respectively, acyclic cofibrations). □

Another purely formal consequence of the definition is that the $\cap$-pairing preserves colimits in both variables. We emphasize that the next proposition does not claim the existence of any kind of colimits; it only says that if a certain colimit exists in the category of cofibrant co-$\Delta$-objects (respectively, the category of finite $\Delta$-sets), then the functor $K \cap -$ (respectively, $- \cap A$) preserves the colimit.

Proposition 3.4. *For every finite $\Delta$-set $K$ the functor $K \cap -$ takes colimits in the category of cofibrant co-$\Delta$-objects to colimits in $\mathcal{C}$. For every cofibrant co-$\Delta$-object $A$ the functor $- \cap A$ takes colimits in the category of finite $\Delta$-sets to colimits in $\mathcal{C}$.*

*Proof.* We show the claim for the functor $- \cap A$, the other case being analogous. Let $I$ be a small category and $F$ be a functor from $I$ to the category of finite $\Delta$-sets that has a colimit $\mathrm{colim}_I F$ (inside the category of finite $\Delta$-sets). We denote by $F \cap A : I \to \mathcal{C}$ the functor given by capping $F$ objectwise with $A$. The representability property of the cap product and the universal property of a colimit combine into natural isomorphisms

$$\mathcal{C}((\mathrm{colim}_I F) \cap A, -) \cong \Delta\text{-set}(\mathrm{colim}_I F, \mathcal{C}(A, -)) \cong \lim_I \Delta\text{-set}(F, \mathcal{C}(A, -))$$
$$\cong \lim_I \mathcal{C}(F \cap A, -).$$

So $(\mathrm{colim}_I F) \cap A$ has the universal property of a colimit of $F \cap A$. □

Proposition 3.5. *Let $i : K \to L$ be a monomorphism between finite $\Delta$-sets and $j : A \to B$ a cofibration between cofibrant co-$\Delta$-objects in $\mathcal{C}$.*

(i) *The pushout of the diagram*

$$L \cap A \xleftarrow{i \cap A} K \cap A \xrightarrow{K \cap j} K \cap B$$

*exists in $\mathcal{C}$ and the pushout product morphism*

$$(L \cap j) \cup (i \cap B) \,:\, (L \cap A) \cup_{(K \cap A)} (K \cap B) \longrightarrow L \cap B$$

*is a cofibration.*

(ii) *If moreover $j$ is a level equivalence, then $(L \cap j) \cup (i \cap B)$ is a weak equivalence in $\mathcal{C}$.*

*Proof.* (i) The morphism $K \cap j$ is a cofibration by Proposition 3.3, so the pushout exists by axiom (C3). For the proof that $(L \cap j) \cup (i \cap B)$ is a cofibration we argue by induction on the number of simplices of $L$ that are not in the image of $i$. If $i$ is bijective, then $(L \cap j) \cup (i \cap B)$ is an isomorphism. Otherwise we choose a sub-$\Delta$-set $L'$ of $L$ such that $i(K) \subseteq L'$ and such that $L'$

has one simplex less than $L$. The morphism $(L \cap j) \cup (i \cap B)$ then factors as the composite

$$(L \cap A) \cup_{(K \cap A)} (K \cap B) \xrightarrow{(L \cap A) \cup (i \cap B)} (L \cap A) \cup_{(L' \cap A)} (L' \cap B) \xrightarrow{(L \cap j) \cup (\mathrm{incl} \cap B)} L \cap B.$$

The first map is a cobase change of

$$(L' \cap j) \cup (i \cap B) : (L' \cap A) \cup_{(K \cap A)} (K \cap B) \longrightarrow L' \cap B,$$

which is a cofibration by induction. The second map is a cobase change of the cofibration $f^n \cup \nu^n : A^n \cup_{\partial^n A} \partial^n B \to B^n$, where $n$ is the dimension of the simplex not in $L'$. So $(L \cap j) \cup (i \cap B)$ is the composite of two cofibrations, hence is itself a cofibration.

(ii) The morphism $K \cap j : K \cap A \to K \cap B$ is an acyclic cofibration by Proposition 3.3. Hence its cobase change $\psi : L \cap A \to (L \cap A) \cup_{(K \cap A)} (K \cap B)$ is an acyclic cofibration. Also by Proposition 3.3, the morphism $L \cap j : L \cap A \to L \cap B$ is an acyclic cofibration. Since $((L \cap j) \cup (i \cap B)) \circ \psi = L \cap j$, the pushout product map is a weak equivalence by the 2-out-of-3 property. □

Definition 3.6. A *frame* in a cofibration category is a cofibrant co-$\Delta$-object $A$ that is homotopically constant, that is, for every morphism $\alpha : [n] \to [m]$ in the category $\Delta$ the morphism $\alpha_* : A^n \to A^m$ is a weak equivalence.

A cofibration category $\mathcal{C}$ is *saturated* if every morphism that becomes an isomorphism under the localization functor $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ is already a weak equivalence. Saturation is no serious restriction since the weak equivalences in any cofibration category can be saturated without changing the cofibrations or the homotopy category (see [6, Proposition 3.16]). If $A$ is a frame, then the functor $- \cap A$ tries hard to turn weak equivalences of finite $\Delta$-sets into weak equivalences in $\mathcal{C}$. This does not work in complete generality, but the next proposition shows, among other things, that $- \cap A$ has this property in all saturated cofibration categories.

Proposition 3.7. *Let $A$ be a frame in a cofibration category $\mathcal{C}$.*

(i) *For every elementary expansion $K \subset L$ the morphism $K \cap A \to L \cap A$ induced by the inclusion is an acyclic cofibration.*

(ii) *If the cofibration category $\mathcal{C}$ is saturated, then the functor $- \cap A$ takes weak equivalences between finite $\Delta$-sets to weak equivalences in $\mathcal{C}$.*

*Proof.* (i) We proceed by induction on the dimension $n$ of the elementary expansion. For $n = 1$ the $\Delta$-set $\Lambda^i[1]$ is isomorphic to $\Delta[0]$ and the inclusion $\Lambda^i[0] \to \Delta[1]$ corresponds to the face map $d_{1-i} : \Delta[0] \to \Delta[1]$. So the map in question is isomorphic to $(d_{1-i})_* : A^0 \to A^1$, hence an acyclic cofibration.

Now we suppose that $n \geqslant 2$. We start with the special (but universal) cases of the horn inclusions $\Lambda^i[n] \to \Delta[n]$ for $0 \leqslant i \leqslant n$. We let $v : [0] \to [n]$ be the map with $v(0) = 0$. The inclusion $\{v\} \to \Lambda^i[n]$ is the composite of a sequence of elementary expansions of dimensions strictly less than $n$. So the induced morphism $\{v\} \cap A \to \Lambda^i[n] \cap A$ is a weak equivalence by induction. The composite of this weak equivalence with the morphism $\Lambda^i[n] \cap A \to \Delta[n] \cap A$ is isomorphic to the structure morphism $v_* : A^0 \to A^n$ and hence a weak equivalence. So the map $\Lambda^i[n] \cap A \to \Delta[n] \cap A$ is also a weak equivalence. This map is also a cofibration by Proposition 3.3, hence an acyclic cofibration. If $K \subset L$ is a general elementary expansion of dimension $n$, then the pushout square (2.3) caps with $A$ to a pushout square in $\mathcal{C}$:

$$\begin{tikzcd} \Lambda^i[n] \cap A \arrow[r] \arrow[d] & \Delta[n] \cap A \arrow[d] \\ K \cap A \arrow[r] & L \cap A. \end{tikzcd}$$

As a cobase change of an acyclic cofibration, the morphism $K \cap A \to L \cap A$ is itself an acyclic cofibration. Part (ii) follows from part (i) by applying Proposition 2.6 to the functor $F = \gamma \circ (- \cap A)$. □

The following proposition is a special case of [**26**, Theorem 9.2.4(1a)] where the indexing category is the direct category $\Delta$. More precisely, part (i) is Radulescu-Banu's axiom (CF4) saying that every morphism $f : A \to B$ from a cofibrant object can be factored as $f = pi$ with $i$ a cofibration and $p$ a pointwise weak equivalence.

Proposition 3.8. *Let $\mathcal{C}$ be a cofibration category.*

(i) *Let $f : A \to Z$ be a morphism of co-$\Delta$-objects in $\mathcal{C}$ such that $A$ is cofibrant. Then there exists a cofibrant co-$\Delta$-object $B$ and a factorization $f = pi$ as a cofibration $i : A \to B$ followed by a level equivalence $p : B \to Z$.*

(ii) *The cofibrations and level equivalences make the category of cofibrant co-$\Delta$-objects into a cofibration category.*

The next proposition is the key step in the proof that the homotopy category of frames in $\mathcal{C}$ is equivalent to the homotopy category of $\mathcal{C}$; see Theorem 3.10.

Proposition 3.9. *Let $A$ be a frame in a cofibration category $\mathcal{C}$, $Z$ be an object of $\mathcal{C}$ and $\varphi : A^0 \to Z$ be a $\mathcal{C}$-morphism. Then there is a homotopically constant co-$\Delta$-object $Y$, a morphism $y : A \to Y$ of co-$\Delta$-objects and a weak equivalence $p : Z \to Y^0$ such that $y^0 = p\varphi$.*

*Proof.* We denote by $P(n)$ the $n$-dimensional $\Delta$-set with a unique $i$-simplex $x_i$ for $i = 0, \dots, n$ and we let $C(n)$ be the cone (see Example 2.4) of $P(n)$. We start by choosing a pushout:

$$\begin{tikzcd} A^0 \arrow[r, "\varphi"] \arrow[d, "(x_0)_*"', "\sim"] & Z \arrow[d, "\sim"', "p"] \\ C(0) \cap A \arrow[r, "\lambda_0"'] & Y^0. \end{tikzcd}$$

The inclusion $\{x_0\} \to C(0)$ is an elementary expansion, so $(x_0)_* : A^0 \cong \{x_0\} \cap A \to C(0) \cap A$ is an acyclic cofibration by Proposition 3.7(i). Then we proceed by induction on $n$ and define $Y^n$ as a pushout:

$$\begin{tikzcd} C(n-1) \cap A \arrow[r, "\lambda_{n-1}"] \arrow[d, "\sim"'] & Y^{n-1} \arrow[d, "i_n"] \\ C(n) \cap A \arrow[r, "\lambda_n"'] & Y^n. \end{tikzcd}$$

The inclusion $C(n-1) \to C(n)$ is an elementary expansion, so the left vertical map is an acyclic cofibration, again by Proposition 3.7(i). For $0 \leqslant j \leqslant n$, we define the structure map $d_j : Y^{n-1} \to Y^n$ as the cobase change $i_n$ of $C(n-1) \cap A \to C(n) \cap A$. So $d_j$ is independent of $j$ and an acyclic cofibration. The co-$\Delta$-object $Y$ is thus homotopically constant (but usually not cofibrant). Because $x_{n+1} d_j = x_n$ the composite maps

$$A^n \xrightarrow{(x_n)_*} C(n) \cap A \xrightarrow{\lambda_n} Y^n$$

constitute a morphism of co-$\Delta$-objects $y : A \to Y$. Moreover, $y^0 = \lambda_0 \circ (x_0)_* = p\varphi$. □

Now we can prove the main result of this section.

THEOREM 3.10. *Let $\mathcal{C}$ be a cofibration category. Then the category $f\mathcal{C}$ of frames in $\mathcal{C}$ forms a cofibration category with respect to cofibrations and level equivalences. The functor $f\mathcal{C} \to \mathcal{C}$ that evaluates in dimension 0 is exact and its derived functor is an equivalence of categories from $\mathbf{Ho}(f\mathcal{C})$ to $\mathbf{Ho}(\mathcal{C})$.*

*Proof.* Frames are closed under level equivalences within the category of cofibrant co-$\Delta$-objects. Also, the initial co-$\Delta$-object is a frame, and for every pushout square (1.2) in which $A, B$ and $C$ are frames, the pushout again is homotopically constant by the gluing lemma. So the class of frames is also closed under pushouts along cofibrations, and hence forms a cofibration category.

In order to show that the derived functor of evaluation in dimension 0 is an equivalence of categories, we use the criterion given by the 'approximation theorem' [**6**, Theorem 3.12]. The necessary hypotheses are satisfied: a morphism of $f : A \to B$ of homotopically constant co-$\Delta$-objects is a level equivalence if and only if $f^0 : A^0 \to B^0$ is a weak equivalence, that is, the evaluation functor satisfies the approximation property (AP1) of [**6**, 3.6]. The second condition (AP2) demands that, for every frame $A$, every $\mathcal{C}$-object $Z$ and every morphism $\varphi : A^0 \to Z$ there should be a frame $B$, a cofibration $i : A \to B$ and weak equivalences $q^0 : B^0 \to Y^0$ and $p : Z \to Y^0$ such that $q^0 i^0 = p\varphi$. Indeed, Proposition 3.9 provides a morphism $y : A \to Y$ of co-$\Delta$-objects, with $Y$ homotopically constant, and a weak equivalence $p : Z \to Y^0$ such that $y^0 = p\varphi$. Proposition 3.8(i) provides a factorization $y = qi$ for a cofibration $i : A \to B$ followed by a level equivalence $q : B \to Y$. Then $B$ is cofibrant (since $A$ is cofibrant and $i$ is a cofibration) and homotopically constant (since $Y$ is homotopically constant and $q$ is a level equivalence). So $B$ is a frame. Moreover, the morphism $q^0$ is a weak equivalence and satisfies $q^0 i^0 = y^0 = p\varphi$. □

REMARK 3.11. As a combination of the previous results we have effectively constructed a natural action of the homotopy category of finite CW-complexes on the homotopy category of any cofibration category. In more detail, the composite functor

$$\Delta\text{-sets}^{\text{fin}} \times f\mathcal{C} \xrightarrow{\cap} \mathcal{C} \xrightarrow{\gamma} \mathbf{Ho}(\mathcal{C})$$

takes weak equivalences of finite $\Delta$-sets and level equivalences of frames to isomorphisms (by Propositions 3.7(ii) and 3.5(ii)). So the functor factors over the localization of the left-hand side through a unique functor

$$\cap^L : \mathbf{Ho}(\Delta\text{-sets}^{\text{fin}}) \times \mathbf{Ho}(f\mathcal{C}) \longrightarrow \mathbf{Ho}(\mathcal{C}). \tag{3.12}$$

Here we denote by $\mathbf{Ho}(\Delta\text{-sets}^{\text{fin}})$ a localization of the category of finite $\Delta$-sets at the class of weak equivalences. One should beware though that finite $\Delta$-sets do *not* form a cofibration category (the factorization axiom (C4) fails), but such a localization can be constructed 'by hand', for example by setting

$$\mathbf{Ho}(\Delta\text{-sets}^{\mathrm{fin}})(K, L) = [|K|, |L|],$$

the set of homotopy classes of continuous maps between the geometric realizations of $K$ and $L$. In fact, the homotopy category $\mathbf{Ho}(\Delta\text{-sets}^{\mathrm{fin}})$ is equivalent, via the functor of geometric realization, to the homotopy category (in the traditional sense) of finite CW-complexes; see, for example, [**5**, Chapter I, Theorem 4.3].

The functor (3.12) is not quite an action of $\mathbf{Ho}(\Delta\text{-sets}^{\mathrm{fin}})$ on $\mathbf{Ho}(\mathcal{C})$ yet, but we can fix that by choosing a 'framing' of $\mathcal{C}$, that is, an inverse $\mathcal{F} : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(f\mathcal{C})$ to the equivalence of Theorem 3.10. One can then show that the composite functor

$$\mathbf{Ho}(\Delta\text{-sets}^{\mathrm{fin}}) \times \mathbf{Ho}(\mathcal{C}) \xrightarrow{\mathrm{Id} \times \mathcal{F}} \mathbf{Ho}(\Delta\text{-sets}^{\mathrm{fin}}) \times \mathbf{Ho}(f\mathcal{C}) \xrightarrow{\cap^L} \mathbf{Ho}(\mathcal{C})$$

is coherently associative and unital with respect to the (derived) geometric product of $\Delta$-sets, and it is natural for exact functors of cofibration categories. We shall not elaborate on this point since we do not need it in the present paper.

There is now a standard way of extending the $\cap$-pairing to a pairing $(K, A) \mapsto K \otimes A$ that takes a finite $\Delta$-set $K$ and a co-$\Delta$-object $A$ to another co-$\Delta$-object $K \otimes A$. In dimension $n$ we set

$$(K \otimes A)^n = (\Delta[n] \otimes K) \cap A, \tag{3.13}$$

where the tensor symbol on the right-hand side is the geometric product of $\Delta$-sets. The structure maps arise via the functoriality in $\Delta[-]$.

The tensor product construction comes with coherent natural isomorphisms

$$K \cap (L \otimes A) \cong (K \otimes L) \cap A \quad \text{and} \quad K \otimes (L \otimes A) \cong (K \otimes L) \otimes A \tag{3.14}$$

in the category $\mathcal{C}$ and the category of co-$\Delta$-objects in $\mathcal{C}$, respectively. Indeed, given a simplex $x \in K_i$ we let $\bar{x} : \Delta[i] \to K$ be the morphism with $\bar{x}(\mathrm{Id}_{[i]}) = x$. As $i$ and $x$ vary, the morphisms

$$x_* = (\bar{x} \otimes L) \cap A \,:\, (L \otimes A)^i = (\Delta[i] \otimes L) \cap A \longrightarrow (K \otimes L) \cap A$$

are compatible, so the universal property of $K \cap (L \otimes A)$ provides a morphism

$$\psi \,:\, K \cap (L \otimes A) \longrightarrow (K \otimes L) \cap A.$$

In the other direction we consider a triple $(x, y; \varphi)$ with $x \in K_i$, $y \in L_j$ and $\varphi : [n] \to [i] \times [j]$ a monotone injection. Then $[\mathrm{Id}_{[i]}, y; \varphi]$ is an $n$-simplex of $\Delta[i] \otimes L$, so we can form the composite

$$A^n \xrightarrow{[\mathrm{Id}_{[i]}, y; \varphi]_*} (\Delta[i] \otimes L) \cap A = (L \otimes A)^i \xrightarrow{x_*} K \cap (L \otimes A),$$

which in fact only depends on the equivalence class $[x, y; \varphi]$ in $(K \otimes L)_n$. As the class $[x, y; \varphi]$ varies, the maps are again compatible, so the universal property of $(K \otimes L) \cap A$ provides a morphism

$$\bar{\psi} \,:\, (K \otimes L) \cap A \longrightarrow K \cap (L \otimes A).$$

The morphisms $\psi$ and $\bar{\psi}$ are natural in all three variables and are mutually inverse isomorphisms. The second isomorphism in (3.14) is then given in dimension $n$ by the composite

$$\begin{aligned}(K \otimes (L \otimes A))^n &= (\Delta[n] \otimes K) \cap (L \otimes A) \cong ((\Delta[n] \otimes K) \otimes L) \cap A \\ &\cong (\Delta[n] \otimes (K \otimes L)) \cap A = ((K \otimes L) \otimes A)^n\end{aligned}$$

that combines the isomorphism $\psi$ with the associativity isomorphism $(\Delta[n] \otimes K) \otimes L \cong \Delta[n] \otimes (K \otimes L)$ of the geometric product of $\Delta$-sets.

Definition 3.15. A $\Delta$-*cofibration category* is a cofibration category $\mathcal{C}$ equipped with a pairing

$$\Delta\text{-sets}^{\text{fin}} \times \mathcal{C} \longrightarrow \mathcal{C}, \quad (K, X) \longmapsto K \otimes X$$

that is coherently associative and unital with respect to the geometric product of $\Delta$-sets and satisfies the following properties.

(1) For every finite $\Delta$-set $K$ the functor $K \otimes - : \mathcal{C} \to \mathcal{C}$ is exact.
(2) For every object $A$ of $\mathcal{C}$ the functor $- \otimes A : \Delta\text{-sets}^{\text{fin}} \to \mathcal{C}$ is exact.
(3) Let $i : K \to L$ be a monomorphism of finite $\Delta$-sets and $j : A \to B$ be a cofibration in $\mathcal{C}$. Then the pushout product morphism

$$(i \otimes B) \cup (L \otimes j) \;:\; K \otimes B \cup_{K \otimes A} L \otimes A \longrightarrow L \otimes B$$

is a cofibration.

Remark 3.16. Suppose that $\mathcal{C}$ is a $\Delta$-cofibration category, $i : K \to L$ is a monomorphism and $j : A \to B$ is a cofibration as in the pushout product property. If in addition $i$ or $j$ is a weak equivalence, then the morphism

$$(i \otimes B) \cup (L \otimes j) \;:\; K \otimes B \cup_{K \otimes A} L \otimes A \longrightarrow L \otimes B$$

is also a weak equivalence. Indeed, if $i$ is also weak equivalence, then so is $i \otimes A$ (since $\otimes A$ is exact), and similarly for $i \otimes B$. So $i \otimes A : K \otimes A \to L \otimes A$ is an acyclic cofibration, and hence so is its cobase change $K \otimes B \to K \otimes B \cup_{K \otimes A} L \otimes A$. The composite of this weak equivalence with $(i \otimes B) \cup (L \otimes j)$ is the weak equivalence $i \otimes B$, so $(i \otimes B) \cup (L \otimes j)$ is a weak equivalence, as claimed. The argument is analogous when $j$ is a weak equivalence.

Theorem 3.17. *Let $\mathcal{C}$ be a saturated cofibration category. Then the $\otimes$-product* (3.13) *makes the category of frames $f\mathcal{C}$ into a $\Delta$-cofibration category.*

*Proof.* We start by showing that, for every finite $\Delta$-set $K$ and every frame $A$, the co-$\Delta$-object $K \otimes A$ is again a frame. Since $A$ is cofibrant, the object $(\partial\Delta[n] \otimes K) \cap A$ exists; by (3.14), this object is isomorphic to

$$\partial\Delta[n] \cap (K \otimes A) = \partial^n (K \otimes A),$$

so the $n$th latching object of $K \otimes A$ exists. The latching morphism $\partial^n(K \otimes A) \to (K \otimes A)^n$ is induced by the inclusion $\partial\Delta[n] \to \Delta[n]$, and thus is isomorphic to $(\text{incl} \otimes K) \cap A : (\partial\Delta[n] \otimes K) \cap A \to (\Delta[n] \otimes K) \cap A$; this is a cofibration by Proposition 3.5(i). In other words, if $A$ is cofibrant, then so is $K \otimes A$.

If $A$ is a frame, then $K \otimes A$ is also homotopically constant: for every morphism $\alpha : [n] \to [m]$ in the category $\Delta$ the induced morphisms $\alpha_* : \Delta[n] \to \Delta[m]$ and $\alpha_* \otimes K : \Delta[n] \otimes K \to \Delta[m] \otimes K$ are sequences of elementary expansions, and so, by Proposition 3.7(i), the induced map

$$(K \otimes A)^n = (\Delta[n] \otimes K) \cap A \longrightarrow (\Delta[m] \otimes K) \cap A = (K \otimes A)^m$$

is an acyclic cofibration.

The $\otimes$-pairing preserves all existing colimits in both variables, in particular initial objects and pushouts along cofibrations. Indeed, colimits in functor categories are objectwise, so we must show that, for every $n \geqslant 0$, the functor $(K, A) \mapsto (\Delta[n] \otimes K) \cap A$ takes all colimits in $K$ and $A$ to colimits in $\mathcal{C}$. Since the geometric product $\Delta[n] \otimes -$ preserves colimits of finite

$\Delta$-sets, this follows from the fact that the $\cap$-pairing preserves colimits in both variables; see Proposition 3.4.

For the pushout product property we consider an inclusion $i : K \to L$ of finite $\Delta$-sets and a cofibration $j : A \to B$ between frames in $\mathcal{C}$. To shorten the notation, we write

$$\langle i, j\rangle = (i \otimes B) \cup (L \otimes j) \,:\, K \otimes B \cup_{K\otimes A} L \otimes A \longrightarrow L \otimes B$$

for a pushout product map. We then have to show that $\langle i, j\rangle$ is a cofibration of co-$\Delta$-objects, and that in turn means showing that the morphism

$$\langle i, j\rangle^n \cup \nu^n \,:\, (K \otimes B \cup_{K\otimes A} L \otimes A)^n \cup_{\partial^n(K\otimes B\cup_{K\otimes A} L\otimes A)} \partial^n(L \otimes B) \longrightarrow (L \otimes B)^n$$

is a cofibration in $\mathcal{C}$. The isomorphism (3.14) and a rearranging of pushouts translates this into the claim that the morphism

$$\langle k, j\rangle \colon (\Delta[n] \otimes K \cup \partial\Delta[n] \otimes L) \cap B \cup_{(\Delta[n]\otimes K\cup\partial\Delta[n]\otimes L)\cap A} (\Delta[n] \otimes L) \cap A \longrightarrow (\Delta[n] \otimes L) \cap B$$

is a cofibration in $\mathcal{C}$, where

$$k \,:\, \Delta[n] \otimes K \cup \partial\Delta[n] \otimes L \longrightarrow \Delta[n] \otimes L$$

is the inclusion (the union in the source is along the intersection $\partial\Delta[n] \otimes K$). So $\langle k, j\rangle$ is a cofibration in $\mathcal{C}$ by Proposition 3.5(i). This completes the proof of the pushout product property.

In the special case when $K$ is the empty $\Delta$-set, the pushout product property shows that the functor $L \otimes -$ preserves cofibrations. Since $(L \otimes A)^n = (\Delta[n] \otimes L) \cap A$, Proposition 3.5(ii) shows that $L \otimes -$ preserves level equivalences.

In the special case when $A$ is the initial object the pushout product property shows that the functor $- \otimes B$ takes monomorphisms to cofibrations. If $f : K \to L$ is a weak equivalence, then so is $\Delta[n] \otimes f : \Delta[n] \otimes K \to \Delta[n] \otimes L$. So the morphism $(f \otimes B)^n = (\Delta[n] \otimes f) \cap B$ is a weak equivalence for every $n \geqslant 0$, by Proposition 3.7 (ii), that is, $- \otimes B$ takes weak equivalences to level equivalences. □

Remark 3.18. If $\mathcal{C}$ has arbitrary coproducts and colimits of sequences of cofibrations, then the functor $\Delta\text{-sets}(K, \mathcal{C}(A, -))$ is representable for every $\Delta$-set $K$. If coproducts and sequential colimits are suitably compatible with cofibrations and weak equivalences, then most of the results of this section carry over from finite to arbitrary $\Delta$-sets.

## 4. *Pointed $\Delta$-cofibration categories*

In this section, we introduce and study actions of based $\Delta$-sets on objects of a pointed $\Delta$-cofibration category. A *based $\Delta$-set* is a contravariant functor from the category $\Delta$ to the category of based sets. So a based $\Delta$-set is a $\Delta$-set equipped with a distinguished basepoint in every dimension, preserved under the face maps. One should beware that, in contrast to the world of simplicial sets, the 0-simplex $\Delta[0]$ is *not* a terminal object in the category of $\Delta$-sets. So, specifying a vertex in a $\Delta$-set does *not* determine a morphism from the terminal $\Delta$-set (which has exactly one simplex in every dimension), and does *not* make a $\Delta$-set based. A based $\Delta$-set is non-empty in every dimension, so it is never finite-dimensional, and never finite.

Now we discuss how to pair objects in a pointed $\Delta$-cofibration category $\mathcal{C}$ with based $\Delta$-sets. If $X$ and $Z$ are objects of $\mathcal{C}$, we can define a $\Delta$-set $\mathrm{map}(X, Z)$ as the composite functor

$$\Delta^{\mathrm{op}} \xrightarrow{[n]\mapsto\Delta[n]} (\Delta\text{-sets})^{\mathrm{op}} \xrightarrow{\mathcal{C}(-\otimes X, Z)} (\text{sets}).$$

Since $\mathcal{C}$ is pointed, this $\Delta$-set is canonically based: the basepoint of $\mathrm{map}(X, Z)_n$ is the zero map from $\Delta[n] \otimes X$ to $Z$. If $K$ is a based $\Delta$-set, then we denote by $K \wedge X$ a representing object, if it exists, of the functor

$$\mathcal{C} \longrightarrow (\text{sets}), \quad Z \longmapsto \Delta\text{-set}_*(K, \text{map}(X, Z)),$$

where the right-hand side is the set of morphisms of based $\Delta$-sets from $K$ to $\text{map}(X, Z)$.

The smash product $K \wedge X$ plays the role of a quotient of $K \otimes X$ by $* \otimes X$ where $* \subseteq K$ denotes the sub-$\Delta$-set consisting only of the various basepoints in all dimensions. However, this is not literally true, because $K$ and $*$ are infinite $\Delta$-sets, and so the expressions $K \otimes X$ and $* \otimes X$ do not individually make sense in a $\Delta$-cofibration category. The following concepts of 'essentially finite' based $\Delta$-sets and 'finite approximation' allow us to deal with the fact that based $\Delta$-sets are never finite.

Definition 4.1. A based $\Delta$-set is *essentially finite* if it has only finitely many non-basepoint simplices. A *finite presentation* of a based $\Delta$-set $K$ is a morphism $r : R \to K$ of (unbased) $\Delta$-sets such that $R$ is finite (in the absolute sense) and every non-basepoint simplex of $K$ has exactly one preimage under $r$.

In other words, a finite presentation is an 'isomorphism away from the basepoints'. So, if a based $\Delta$-set has a finite presentation, then it must be essentially finite. A based $\Delta$-set $K$ is essentially finite if and only if each $K_n$ is finite and almost all $K_n$ consist only of the basepoint. So, if $m$ is the maximum of the dimensions of the non-basepoint simplices, then the inclusion $K^{(m)} \to K$ of the $m$-skeleton is a finite presentation. (This particular finite presentation is also injective, but that is not required of finite presentations in general.) We conclude that a based $\Delta$-set has a finite presentation if and only if it is essentially finite.

Proposition 4.2. *Let $r : R \to K$ be a finite presentation of a based $\Delta$-set and denote by $r^{-1}(*)$ the sub-$\Delta$-set of $R$ consisting of all preimages of the respective basepoints. Then, for every object $X$ of a pointed $\Delta$-cofibration category, every cokernel of the morphism*

$$\iota \otimes X \;:\; r^{-1}(*) \otimes X \longrightarrow R \otimes X$$

*is an object $K \wedge X$, where $\iota : r^{-1}(*) \to R$ is the inclusion. In particular, the object $K \wedge X$ exists for every essentially finite based $\Delta$-set $K$.*

*Proof.* We let $Y$ be another based $\Delta$-set. Then precomposition with $r$ induces a bijection from the set $\Delta\text{-set}_*(K, Y)$ to the subset of $\Delta\text{-set}(R, Y)$ consisting of those morphisms $\varphi : R \to Y$ that send the entire sub-$\Delta$-set $r^{-1}(*)$ to the basepoints. In the special case of the based $\Delta$-set $Y = \text{map}(X, Z)$ this shows that $\Delta\text{-set}_*(K, \text{map}(X, Z))$ is the fiber, over the constant morphism with zero map values, of the restriction map

$$\Delta\text{-set}(R, \text{map}(X, Z)) \longrightarrow \Delta\text{-set}(r^{-1}(*), \text{map}(X, Z)).$$

This restriction map is isomorphic to

$$\mathcal{C}(\iota \otimes X, Z) : \mathcal{C}(R \otimes X, Z) \longrightarrow \mathcal{C}(r^{-1}(*) \otimes X, Z),$$

so its fiber is represented by any cokernel of the morphism $\iota \otimes X$. In other words, any cokernel represents the functor $\Delta\text{-set}_*(K, \text{map}(X, -))$ and is thus a possible choice of $K \wedge X$. Since every essentially finite based $\Delta$-set has a finite presentation, the object $K \wedge X$ always exists. $\square$

The same kind of arguments as for the $\cap$-pairing in Section 3 show that the smash product pairing $K \wedge X$ canonically extends to a functor of two variables, and that it preserves colimits in each variable.

PROPOSITION 4.3. *Let $f : K \to L$ be a weak equivalence between essentially finite based $\Delta$-sets and $X$ be an object of a pointed $\Delta$-cofibration category. Then the morphism*

$$f \wedge X : K \wedge X \longrightarrow L \wedge X$$

*is a weak equivalence.*

*Proof.* We let $m$ be any even number at least as large as the maximum dimension of a non-basepoint simplex of $K$. We claim that the inclusion $K^{(m)} \to K$ of the $m$-skeleton is a weak equivalence of $\Delta$-sets. Indeed, the geometric realization of $K^{(m)}$ is the $m$-skeleton of the canonical CW-structure on $|K|$. Since $K$ consists only of basepoints above dimension $m$, from $|K^{(m)}|$ to $|K^{(m+2)}|$ two cells of dimension $m+1$ and $m+2$ are attached; since $m$ is even, the $(m+2)$-cell is attached to the $(m+1)$-cell by a map of degree 1. So, for all large enough even $m$ the skeleton inclusion $|K^{(m)}| \to |K^{(m+2)}|$ is a homotopy equivalence. Hence the skeleton inclusion $|K^{(m)}| \to |K|$ is also a homotopy equivalence.

Now we let $m$ be even and at least as large as the maximum dimension of a non-basepoint simplex of both $K$ and $L$. Since $f$ is a weak equivalence, so is $f^{(m)} : K^{(m)} \to L^{(m)}$, by the previous paragraph. These skeleta are finite $\Delta$-sets (in the absolute sense). We let $P$ denote a $\Delta$-set with a unique simplex in each dimension up to dimension $m$, and no simplices above dimension $m$. We let $j : P \to K^{(m)}$ be the morphism that hits the basepoints. In the commutative diagram

$$\begin{tikzcd}
* \arrow[d, equal] & P \otimes X \arrow[l] \arrow[r, "j\otimes X"] \arrow[d, equal] & K^{(m)} \otimes X \arrow[d, "f^{(m)}\otimes X", "\sim"'] \\
* & P \otimes X \arrow[l] \arrow[r, "f^{(m)}j\otimes X"'] & L^{(m)} \otimes X
\end{tikzcd}$$

the right vertical morphism is a weak equivalence since $f^{(m)}$ is. Since $f \wedge X : K \wedge X \to L \wedge X$ can be obtained from this diagram by passage to horizontal pushouts (by Proposition 4.2), it is a weak equivalence by the gluing lemma. □

EXAMPLE 4.4. The smash product with certain based $\Delta$-sets $I$ respectively $S^1$ provides functorial cones and suspensions in any pointed $\Delta$-cofibration category $\mathcal{C}$. We let $I$ be the essentially finite based $\Delta$-set with

$$I_1 = \{z, *\}, \quad I_0 = \{zd_0, zd_1\}$$

and with $I_k$ consisting only of the basepoint for $k \geqslant 2$. The basepoint in dimension 0 is the vertex $zd_0$. The morphism $r : \Delta[1] \to I$ that hits the 1-simplex $z$ is a finite presentation with $r^{-1}(*) = \{d_0\}$. Proposition 4.2 provides a pushout square:

$$\begin{tikzcd}
X \arrow[r, "(d_0)_*"] \arrow[d] & \Delta[1] \otimes X \arrow[d, "q"] \\
* \arrow[r] & I \wedge X.
\end{tikzcd}$$

Since $(d_0)_* : X \to \Delta[1] \otimes X$ is an acyclic cofibration, the object $I \wedge X$ is weakly contractible.

The object $\partial\Delta[1] \otimes X$ is a coproduct of two copies of $X$, and the composite

$$\partial\Delta[1] \otimes X \xrightarrow{\mathrm{incl}\otimes X} \Delta[1] \otimes X \xrightarrow{q} I \wedge X$$

is zero on the copy of $X$ indexed by $d_0$. So the commutative square

$$\begin{tikzcd} \partial\Delta[1] \otimes X \arrow[r, "\mathrm{incl}\otimes X"] \arrow[d, "p"'] & \Delta[1] \otimes X \arrow[d, "q"] \\ X \arrow[r, "i_X"'] & I \wedge X \end{tikzcd}$$

is a pushout, where $i_X = q \circ (d_1)_*$, and where $p$ is the morphism such that $p \circ (d_0)_*$ is zero and $p \circ (d_1)_* = \mathrm{Id}_X$. So $i_X : X \to I \wedge X$ is a cofibration with weakly contractible target, that is, a functorial cone of $X$.

We let $S^1$ be the based $\Delta$-set with a unique non-basepoint simplex $z$ of dimension 1. The morphism $r : \Delta[1] \to S^1$ that hits $z$ is a finite presentation, and it satisfies $r^{-1}(*) = \partial\Delta[1]$. By Proposition 4.2, the object $S^1 \wedge X$ is then a cokernel of the morphism $\partial\Delta[1] \otimes X \to \Delta[1] \otimes X$, hence also a cokernel of the cone inclusion $i_X : X \to I \wedge X$. So $S^1 \wedge X$ is isomorphic in $\mathbf{Ho}(\mathcal{C})$ to the suspension of $X$.

We define the *mapping cone* $Cf$ of a morphism $f : X \to Y$ as a pushout:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "f"'] & I \wedge X \arrow[d] \\ Y \arrow[r, "j"'] & Cf. \end{tikzcd} \tag{4.5}$$

The pushout exists because the cone inclusion $i_X$ is a cofibration. We can compare the elementary distinguished triangles of the two cofibrations $i_X$ and $j$ in $\mathbf{Ho}(\mathcal{C})$:

$$\begin{tikzcd} X \arrow[r, "\gamma(i_X)"] \arrow[d, "\gamma(f)"'] & I \wedge X \arrow[r] \arrow[d] & (I \wedge X)/X \arrow[r, "\delta(i_X)", "\cong"'] \arrow[d, "\cong"] & \Sigma X \arrow[d, "\Sigma\gamma(f)"] \\ Y \arrow[r, "\gamma(j)"'] & Cf \arrow[r] & Cf/Y \arrow[r, "\delta(j)"'] & \Sigma Y. \end{tikzcd}$$

Since the square (4.5) is a pushout, the induced map $(I \wedge X)/X \to Cf/Y$ is an isomorphism in $\mathcal{C}$. Since $I \wedge X$ is weakly contractible, the connecting morphism $\delta(i_X)$ is an isomorphism in $\mathbf{Ho}(\mathcal{C})$. In the lower distinguished triangle we can thus replace $Cf/Y$ by the isomorphic object $\Sigma X$ and obtain a distinguished triangle

$$Y \xrightarrow{\gamma(j)} Cf \longrightarrow \Sigma X \xrightarrow{\Sigma\gamma(f)} \Sigma Y.$$

We rotate this triangle to the left and compensate for the sign by changing the unnamed morphism into its negative; the result is a distinguished triangle

$$X \xrightarrow{\gamma(f)} Y \xrightarrow{\gamma(j)} Cf \longrightarrow \Sigma X. \tag{4.6}$$

For $n \geqslant 2$ we define an essentially finite based $\Delta$-set $S\langle n\rangle$ by

$$S\langle n\rangle_0 = \{e_i \mid i \in \mathbb{Z}/n\} \quad \text{and} \quad S\langle n\rangle_1 = \{f_i \mid i \in \mathbb{Z}/n\} \cup \{*\},$$

and with $S\langle n\rangle_k$ consisting only of the basepoint for $k \geqslant 2$. We take the vertex $e_0$ as the basepoint in $S\langle n\rangle_0$. The face maps are given by $\partial f_i = (e_i, e_{i+1})$ (to be read modulo $n$). As in Example 4.4, we let $S^1$ be the based $\Delta$-set with exactly one non-basepoint simplex $z$ of dimension 1.

We now define morphisms of based $\Delta$-sets

$$\psi_i : S\langle n\rangle \longrightarrow S^1 \quad \text{and} \quad \nabla : S\langle n\rangle \longrightarrow S^1.$$

The morphism $\psi_i$ is determined by $\psi_i(f_i) = z$ and $\psi_i(f_k) = *$ for $i \not\equiv k$ modulo $n$. The morphism $\nabla$ is determined by $\nabla(f_i) = z$ for all $i \in \mathbb{Z}/n$. The square

$$\begin{tikzcd} S\langle n\rangle - \{f_i\} \arrow[r, "\psi_i"] \arrow[d, "\mathrm{incl}"'] & * \arrow[d] \\ S\langle n\rangle \arrow[r, "\psi_i"'] & S^1 \end{tikzcd}$$

is a pushout of based $\Delta$-sets and both $S\langle n\rangle - \{f_i\}$ and $*$ are weakly contractible, so $\psi_i$ is a weak equivalence.

Proposition 4.7. *Let $X$ be an object in a pointed $\Delta$-cofibration category $\mathcal{C}$.*

(i) *The relation $\gamma(\psi_i \wedge X) = \gamma(\psi_{i+1} \wedge X)$ holds as morphism from $S\langle n\rangle \wedge X$ to $S^1 \wedge X$ in $\mathbf{Ho}(\mathcal{C})$.*

(ii) *If $\mathcal{C}$ is stable, then the relation*

$$\gamma(\nabla \wedge X) = n \cdot \gamma(\psi_1 \wedge X)$$

*holds as morphisms from $S\langle n\rangle \wedge X$ to $S^1 \wedge X$ in $\mathbf{Ho}(\mathcal{C})$.*

*Proof.* (i) We let $\hat{S}^1$ be the extension of the $\Delta$-set $S^1$ given by

$$\hat{S}^1_0 = \{*\}, \quad \hat{S}^1_1 = \{*, z, g, g'\} \quad \text{and} \quad \hat{S}^1_2 = \{*, c, c'\}$$

and $\hat{S}^1_k = \{*\}$ for $k \geqslant 3$. The faces of the additional 2-simplices are $\partial c = (*, g, z)$ and $\partial c' = (z, g', *)$. The inclusion $j : S^1 \to \hat{S}^1$ is a sequence of two elementary expansions, hence a weak equivalence. The morphism $\gamma(j \wedge X) : S^1 \wedge X \to \hat{S}^1 \wedge X$ is then an isomorphism in $\mathbf{Ho}(\mathcal{C})$.

We let $S \subseteq S\langle n\rangle$ be the (unbased) one-dimensional sub-$\Delta$-set consisting of the simplices $e_i$ and $f_i$ for all $i \in \mathbb{Z}/n$. The inclusion $\iota : S \to S\langle n\rangle$ is a finite presentation with $\iota^{-1}(*) = \{e_0\}$, the basepoint in dimension 0. By Proposition 4.2, the object $S\langle n\rangle \wedge X$ is then a cokernel of the cofibration $\iota \otimes X : \{e_0\} \otimes X \to S \otimes X$, and we claim that the projection $q : S \otimes X \to S\langle n\rangle \wedge X$ becomes a split epimorphism in the homotopy category $\mathbf{Ho}(\mathcal{C})$. Indeed, the cofibration $\iota \otimes X$ gives rise to an elementary distinguished triangle in $\mathbf{Ho}(\mathcal{C})$:

$$\{e_0\} \otimes X \xrightarrow{\gamma(\iota\otimes X)} S \otimes X \xrightarrow{\gamma(q)} S\langle n\rangle \wedge X \xrightarrow{\delta(\iota\otimes X)} \Sigma(\{e_0\} \otimes X).$$

The composite of the inclusion $\{e_0\} \to S$ with the cone inclusion $i_S : S \to CS$ (compare Example 2.4) is a weak equivalence, hence the composite

$$\{e_0\} \otimes X \xrightarrow{\gamma(\iota\otimes X)} S \otimes X \xrightarrow{\gamma(i_S\otimes X)} CS \otimes X$$

is an isomorphism. So $\gamma(\iota \otimes X)$ is a split monomorphism, and so $\gamma(q)$ is a split epimorphism.

We will now define a combinatorial homotopy, that is, a morphism of $\Delta$-sets $H : S \otimes \Delta[1] \to \hat{S}^1$. We let $\varphi, \varphi' : [2] \to [1] \times [1]$ be the two monotone injective maps defined by

$$\varphi(0) = \varphi'(0) = (0, 0), \quad \varphi(1) = (0, 1), \quad \varphi'(1) = (1, 0) \quad \text{and} \quad \varphi(2) = \varphi'(2) = (1, 1).$$

The $\Delta$-set $S \otimes \Delta[1]$ is generated by the 2-simplices

$$A_k = [f_k, \mathrm{Id}_{[1]}; \varphi] \quad \text{and} \quad B_k = [f_k, \mathrm{Id}_{[1]}; \varphi']$$

for $k \in \mathbb{Z}/n$, subject only to the relations

$$A_k d_1 = B_k d_1 \quad \text{and} \quad A_k d_2 = B_{k+1} d_0$$

(to be read modulo $n$). The homotopy $H : S \otimes \Delta[1] \to \hat{S}^1$ is determined by

$$H(A_k) = H(B_k) = \begin{cases} c & \text{if } k \equiv i \text{ modulo } n, \\ c' & \text{if } k \equiv i+1 \text{ modulo } n, \text{ and} \\ * & \text{else.} \end{cases}$$

We let $i_0, i_1 : S \to S \otimes \Delta[1]$ be the 'front and back inclusion', that is, the morphisms determined by

$$i_0(f_k) = B_k d_2 \quad \text{respectively } i_1(f_k) = A_k d_0.$$

We let $P$ be a two-dimensional $\Delta$-set with a unique simplex of dimension 0, 1 and 2. The unique morphism $u : \Delta[1] \to P$ satisfies $(S \otimes u)i_0 = (S \otimes u)i_1$ as morphisms $S \to S \otimes P$. So we also have $\gamma(S \otimes u \otimes X) \circ \gamma(i_0 \otimes X) = \gamma(S \otimes u \otimes X) \circ \gamma(i_1 \otimes X)$. Since $P$ is weakly contractible, the morphism $u$, and hence also the morphism $S \otimes u \otimes X : S \otimes \Delta[1] \otimes X \to S \otimes P \otimes X$, is a weak equivalence. So $\gamma(S \otimes u \otimes X)$ is invertible and we conclude that $\gamma(i_0 \otimes X) = \gamma(i_1 \otimes X)$.

The homotopy $H$ has image in the 2-skeleton $(\hat{S}^1)^{(2)}$ of $\hat{S}^1$. The morphisms $Hi_0$ and $Hi_1$ are lifts of the based morphism $j\psi_i$ and $j\psi_{i+1}$, respectively, to morphisms of unbased $\Delta$-sets; that is, the squares

$$\begin{tikzcd} S \arrow[r, "Hi_0"] \arrow[d, "\text{incl}"'] & (\hat{S}^1)^{(2)} \arrow[d, "\text{incl}"] \\ S\langle n\rangle \arrow[r, "j\psi_i"'] & \hat{S}^1 \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "Hi_1"] \arrow[d, "\text{incl}"'] & (\hat{S}^1)^{(2)} \arrow[d, "\text{incl}"] \\ S\langle n\rangle \arrow[r, "j\psi_{i+1}"'] & \hat{S}^1 \end{tikzcd}$$

commute. Since the vertical maps are finite presentations, the squares

$$\begin{tikzcd} S \otimes X \arrow[r, "Hi_0 \otimes X"] \arrow[d, "q"'] & (\hat{S}^1)^{(2)} \otimes X \arrow[d, "\text{proj}"] \\ S\langle n\rangle \wedge X \arrow[r, "j\psi_i \wedge X"'] & \hat{S}^1 \wedge X \end{tikzcd} \qquad \begin{tikzcd} S \otimes X \arrow[r, "Hi_1 \otimes X"] \arrow[d, "q"'] & (\hat{S}^1)^{(2)} \otimes X \arrow[d, "\text{proj}"] \\ S\langle n\rangle \wedge X \arrow[r, "j\psi_{i+1} \wedge X"'] & \hat{S}^1 \wedge X \end{tikzcd}$$

commute in $\mathcal{C}$. So we conclude that

$$\begin{aligned} \gamma(j \wedge X) \circ \gamma(\psi_i \wedge X) \circ \gamma(q) &= \gamma(\text{proj}) \circ \gamma(H \otimes X) \circ \gamma(i_0 \otimes X) \\ &= \gamma(\text{proj}) \circ \gamma(H \otimes X) \circ \gamma(i_1 \otimes X) \\ &= \gamma(j \wedge X) \circ \gamma(\psi_{i+1} \wedge X) \circ \gamma(q). \end{aligned}$$

Since $\gamma(j \wedge X)$ is an isomorphism and $\gamma(q)$ is a split epimorphism, this implies the desired relation $\gamma(\psi_i \wedge X) = \gamma(\psi_{i+1} \wedge X)$.

(ii) We define a morphism of based $\Delta$-sets $\kappa : S\langle n\rangle \to \bigvee_{j=1}^n S^1$ by sending $f_i$ to the $i$th copy of $z$. This morphism makes the diagram

$$\begin{tikzcd} & S\langle n\rangle \wedge X \arrow[dl, "\nabla \wedge X"'] \arrow[d, "\kappa \wedge X"] \arrow[dr, "\psi_i \wedge X"] & \\ S^1 \wedge X & \bigvee_{j=1}^n S^1 \wedge X \arrow[l, "\text{fold}"] \arrow[r, "p_i"'] & S^1 \wedge X \end{tikzcd}$$

commute in $\mathcal{C}$, where $p_i$ is the projection to the $i$th wedge summand. Since coproducts in $\mathcal{C}$ become sums in the homotopy category of $\mathcal{C}$, the fold map occurring in the diagram becomes the sum of the $n$ projections in $\mathbf{Ho}(\mathcal{C})$. So we obtain the desired relation

$$\gamma(\nabla \wedge X) = \sum_{i=1}^{n} \gamma(p_i) \circ \gamma(\kappa \wedge X) = \sum_{i=1}^{n} \gamma(\psi_i) = n \cdot \gamma(\psi_1). \qquad \square$$

Our next aim is to define 'actions' of a mod-$n$ Moore space (that is, $\Delta$-set) on objects of a pointed $\Delta$-cofibration category. For the arguments below we need a $\Delta$-set model for a Moore space whose reduced homology is concentrated in an *even* dimension. The easiest way to construct one is to cone off a suspension of the morphism $\nabla : S\langle n\rangle \to S^1$. To make this precise, we need the following 'reduced' version of the geometric product for based $\Delta$-sets. A based $\Delta$-set $K$ admits a unique based morphism $* \to K$ from the terminal $\Delta$-set. Given two based $\Delta$-sets $K$ and $L$, we define their *geometric smash product* $K \wedge L$ as the pushout

$$\begin{tikzcd}
(K \otimes *) \cup_{(*\otimes *)} (* \otimes L) \arrow[r, "\mathrm{incl}"] \arrow[d] & K \otimes L \arrow[d] \\
* \arrow[r] & K \wedge L.
\end{tikzcd}$$

If both $K$ and $L$ are essentially finite, then the smash product $K \wedge L$ is again essentially finite. The associativity isomorphism in a pointed $\Delta$-cofibration category then passes to coherent isomorphisms

$$(K \wedge L) \wedge X \;\cong\; K \wedge (L \wedge X).$$

Remark 4.8. Since the geometric realization of a terminal $\Delta$-set is *not* a point (but rather an infinite-dimensional contractible CW-complex), the geometric realization does *not* take the geometric smash product to the smash product of spaces. However, we can consider the *reduced realization* of a based $\Delta$-set $K$

$$|K|_\bullet = |K|/|*|,$$

where $*$ denotes the sub-$\Delta$-space of $K$ consisting of the basepoints in the various dimensions. Since a terminal $\Delta$-set is weakly contractible, the projection $|K| \to |K|_\bullet$ is a homotopy equivalence. Thus a morphism of based $\Delta$-sets is a weak equivalence if and only if it induces a weak equivalence of reduced realizations. Moreover, the homeomorphism (2.5) from $|K \otimes L|$ to $|K| \times |L|$ passes to a homeomorphism

$$|K \wedge L|_\bullet \cong |K|_\bullet \wedge |L|_\bullet.$$

For example, the reduced realization of the based $\Delta$-set $S^1$ is a circle, so $|S^1 \wedge L|_\bullet$ is homeomorphic to the reduced suspension of $|L|_\bullet$.

From Example 4.4 we recall the essentially finite based $\Delta$-set $I$ with

$$I_0 = \{*, zd_1\}, \quad I_1 = \{*, z\}, \quad I_k = \{*\} \quad \text{for } k \geqslant 2,$$

and $zd_0 = *$. For every based $\Delta$-set $K$ the smash product $I \wedge K$ is weakly contractible and comes with a monomorphism $i_K : K \to I \wedge K$. Now we can define an essentially finite based $\Delta$-set $M$ as the pushout:

$$\begin{tikzcd}
S^1 \wedge S\langle n\rangle \arrow[r, "i_{S^1 \wedge S\langle n\rangle}"] \arrow[d, "S^1 \wedge \nabla"'] & I \wedge S^1 \wedge S\langle n\rangle \arrow[d] \\
S^2 \arrow[r, "\iota"'] & M
\end{tikzcd} \tag{4.9}$$

where we use the abbreviation $S^2 = S^1 \wedge S^1$. The geometric realization of $M$ is then a mod-$n$ Moore space, that is, a simply connected CW-complex whose reduced homology is $\mathbb{Z}/n$ concentrated in dimension 2. Moreover, the inclusion $\iota : S^2 \to M$ induces an epimorphism in integral homology.

If we smash the pushout square (4.9) with an object $X$ of a stable $\Delta$-cofibration category $\mathcal{C}$, we obtain a pushout in $\mathcal{C}$:

$$\begin{tikzcd} S^1 \wedge S\langle n\rangle \wedge X \arrow[r, "i_{S^1\wedge S\langle n\rangle\wedge X}"] \arrow[d, "S^1\wedge\nabla\wedge X"'] & I \wedge S^1 \wedge S\langle n\rangle \wedge X \arrow[d] \\ S^2 \wedge X \arrow[r, "\iota\wedge X"'] & M \wedge X. \end{tikzcd}$$

This shows that $M \wedge X$ is the mapping cone, as defined in (4.5), of the morphism $S^1 \wedge \nabla \wedge X : S^1 \wedge S\langle n\rangle \wedge X \to S^2 \wedge X$. The distinguished triangle (4.6) then becomes a distinguished triangle

$$S^1 \wedge S\langle n\rangle \wedge X \xrightarrow{\gamma(S^1\wedge\nabla\wedge X)} S^2 \wedge X \xrightarrow{\gamma(\iota\wedge X)} M \wedge X \longrightarrow \Sigma(S^1 \wedge S\langle n\rangle \wedge X).$$

We use the isomorphism $\gamma(S^1 \wedge \psi_1 \wedge X) : S^1 \wedge S\langle n\rangle \wedge X \to S^1 \wedge S^1 \wedge X = S^2 \wedge X$ in $\mathbf{Ho}(\mathcal{C})$ to replace the first and last objects in this triangle. Since $\gamma(\nabla \wedge X) = n \cdot \gamma(\psi_1)$ by Proposition 4.7(ii), the morphism $\gamma(S^1 \wedge \nabla \wedge X)$ then turns into the $n$-fold multiple of the identity. The upshot is a distinguished triangle

$$S^2 \wedge X \xrightarrow{n\cdot(S^2\wedge X)} S^2 \wedge X \xrightarrow{\gamma(\iota\wedge X)} M \wedge X \longrightarrow \Sigma(S^2 \wedge X). \tag{4.10}$$

DEFINITION 4.11. We denote by $P^i$ the $i$th *symmetric power* of the based $\Delta$-set $M$,

$$P^i = M^{\wedge i}/\Sigma_i.$$

Here the symmetric group $\Sigma_i$ permutes the factors of the $i$th geometric smash power. The associativity isomorphism $M^{\wedge i} \wedge M^{\wedge j} \cong M^{\wedge(i+j)}$ passes to a quotient morphism of symmetric powers

$$\mu_{i,j} \,:\, P^i \wedge P^j = (M^{\wedge i}/\Sigma_i) \wedge (M^{\wedge j}/\Sigma_j) \longrightarrow M^{\wedge(i+j)}/\Sigma_{i+j} = P^{i+j},$$

which we refer to as the *canonical projection*. The canonical projections are associative in the sense that the following diagram commutes:

$$\begin{tikzcd} P^i \wedge P^j \wedge P^k \arrow[r, "P^i\wedge\mu_{j,k}"] \arrow[d, "\mu_{i,j}\wedge P^k"'] & P^i \wedge P^{j+k} \arrow[d, "\mu_{i,j+k}"] \\ P^{i+j} \wedge P^k \arrow[r, "\mu_{i+j,k}"'] & P^{i+j+k} \end{tikzcd}$$

for all $i, j, k \geqslant 1$.

The Moore 'space' $M$ is an essentially finite based $\Delta$-set, hence all its smash powers $M^{\wedge i}$ and its symmetric powers $P^i$ are essentially finite. So it makes sense to smash $P^i$ with objects in any pointed $\Delta$-cofibration category.

DEFINITION 4.12. Let $\mathcal{C}$ be a pointed $\Delta$-cofibration category. An *$M$-module* $X$ consists of an infinite sequence

$$X_{(1)}, X_{(2)}, \ldots, X_{(k)}, \ldots$$

of objects of $\mathcal{C}$, together with morphisms in $\mathcal{C}$

$$\alpha_{i,j} \,:\, P^i \wedge X_{(j)} \longrightarrow X_{(i+j)}$$

for $i, j \geqslant 1$ such that the associativity diagram

$$\begin{tikzcd} P^i \wedge P^j \wedge X_{(k)} \arrow[r, "\mu_{i,j} \wedge X_{(k)}"] \arrow[d, "P^i \wedge \alpha_{j,k}"'] & P^{i+j} \wedge X_{(k)} \arrow[d, "\alpha_{i+j,k}"] \\ P^i \wedge X_{(j+k)} \arrow[r, "\alpha_{i,j+k}"'] & X_{(i+j+k)} \end{tikzcd}$$

commutes for all $i, j, k \geqslant 1$. The *underlying object* of an $M$-module $X$ is the object $X_{(1)}$ of $\mathcal{C}$. A *morphism* $f : X \to Y$ of $M$-modules consists of $\mathcal{C}$-morphisms $f_{(j)} : X_{(j)} \to Y_{(j)}$ for $j \geqslant 1$, such that the diagrams

$$\begin{tikzcd} P^i \wedge X_{(j)} \arrow[r, "P^i \wedge f_{(j)}"] \arrow[d, "\alpha_{i,j}"'] & P^i \wedge Y_{(j)} \arrow[d, "\alpha_{i,j}"] \\ X_{(i+j)} \arrow[r, "f_{(i+j)}"'] & Y_{(i+j)} \end{tikzcd}$$

commute for $i, j \geqslant 1$.

An $M$-module $X$ is *$k$-coherent* for a natural number $k \geqslant 1$ if the composite

$$S^2 \wedge X_{(j-1)} \xrightarrow{\iota \wedge X_{(j-1)}} M \wedge X_{(j-1)} \xrightarrow{\alpha_{1,j-1}} X_{(j)}$$

is a weak equivalence for all $2 \leqslant j \leqslant k$ (where we use that $M = P^1$).

For example, every $M$-module $X$ is 1-coherent, and if $X$ is 2-coherent, then in the homotopy category of $\mathcal{C}$ the composite map

$$M \wedge X_{(1)} \xrightarrow{\gamma(\alpha_{1,1})} X_{(2)} \xleftarrow{\gamma(\alpha_{1,1} \circ (\iota \wedge X_{(1)}))^{-1}} S^2 \wedge X_{(1)}$$

is a retraction to $\gamma(\iota \wedge X_{(1)}) : S^2 \wedge X_{(1)} \to M \wedge X_{(1)}$. So, if the cofibration category is stable, then by the distinguished triangle (4.10) the identity map of $X_{(1)}$ is annihilated by $n$ in the group $\mathbf{Ho}(\mathcal{C})(X_{(1)}, X_{(1)})$.

Now we show that, for every prime $p$ and for all $i$ that are strictly less than $p$, the $i$th reduced symmetric power $P^i$ of a mod-$p$ Moore space is again a mod-$p$ Moore space. This ought to be well known, but the author was unable to find a reference.

PROPOSITION 4.13. *For every odd prime $p$ and every $2 \leqslant i \leqslant p-1$ the composite morphism of based $\Delta$-sets*

$$S^2 \wedge P^{i-1} \xrightarrow{\iota \wedge P^{i-1}} M \wedge P^{i-1} \xrightarrow{\mu_{1,i-1}} P^i$$

*is a weak equivalence.*

*Proof.* We start by showing that the projection $q : M^{\wedge i} \to M^{\wedge i}/\Sigma_i = P^i$ induces an epimorphism of fundamental groups. Since the geometric realization of $M^{\wedge i}$ is simply connected, the geometric realization of $P^i$ is then also simply connected. Indeed, every element of the fundamental group of $|P^i|$ can be represented by a closed path of 1-simplices in $P^i$. Such a path can be lifted to a path of 1-simplices in $M^{\wedge i}$, which a priori need not close up. However, the basepoint vertex in $M^{\wedge i}$ is $\Sigma_i$-fixed, so it is the only preimage of the basepoint vertex in $P^i$, and any closed path at the basepoint in $P^i$ lifts to a closed path in $M^{\wedge i}$.

Now we determine the integral homology of $|P^i|$, using that the homology of the realization of a $\Delta$-set can be calculated directly from the chain complex freely generated by the simplices with differential the alternating sum of the face maps. We start from the reduced mod-$p$ homology of $M$, which has a basis consisting of a three-dimensional class $v$ and its image $u = \beta(v) \in H_2(M, \mathbb{F}_p)$ under the Bockstein homomorphism. The Künneth theorem identifies the reduced mod-$p$ homology of $M^{\wedge i}$ with an $i$-fold tensor power of $\tilde{H}_*(M, \mathbb{F}_p)$, and the $\Sigma_i$-invariant subspace of $\tilde{H}_*(M^{\wedge i}, \mathbb{F}_p)$ is generated by the two classes $u^{\times i} \in H_{2i}(M^{\wedge i}, \mathbb{F}_p)$ and

$$w = \sum_{k=1}^{i} u^{\times(k-1)} \times v \times u^{\times(i-k)}$$

of dimension $2i+1$. Because $\beta(w) = i \cdot u^{\times i}$, the kernel of the Bockstein homomorphism on the $\Sigma_i$-invariants $\tilde{H}_*(M^{\wedge i}, \mathbb{F}_p)^{\Sigma_i}$ is generated by $u^{\times i}$ alone. Since the reduced integral homology of $M^{\wedge i}$ is an $\mathbb{F}_p$-vector space, the coefficient reduction $\tilde{H}_*(M^{\wedge i}, \mathbb{Z}) \to \tilde{H}_*(M^{\wedge i}, \mathbb{F}_p)$ identifies it with the kernel of the Bockstein. We conclude that the $\Sigma_i$-invariant subgroup of the reduced integral homology of $M^{\wedge i}$ is cyclic of order $p$ generated by the class

$$x \times \cdots \times x \in H_{2i}(M^{\wedge i}, \mathbb{Z}),$$

where $x \in H_2(M, \mathbb{Z})$ is any generator.

Since $M$ is a mod-$p$ Moore space with homology in dimension 2, its infinite symmetric product $SP(M)$ is a mod-$p$ Eilenberg–MacLane space with homotopy in dimension 2, by the Dold–Thom theorem [**7**, Satz 6.10]. In particular, all reduced integral homology groups of $SP(M)$ are $\mathbb{F}_p$-vector spaces. Since $\tilde{H}_*(P^i, \mathbb{Z})$ is a direct summand of $H_*(SP(M), \mathbb{Z})$ (see, for example, [**24**, Theorem 2.9]), the groups $\tilde{H}_*(P^i, \mathbb{Z})$ are also annihilated by $p$. Both composites of the norm map

$$\tilde{H}_*(P^i, \mathbb{Z}) = \tilde{H}_*(M^{\wedge i}/\Sigma_i, \mathbb{Z}) \longrightarrow \tilde{H}_*(M^{\wedge i}, \mathbb{Z})^{\Sigma_i}, \quad [x \cdot \Sigma_i] \longmapsto \left[\sum_{\sigma \in \Sigma_i} x\sigma\right]$$

with the composite

$$\mathbb{Z}/p\{x \times \cdots \times x\} = (\tilde{H}_*(M^{\wedge i}, \mathbb{Z}))^{\Sigma_i} \xrightarrow{\text{inclusion}} \tilde{H}_*(M^{\wedge i}, \mathbb{Z}) \xrightarrow{\tilde{H}_*(q, \mathbb{Z})} \tilde{H}_*(P^i, \mathbb{Z}) \tag{4.14}$$

are multiplication by the order $i!$ of the group $\Sigma_i$. Since $i < p$ and the reduced integral homology of $P^i$ and $M^{\wedge i}$ are graded $\mathbb{F}_p$-vector spaces, the map (4.14) is an isomorphism. We conclude that the reduced integral homology of $P^i$ is concentrated in dimension $2i$, and $H_{2i}(P^i, \mathbb{Z})$ is cyclic of order $p$, generated by the image of the class $x \times \cdots \times x$.

Now we put everything together: the spaces $|S^2 \wedge P^{i-1}|$ and $|P^i|$ are simply connected and the morphism $\mu_{1,i-1}(\iota \wedge P^{i-1}) : S^2 \wedge P^{i-1} \to P^i$ induces an isomorphism in integral homology, so it is a weak equivalence. □

Example 4.15. For every object $K$ of $\mathcal{C}$ we define the *free $M$-module* $M \triangle K$ generated by $K$. The $j$th term of this $M$-module is

$$(M \triangle K)_{(j)} = P^j \wedge K$$

and the structure map $\alpha_{i,j} : P^i \wedge (M \triangle K)_{(j)} \to (M \triangle K)_{(i+j)}$ is the morphism

$$\mu_{i,j} \wedge K \; : \; P^i \wedge P^j \wedge K \longrightarrow P^{i+j} \wedge K.$$

The associativity condition is then a consequence of the associativity of the projection maps. For all $2 \leqslant i \leqslant p-1$, the composite

$$S^2 \wedge P^{i-1} \wedge K \xrightarrow{\iota \wedge P^{i-1} \wedge K} M \wedge P^{i-1} \wedge K \xrightarrow{\mu_{1,i-1} \wedge K} P^i \wedge K$$

is a weak equivalence by Proposition 4.13. So the free $M$-module $M \triangle K$ is $(p-1)$-coherent.

Now we let $X$ be any $M$-module and $f : K \to X_{(1)}$ be a morphism in $\mathcal{C}$. We denote by $X_{\bullet+1}$ the $M$-module obtained from $X$ by forgetting the object $X_{(1)}$ and reindexing the rest of the data, that is, $(X_{\bullet+1})_{(i)} = X_{(i+1)}$, and similarly for the action maps. As $i$ varies, the action maps $\alpha_{i,1}$ of $X$ assemble into a morphism of $M$-modules

$$\alpha_{\bullet,1} : M \triangle X_{(1)} \longrightarrow X_{\bullet+1}.$$

The *free extension* of $f : K \to X_{(1)}$ is the composite morphism of $M$-modules

$$M \triangle K \xrightarrow{M \triangle f} M \triangle X_{(1)} \xrightarrow{\alpha_{\bullet,1}} X_{\bullet+1}. \tag{4.16}$$

CONSTRUCTION 4.17. The mapping cone construction of Example 4.4 gives a way to make new $M$-modules from old ones. Given a morphism $f : X \to Y$ of $M$-modules, we construct another $M$-module $Cf$, the *mapping cone* of $f$ as follows. For $j \geqslant 1$ we set $(Cf)_{(j)} = C(f_{(j)})$, that is, the $j$th object $(Cf)_{(j)}$ is the mapping cone of the morphism $f_{(j)} : X_{(j)} \to Y_{(j)}$. The action map $\alpha_{i,j} : P^i \wedge Cf_{(j)} \to Cf_{(i+j)}$ is obtained by taking horizontal pushouts in the commutative diagram:

$$\begin{tikzcd}
P^i \wedge Y_{(j)} \arrow[d, "\alpha_{i,j}"'] & P^i \wedge X_{(j)} \arrow[l, "P^i \wedge f_{(j)}"'] \arrow[r, "P^i \wedge i_{X_{(j)}}"] \arrow[d, "\alpha_{i,j}"'] & P^i \wedge CX_{(j)} \arrow[d, "\bar{\alpha}_{i,j}"] \\
Y_{(i+j)} & X_{(i+j)} \arrow[l, "f_{(i+j)}"] \arrow[r, "i_{X_{(i+j)}}"'] & CX_{(i+j)}
\end{tikzcd}$$

where the right vertical map is the composite

$$P^i \wedge I \wedge X_{(j)} \xrightarrow{\text{symmetry} \wedge X_{(j)}} I \wedge P^i \wedge X_{(j)} \xrightarrow{I \wedge \alpha_{i,j}} I \wedge X_{(i+j)}.$$

Associativity is inherited from the associativity of the actions on $X$ and $Y$.

PROPOSITION 4.18. *Let $\mathcal{C}$ be a pointed $\Delta$-cofibration category.*

(i) *Let $f : X \to Y$ be a morphism of $M$-modules. If $X$ and $Y$ are $k$-coherent, then the mapping cone $Cf$ is again $k$-coherent.*

(ii) *Let $p$ be a prime and $X$ be a $(k+1)$-coherent $M$-module for some $1 \leqslant k \leqslant p-1$. Then, for every $\mathcal{C}$-morphism $f : K \to X_{(1)}$, the mapping cone of the free extension (4.16) $\hat{f} : M \triangle K \to X_{\bullet+1}$ is $k$-coherent.*

*Proof.* (i) The morphism $\alpha_{1,j-1}(\iota \wedge Cf_{(j-1)}) : S^2 \wedge Cf_{(j-1)} \to Cf_{(j)}$ is obtained by passage to horizontal pushouts in the commutative diagram

$$\begin{tikzcd}
S^2 \wedge Y_{(j-1)} \arrow[d, "\alpha_{1,j-1}(\iota \wedge Y_{(j-1)})"'] & S^2 \wedge X_{(j-1)} \arrow[l, "S^2 \wedge f_{(j-1)}"'] \arrow[r, "S^2 \wedge i_{X_{(j-1)}}"] \arrow[d, "\alpha_{1,j-1}(\iota \wedge X_{(j-1)})"] & S^2 \wedge CX_{(j-1)} \arrow[d, "\bar{\alpha}_{1,j-1}(\iota \wedge CX_{(j-1)})"] \\
Y_{(j)} & X_{(j)} \arrow[l, "f_{(j)}"] \arrow[r, "i_{X_{(j)}}"'] & CX_{(j)}.
\end{tikzcd}$$

The two left horizontal morphisms are cofibrations, and for $j \leqslant k$ all vertical morphisms are weak equivalences. So, by the gluing lemma, the induced map on pushouts is a weak equivalence for $j \leqslant k$.

(ii) If $X$ is $(k+1)$-coherent, then the shifted $M$-module $X_{\bullet+1}$ is $k$-coherent. The free module $M \triangle K$ is $(p-1)$-coherent, hence $k$-coherent. So the mapping cone of the free extension $\hat{f} : M \triangle K \to X_{\bullet+1}$ is $k$-coherent by part (i). □

We come to a final useful property of $M$-modules, needed in the next section.

Lemma 4.19. *For every $M$-module $X$ and every acyclic cofibration $\varphi_{(1)} : X_{(1)} \to Z_{(1)}$ there exists an $M$-module $Z$ with underlying object $Z_{(1)}$ and a morphism of $M$-modules $\varphi : X \to Z$ that extends $\varphi_{(1)}$ and such that every component $\varphi_{(j)} : X_{(j)} \to Z_{(j)}$ is an acyclic cofibration.*

*Proof.* For $j \geqslant 2$ we define the object $Z_{(j)}$ and the morphism $\varphi_{(j)}$ as the pushout:

$$\begin{tikzcd} P^{j-1} \wedge X_{(1)} \arrow[r, "P^{j-1} \wedge \varphi_{(1)}"] \arrow[d, "\alpha_{j-1,1}"'] & P^{j-1} \wedge Z_{(1)} \arrow[d, "\alpha_{j-1,1}"] \\ X_{(j)} \arrow[r, "\varphi_{(j)}"'] & Z_{(j)}. \end{tikzcd}$$

Since the morphism $\varphi_{(1)}$ is an acyclic cofibration, so is $P^{j-1} \wedge \varphi_{(1)}$; hence the pushout exists and the morphism $\varphi_{(j)}$ is an acyclic cofibration. The structure maps $\alpha_{i,j} : P^i \wedge Z_{(j)} \to Z_{(i+j)}$ are induced on pushouts by the commutative diagram:

$$\begin{tikzcd} P^i \wedge X_{(j)} \arrow[d, "\alpha_{i,j}"'] & P^i \wedge P^{j-1} \wedge X_{(1)} \arrow[l, "P^i \wedge \alpha_{j-1,1}"'] \arrow[r, "P^i \wedge P^{j-1} \wedge \varphi_{(1)}"] \arrow[d, "\mu_{i,j-1} \wedge X_{(1)}"] & P^i \wedge P^{j-1} \wedge Z_{(1)} \arrow[d, "\mu_{i,j-1} \wedge Z_{(1)}"] \\ X_{(i+j)} & P^{i+j-1} \wedge X_{(1)} \arrow[l, "\alpha_{i+j-1,1}"] \arrow[r, "P^{i+j-1} \wedge \varphi_{(1)}"'] & P^{i+j-1} \wedge Z_{(1)}. \end{tikzcd}$$

The associativity condition follows. □

## 5. *The $p$-order in topological triangulated categories*

In this section, we prove our main result: every topological triangulated category has $p$-order at least $p-1$ for any prime $p$. The proof relies on the techniques developed in the last two sections. A key ingredient is the link, established in Proposition 5.2, between the concepts of coherent $M$-action (which takes place in a cofibration category) and the notion of $p$-order (which takes place in the triangulated homotopy category).

We recall from [**29**] the notion of *n-order* of an object in a triangulated category $\mathcal{T}$. For an object $K$ of $\mathcal{T}$ and a natural number $n$, we write $n \cdot K$ for the $n$-fold multiple of the identity morphism in the abelian group of endomorphisms in $\mathcal{T}$. We let $K/n$ denote any cone of $n \cdot K$, that is, an object that is part of a distinguished triangle

$$K \xrightarrow{n\cdot} K \xrightarrow{\pi} K/n \longrightarrow \Sigma K.$$

In the following definition, an *extension* of a morphism $f : K \to X$ is any morphism $\bar{f} : K/n \to X$ satisfying $\bar{f}\pi = f$.

DEFINITION 5.1. We consider an object $X$ of a triangulated category $\mathcal{T}$ and a natural number $n \geqslant 1$. We define the *$n$-order* of $X$ inductively.

(1) Every object has $n$-order greater or equal to 0.

(2) For $k \geqslant 1$, the object $X$ has $n$-order greater or equal to $k$ if and only if, for every object $K$ of $\mathcal{T}$ and every morphism $f : K \to X$, there exists an extension $\bar{f} : K/n \to X$ such that some (hence any) cone of $\bar{f}$ has $n$-order greater than or equal to $k - 1$.

The *$n$-order* of the triangulated category $\mathcal{T}$ is the $n$-order of some (hence any) zero object.

As a slogan, the $n$-order measures 'how strongly' the relation $n \cdot K = 0$ holds. For more background, examples and elementary properties of $n$-order, we refer the reader to [**29**] or [**30**, Section 1]. In [**30**, Proposition 1.5], we prove a sufficient criterion, the existence of a 'mod-$n$ reduction', so that a triangulated category has infinite $n$-order. This criterion is then used to show that the derived categories of certain structured ring spectra have infinite $n$-order [**30**, Proposition 1.6], and to show that every algebraic triangulated category has infinite $n$-order [**30**, Theorem 2.2].

The next proposition relates coherent $M$-actions on an object of a cofibration category to its $p$-order in the homotopy category. In any pointed $\Delta$-cofibration category the functor $S^2 \wedge - : \mathcal{C} \to \mathcal{C}$ is exact and so it descends to an exact functor of triangulated categories on $\mathbf{Ho}(\mathcal{C})$ (see Proposition A.14). We denote the derived functor again by $S^2 \wedge -$; it is naturally isomorphic to the double suspension functor. We define an *$M$-extension* of a morphism $f : K \to X$ in $\mathbf{Ho}(\mathcal{C})$ to be a morphism $\Phi : M \wedge K \to S^2 \wedge X$ in $\mathbf{Ho}(\mathcal{C})$ such that $\Phi \circ \gamma(\iota \wedge K) = S^2 \wedge f : S^2 \wedge K \to S^2 \wedge X$.

PROPOSITION 5.2. *Let $\mathcal{C}$ be a stable $\Delta$-cofibration category and $p$ be a prime. Let $X$ be the underlying object of a $(k+1)$-coherent $M$-module for some $k < p$.*

(i) *For every object $K$ of $\mathcal{C}$ and every morphism $f : K \to X$ in $\mathbf{Ho}(\mathcal{C})$ there exists an $M$-extension $\Phi : M \wedge K \to S^2 \wedge X$ of $f$ and a distinguished triangle*

$$M \wedge K \xrightarrow{\Phi} S^2 \wedge X \longrightarrow C \longrightarrow \Sigma(M \wedge K)$$

*such that the object $C$ is underlying a $k$-coherent $M$-module.*

(ii) *The $p$-order of $X$ is at least $k$.*

*Proof.* (i) By assumption there is a $(k+1)$-coherent $M$-module $Y$ such that $Y_{(1)} = X$. The morphism $f : K \to X = Y_{(1)}$ in the homotopy category can be written as a fraction $f = \gamma(s_{(1)})^{-1}\gamma(a)$ where $a : K \to Z_{(1)}$ and $s_{(1)} : Y_{(1)} \to Z_{(1)}$ are morphisms in $\mathcal{C}$ and $s$ is an acyclic cofibration. By Lemma 4.19 the object $Z_{(1)}$ can be extended to an $M$-module $Z$ and $s$ can be extended to a morphism $s : Y \to Z$ of $M$-modules all of whose components $s_{(i)} : Y_{(i)} \to Z_{(i)}$ are acyclic cofibrations. Since $Y$ is $(k+1)$-coherent, $Z$ is then also $(k+1)$-coherent.

We let $\hat{a}_{(1)} : M \wedge K \to Z_{(2)}$ be the first component of the free extension (4.16) of $a$, that is, the composite

$$M \wedge K \xrightarrow{M \wedge a} M \wedge Z_{(1)} \xrightarrow{\alpha_{1,1}} Z_{(2)}.$$

The square in $\mathcal{C}$

$$\begin{tikzcd} S^2\wedge K \arrow[r, "S^2\wedge a"] \arrow[d, "\iota\wedge K"'] & S^2\wedge Z_{(1)} \arrow[d, "\sim"', "\alpha_{1,1}(\iota\wedge Z_{(1)})"] \\ M\wedge K \arrow[r, "\hat{a}_{(1)}"'] & Z_{(2)} \end{tikzcd}$$

commutes and the right vertical morphism is a weak equivalence. The composite in $\mathbf{Ho}(\mathcal{C})$

$$M\wedge K \xrightarrow{\gamma(\hat{a}_{(1)})} Z_{(2)} \xrightarrow{\gamma(\alpha_{1,1}(\iota\wedge s_{(1)}))^{-1}} S^2\wedge X$$

is thus an $M$-extension $\Phi$ of $f$. The diagram

$$\begin{tikzcd} M\wedge K \arrow[r, "\Phi"] \arrow[d, equal] & S^2\wedge X \arrow[r] \arrow[d, "\cong"', "\gamma(\alpha_{1,1}(\iota\wedge s_{(1)}))"] & C\hat{a}_{(1)} \arrow[d, equal] \\ M\wedge K \arrow[r, "\gamma(\hat{a}_{(1)})"'] & Z_{(2)} \arrow[r] & C\hat{a}_{(1)} \end{tikzcd}$$

commutes in $\mathbf{Ho}(\mathcal{C})$; since the lower row is part of a distinguished triangle by (4.6), so is the upper row. The mapping cone $C\hat{a}_{(1)}$ is the underlying object of the $M$-module $C\hat{a}$ which is $k$-coherent by Proposition 4.18(ii).

(ii) We proceed by induction on $k$. For $k=0$ there is nothing to show, so we assume $k\geqslant 1$. Given any object $K$ of $\mathcal{C}$, we let

$$K \xrightarrow{p\cdot} K \xrightarrow{\pi} K/p \longrightarrow \Sigma K$$

be a distinguished triangle that provides a cone of multiplication by $p$ on $K$. We obtain a distinguished triangle by smashing this triangle from the left with $S^2$, and another one from (4.10) (with $n=p$ and with $K$ instead of $X$). So there is an isomorphism $\psi : M\wedge K\cong S^2\wedge (K/p)$ in $\mathbf{Ho}(\mathcal{C})$ such that $S^2\wedge\pi = \psi\circ\gamma(\iota\wedge K)$ as morphisms $S^2\wedge K\to S^2\wedge (K/p)$.

Now we let $f : K\to X$ be any morphism in $\mathbf{Ho}(\mathcal{C})$. Part (i) provides an $M$-extension $\Phi : M\wedge K\to S^2\wedge X$ of $f$ and a cone $C$ of $\Phi$ that admits a $k$-coherent $M$-action. By induction, this cone $C$ has $p$-order at least $k-1$. Since smashing with $S^2$ is isomorphic to double suspension, and thus an equivalence of categories, the map

$$\mathbf{Ho}(\mathcal{C})(K/p,X) \longrightarrow \mathbf{Ho}(\mathcal{C})(M\wedge K, S^2\wedge X), \quad \varphi\longmapsto (S^2\wedge\varphi)\circ\psi$$

is a bijection; so there is a unique morphism $\bar{f} : K/p\to X$ such that $(S^2\wedge\bar{f})\circ\psi=\Phi$, and the morphism $\bar{f}$ is an extension of $f$. Finally, if $C\bar{f}$ is a cone of the extension $\bar{f}$, then $S^2\wedge C\bar{f}$ is a cone of $\Phi = (S^2\wedge\bar{f})\circ\psi : M\wedge K\to S^2\wedge X$. The $p$-order is invariant under suspension and isomorphism, so the cone of $\bar{f}$ has $p$-order at least $k-1$. Hence $X$ has $p$-order at least $k$. □

Now we can prove our main result.

THEOREM 5.3. *Let $\mathcal{T}$ be a topological triangulated category and $p$ be an odd prime. Then, for every object $X$ of $\mathcal{T}$, the object $X/p$ has $p$-order at least $p-2$. In particular, the $p$-order of $\mathcal{T}$ is at least $p-1$.*

*Proof.* It suffices to treat the case where $\mathcal{T}=\mathbf{Ho}(\mathcal{C})$ is the homotopy category of a stable cofibration category $\mathcal{C}$. We may assume, without loss of generality, that $\mathcal{C}$ is *saturated*, that is, every morphism that becomes an isomorphism in $\mathbf{Ho}(\mathcal{C})$ is a weak equivalence. Indeed, for an arbitrary cofibration category we can define another cofibration structure $\mathcal{C}^{\mathrm{sat}}$ on the same category $\mathcal{C}$ with the same class of cofibrations as before, but with new weak equivalences those morphisms that become isomorphisms in the (old) homotopy category. By Cisinski [**6**, Proposition 3.16], this is indeed a saturated cofibration structure and the identity functor $\mathcal{C} \to \mathcal{C}^{\mathrm{sat}}$ induces an isomorphism of homotopy categories

$$\mathbf{Ho}(\mathcal{C}) \longrightarrow \mathbf{Ho}(\mathcal{C}^{\mathrm{sat}}).$$

Assuming now that the cofibration category $\mathcal{C}$ is saturated, Theorem 3.10 lets us replace it by the category $f\mathcal{C}$ of frames in $\mathcal{C}$ without changing the homotopy category; the cofibration category $f\mathcal{C}$ of frames is then a $\Delta$-cofibration category by Theorem 3.17. The upshot is that we can assume without loss of generality that $\mathcal{C}$ is a stable $\Delta$-cofibration category.

Now we let $X$ be an arbitrary object of $\mathcal{C}$. We choose an object $Y$ whose 2-fold suspension is isomorphic to $X$ in $\mathbf{Ho}(\mathcal{C})$. Then the smash product $M \wedge Y$ with the mod-$p$ Moore space is isomorphic to $X/p$, by the distinguished triangle (4.10). This smash product $M \wedge Y$ is underlying the free $M$-module $M \triangle Y$ that is $(p-1)$-coherent. So $M \wedge Y$ has $p$-order at least $p-2$ by Proposition 5.2(ii). The last claim follows because the $p$-order of a triangulated category is one larger than the minimum of the $p$-orders of all objects of the form $K/p$. □

The stable homotopy category and its $p$-localization are topological, so Theorem 5.3 shows that their $p$-orders are at least $p-1$. In combination with [**30**, Theorem 3.1], we conclude that the $p$-local stable homotopy category has $p$-order exactly $p-1$.

It is an open question whether there is an odd prime $p$, a triangulated category $\mathcal{T}$ (necessarily non-topological) and an object $X$ of $\mathcal{T}$ such that the $p$-order of $X/p$ is less than $p-2$, or even $p \cdot X/p \neq 0$.

Remark 5.4. Theorem 5.3 can be improved to give a sufficient condition for when the $p$-order of $X/p$ in a topological triangulated category is at least $p-1$. A choice of model for $\mathcal{T}$ (that is, a stable cofibration category $\mathcal{C}$ and an exact equivalence $\mathcal{T} \cong \mathbf{Ho}(\mathcal{C})$) provides an action of the homotopy category of finite $\Delta$-sets on $\mathcal{T}$; see Remark 3.11. The obstruction for the object $X/p$ to have $p$-order strictly greater than $p-2$ can be expressed in terms of this action as follows.

We let $\alpha_1 : S^{2p} \to S^3$ denote a morphism of finite $\Delta$-sets such that the geometric realizations of $S^{2p}$ and $S^3$ are homotopy equivalent to a $2p$-sphere and a 3-sphere, respectively, and such that the realization of $\alpha_1$ is a generator of the $p$-torsion in the homotopy group $\pi_{2p}(S^3)$; here $p=2$ is allowed, and then $\alpha_1$ realizes the homotopy class of the Hopf map $\eta$. The arguments of [**28**, Theorem 2.5] can be generalized from simplicial stable model categories to stable $\Delta$-cofibration categories to show that if the morphism $\alpha_1 \wedge X : S^{2p} \wedge X \to S^3 \wedge X$ is zero (or divisible by $p$) in $\mathbf{Ho}(\mathcal{C})$, then the object $X/p$ admits a $p$-coherent $M$-action. By Proposition 5.2(ii), the object $X/p$ then has $p$-order at least $p-1$.

We want to emphasize that the morphism $\alpha_1 \wedge X : S^{2p} \wedge X \to S^3 \wedge X$ does not have an intrinsic meaning in a topological triangulated category $\mathcal{T}$, and depends on the choice of model for $\mathcal{T}$. The $K_{(p)}$-local stable homotopy category at an odd prime $p$ is an explicit example where different models lead to different morphisms $\alpha_1 \wedge X$: there is the 'natural model', that is, the category of sequential spectra [**3**, Definition 2.1] with the $K_{(p)}$-localization of the stable model structure of Bousfield and Friedlander. The class $\alpha_1$ maps non-trivially to the $(2p-3)$th homotopy group of the localized sphere spectrum $L_{K_{(p)}}\mathbb{S}$, so it acts non-trivially on $L_{K_{(p)}}\mathbb{S}$. However, Franke's theorem [**10**, Section 2.2, Theorem 5] provides an 'exotic' algebraic model for $\mathbf{Ho}(K_{(p)}\text{-local})$, and in any algebraic model, all positive-dimensional stable homotopy classes act trivially on all objects.

*Appendix. Homotopy category, suspension and triangulation*

In this appendix, we recall certain facts about the homotopy category of a cofibration category that we need in the body of the paper. In particular, we introduce the suspension functor on the homotopy category of a pointed cofibration category and show in Theorem A.12 that the homotopy category of a stable cofibration category is naturally triangulated.

For us the *homotopy category* of a cofibration category is any localization of $\mathcal{C}$ at the class of weak equivalences. Hence the homotopy category consists of a category $\mathbf{Ho}(\mathcal{C})$ with the same objects as $\mathcal{C}$ and a functor $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ that is the identity on objects, takes all weak equivalences to isomorphisms and satisfies the following universal property: for every category $\mathcal{D}$ and every functor $F : \mathcal{C} \to \mathcal{D}$ that takes all weak equivalences to isomorphisms there is a unique functor $\bar{F} : \mathbf{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $\bar{F}\gamma = F$.

A *cylinder object* for an object $A$ in a cofibration category is a quadruple $(I, i_0, i_1, p)$ consisting of an object $I$, morphisms $i_0, i_1 : A \to I$ and a weak equivalence $p : I \to A$ satisfying $pi_0 = pi_1 = \mathrm{Id}_A$ and such that $i_0 + i_1 : A \vee A \to I$ is a cofibration. Every object has a cylinder object: axiom (C4) allows us to factor the fold map $\mathrm{Id} + \mathrm{Id} : A \vee A \to A$ as a cofibration $i_0 + i_1 : A \vee A \to I$ followed by a weak equivalence $p : I \to A$.

Two morphisms $f, g : A \to Z$ in a cofibration category are *homotopic* if there exists a cylinder object $(I, i_0, i_1, p)$ for $A$ and a morphism $H : I \to Z$ (the *homotopy*) such that $f = Hi_0$ and $g = Hi_1$. Since the morphism $p$ in a cylinder object is a weak equivalence, $\gamma(p)$ is an isomorphism in $\mathbf{Ho}(\mathcal{C})$ and so $\gamma(i_0) = \gamma(i_1)$ since they share $\gamma(p)$ as common left inverse. So, if $f$ and $g$ are homotopic via $H$, then $\gamma(f) = \gamma(H)\gamma(i_0) = \gamma(H)\gamma(i_1) = \gamma(g)$. In other words: homotopic morphisms become equal in the homotopy category. The converse is not true in general, but part (ii) of the following theorem says that the converse is true up to postcomposition with a weak equivalence.

Parts (i) and (ii) of the following theorem are the dual statements to Theorem 1 and Remark 2 of [**4**, part I.2]. The results can also be found, with more detailed proofs and slightly different terminology, as Theorem 6.4.5(1a) and Theorem 6.4.4(1c) in [**26**], respectively. Part (iii) (or rather the dual statement) is a special case of [**6**, Corollary 2.9].

THEOREM A.1. *Let $\mathcal{C}$ be a cofibration category and $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ be a localization at the class of weak equivalences.*

(i) *Every morphism in $\mathbf{Ho}(\mathcal{C})$ is a 'left fraction', that is, of the form $\gamma(s)^{-1}\gamma(f)$, where $f$ and $s$ are $\mathcal{C}$-morphisms with the same target and $s$ is an acyclic cofibration.*

(ii) *Given two morphisms $f, g : A \to B$ in $\mathcal{C}$, we have $\gamma(f) = \gamma(g)$ in $\mathbf{Ho}(\mathcal{C})$ if and only if there is an acyclic cofibration $s : B \to \bar{B}$ such that $sf$ and $sg$ are homotopic.*

(iii) *The localization functor $\gamma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$ preserves coproducts. In particular, the homotopy category $\mathbf{Ho}(\mathcal{C})$ has finite coproducts.*

REMARK A.2. On the face of it, the homotopy category of a cofibration category raises set-theoretic issues: in general the hom-'sets' in $\mathbf{Ho}(\mathcal{C})$ may not be small, but rather proper classes. One way to deal with this is to work with universes in the sense of Grothendieck; the homotopy category of a cofibration category then always exists in a larger universe.

Another way to address the set theory issues is to restrict attention to those cofibration categories that have 'enough fibrant objects'. An object of a cofibration category $\mathcal{C}$ is *fibrant* if every acyclic cofibration out of it has a retraction. If the object $Z$ is fibrant, then the map

$$\mathcal{C}(A, Z) \longrightarrow \mathbf{Ho}(\mathcal{C})(A, Z), \quad f \longmapsto \gamma(f)$$

is surjective: an arbitrary morphism from $A$ to $Z$ in $\mathbf{Ho}(\mathcal{C})$ is of the form $\gamma(s)^{-1}\gamma(a)$ for some acyclic cofibration $s : Z \to Z'$. Since $Z$ is fibrant, there is a retraction $r$ with $rs = \mathrm{Id}_Z$, and then

$\gamma(s)^{-1}\gamma(a) = \gamma(ra)$. Moreover, if two $\mathcal{C}$-morphisms $f, g : A \to Z$ become equal after applying the functor $\gamma$, then there is an acyclic cofibration $s : Z \to Z'$ such that $sf$ is homotopic to $sg$. Composing with any retraction to $s$ shows that $f$ is already homotopic to $g$. So 'homotopy' for morphisms into a fibrant object $Z$ is an equivalence relation and the map

$$\mathcal{C}(A, Z)/_{\text{homotopy}} \longrightarrow \mathbf{Ho}(\mathcal{C})(A, Z), \quad [f] \longmapsto \gamma(f)$$

is bijective.

We say that the cofibration category $\mathcal{C}$ *has enough fibrant objects* if, for every object $X$, there is a weak equivalence $r : X \to Z$ with fibrant target. For example, if $\mathcal{C}$ is the collection of cofibrant objects in an ambient Quillen model category, then it has enough fibrant objects. In the cofibration structure of chain complexes in an additive category discussed in Example 1.6, every object is fibrant (and so, in particular, there are enough fibrant objects).

If $r : X \to Z$ a weak equivalence with fibrant target, then, for every other object $A$ the two maps

$$\mathbf{Ho}(\mathcal{C})(A, X) \xrightarrow{\gamma(r)_*} \mathbf{Ho}(\mathcal{C})(A, Z) \xleftarrow{\gamma} \mathcal{C}(A, Z)/_{\text{homotopy}}$$

are bijective, so the morphisms $\mathbf{Ho}(\mathcal{C})(A, X)$ form a set (as opposed to a proper class). So, if $\mathcal{C}$ has enough fibrant objects, then the homotopy category $\mathbf{Ho}(\mathcal{C})$ has small hom-sets (or is 'locally small').

Now we proceed to the construction of the cone and suspension functor. From now on, $\mathcal{C}$ is a *pointed* cofibration category, that is, every initial object is also terminal, hence a zero object. The following construction of the suspension functor is isomorphic to the construction by Quillen [**25**, I.2] and (the dual of) Brown [**4**, Part I, Theorem 3], although our exposition is somewhat different.

A *cone* in a pointed cofibration category $\mathcal{C}$ is a cofibration $i : A \to C$ whose target $C$ is weakly contractible. The unique morphism from any given object to the terminal object can be factored as a cofibration followed by a weak equivalence; so every object is the source of a cone. Cones of objects in a pointed cofibration category are unique up to homotopy in a rather strong sense. Indeed, the category $\mathrm{Cone}\,\mathcal{C}$ of cones in $\mathcal{C}$ has a natural structure of cofibration category and the source functor that sends a cone $i : A \to C$ to $A$ is exact and passes to an equivalence $\mathbf{Ho}(\mathrm{Cone}\,\mathcal{C}) \to \mathbf{Ho}(\mathcal{C})$ of homotopy categories.

Let us now choose a cone for every object $A$ of $\mathcal{C}$, that is, a cofibration $i_A : A \to CA$ with weakly contractible target. The *suspension* $\Sigma A$ of $A$ is then a cokernel of the chosen cone inclusion, that is, a pushout:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d] & CA \arrow[d, "p"] \\ * \arrow[r] & \Sigma A. \end{tikzcd}$$

Lemma A.3. *Let $i : A \to C$ be a cone and $\alpha : A \to B$ be a morphism in a pointed cofibration category $\mathcal{C}$. Then there exists* a cone extension *of $\alpha$, that is, a pair $(\bar\alpha, s)$ consisting of a morphism $\bar\alpha : C \to \bar C$ and an acyclic cofibration $s : CB \to \bar C$ such that $\bar\alpha i = s i_B \alpha$ and such that the induced morphism $\bar\alpha \cup s : C \cup_A CB \to \bar C$ is a cofibration, where the source is a pushout of $i$ and $i_B\alpha$. Moreover, the composite morphism in $\mathbf{Ho}(\mathcal{C})$*

$$C/A \xrightarrow{\gamma(\bar\alpha/\alpha)} \bar C/B \xrightarrow{\gamma(s/B)^{-1}} CB/B = \Sigma B$$

*is independent of the cone extension $(\bar\alpha, s)$.*

*Proof.* Since $i$ is a cofibration, we can choose a pushout:

$$\begin{tikzcd} A \arrow[r, "i_B\alpha"] \arrow[d, "i"'] & CB \arrow[d, "i'"] \\ C \arrow[r, "k"'] & C \cup_A CB. \end{tikzcd}$$

Then we choose a cone $l : C \cup_A CB \to \bar{C}$, and $\bar{\alpha} = lk$ and $s = li'$ have the desired properties.

Suppose that $(\bar{\alpha}' : C \to \bar{C}', s' : CB \to \bar{C}')$ is another cone extension. Let us first suppose that there is a morphism $\varphi : \bar{C} \to \bar{C}'$ (necessarily a weak equivalence) such that $\varphi\bar{\alpha} = \bar{\alpha}'$ and $\varphi s = s'$. Then we have

$$\begin{aligned} \gamma(s/B)^{-1} \circ \gamma(\bar{\alpha}/\alpha) &= \gamma(s/B)^{-1} \circ \gamma(\varphi/B)^{-1} \circ \gamma(\varphi/B) \circ \gamma(\bar{\alpha}/\alpha) \\ &= \gamma(\varphi s/B)^{-1} \circ \gamma(\varphi\bar{\alpha}/\alpha) = \gamma(s'/B)^{-1} \circ \gamma(\bar{\alpha}'/\alpha). \end{aligned}$$

In the general case we choose a pushout:

$$\begin{tikzcd} C \cup_A CB \arrow[r, "\bar{\alpha}\cup s"] \arrow[d, "\bar{\alpha}'\cup s'"'] & \bar{C} \arrow[d, "k'"] \\ \bar{C}' \arrow[r, "k"'] & P. \end{tikzcd}$$

Then $(i_P k'\bar{\alpha}, i_P k's)$ is yet another cone extension of $\alpha$, where $i_P : P \to CP$ is a cone on $P$. Moreover, this new cone extension receives morphisms from both $(\bar{\alpha}, s)$ and $(\bar{\alpha}', s')$, and so, by the special case, all three cone extensions give rise to the same morphism $C/A \to \Sigma B$ in $\mathbf{Ho}(\mathcal{C})$. □

Now we can define suspension on morphisms, thereby extending it to a functor $\Sigma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$. Given a $\mathcal{C}$-morphism $\alpha : A \to B$, we choose a cone extension $(\bar{\alpha}, s)$ with respect to the chosen cone $i_A : A \to CA$, as in Lemma A.3. We define $\Sigma\alpha$ as the composite in $\mathbf{Ho}(\mathcal{C})$

$$\Sigma A = CA/A \xrightarrow{\gamma(\bar{\alpha}/\alpha)} \bar{C}/B \xrightarrow{\gamma(s/B)^{-1}} CB/B = \Sigma B.$$

Lemma A.3 guarantees that this definition is independent of the cone extension.

Proposition A.4. *The suspension construction is a functor $\Sigma : \mathcal{C} \to \mathbf{Ho}(\mathcal{C})$. The suspension functor takes weak equivalences to isomorphisms and preserves coproducts.*

*Proof.* The pair $(\mathrm{Id}_{CA}, \mathrm{Id}_{CA})$ is a cone extension of the identity of $A$, so we have $\Sigma\,\mathrm{Id}_A = \mathrm{Id}_{\Sigma A}$. Given another morphism $\beta : B \to D$ and a cone extension $(\bar{\beta}, t)$ of $\beta$, we choose a pushout:

$$\begin{tikzcd} CB \arrow[r, "s"] \arrow[d, "\bar{\beta}"'] & \bar{C} \arrow[d, "\bar{\beta}'"] \\ \bar{C}' \arrow[r, "\bar{s}"'] & E. \end{tikzcd}$$

The morphism $\bar{s}$ is an acyclic cofibration since $s$ is. So the pair $(\bar{\beta}'\bar{\alpha}, \bar{s}t)$ is a cone extension of $\beta\alpha : A \to D$ and we obtain

$$\begin{aligned} \Sigma(\beta\alpha) &= \gamma(\bar{s}t/D)^{-1} \circ \gamma(\bar{\beta}'\bar{\alpha}/\beta\alpha) = \gamma(t/D)^{-1} \circ \gamma(\bar{s}/D)^{-1} \circ \gamma(\bar{\beta}'/\beta) \circ \gamma(\bar{\alpha}/\alpha) \\ &= \gamma(t/D)^{-1} \circ \gamma(\bar{\beta}/\beta) \circ \gamma(s/B)^{-1} \circ \gamma(\bar{\alpha}/\alpha) = (\Sigma\beta) \circ (\Sigma\alpha). \end{aligned}$$

The third equation uses that $\bar{\beta}'s = \bar{s}\bar{\beta}$. So the suspension construction is functorial. If $\alpha : A \to B$ is a weak equivalence and $(\bar{\alpha}, s)$ a cone extension, then $\bar{\alpha}/\alpha : CA/A \to \bar{C}/B$ is a weak equivalence by the gluing lemma. So $\gamma(\bar{\alpha}/\alpha)$, and hence $\Sigma\alpha$ is an isomorphism in $\mathbf{Ho}(\mathcal{C})$.

It remains to show that the suspension functor preserves coproducts. Indeed, if $i_A : A \to CA$, $i_B : B \to CB$ and $i_{A\vee B} : A \vee B \to C(A \vee B)$ are the chosen cones for two objects $A$, $B$ and a coproduct $A \vee B$, then $i_A \vee i_B : A \vee B \to CA \vee CB$ is another cone, so Lemma A.3 provides a morphism $\bar{\alpha} : CA \vee CB \to \bar{C}$ and an acyclic cofibration $s : C(A \vee B) \to \bar{C}$ such that $\bar{\alpha} \circ (i_A \vee i_B) = s \circ i_{A\vee B}$. Passing to quotients and applying the localization functor produces an isomorphism

$$\begin{aligned}\Sigma A \vee \Sigma B = (CA \vee CB)/(A \vee B) &\xrightarrow{\gamma(\bar{\alpha}/A\vee B)} \bar{C}/(A \vee B) \\ &\xrightarrow{\gamma(s/A\vee B)^{-1}} C(A \vee B)/(A \vee B) = \Sigma(A \vee B),\end{aligned}$$

which is in fact the canonical morphism. □

Since the suspension functor takes weak equivalences to isomorphisms, it descends to a unique functor

$$\Sigma : \mathbf{Ho}(\mathcal{C}) \longrightarrow \mathbf{Ho}(\mathcal{C})$$

such that $\Sigma \circ \gamma = \Sigma$. Since coproducts in $\mathcal{C}$ are coproducts in $\mathbf{Ho}(\mathcal{C})$, this induced suspension functor again preserves coproducts.

Remark A.5. In many examples, cones (and hence suspensions) can be chosen functorially already on the level of the cofibration category. However, the punchline of the previous construction is that even without functorial cones in $\mathcal{C}$, suspension becomes functorial after passage to the homotopy category. On the other hand, it would not be a serious loss of generality to assume functorial cones and suspensions. Indeed, Theorems 3.10 and 3.17 together say that, for every saturated cofibration category $\mathcal{C}$, the category $f\mathcal{C}$ of frames in $\mathcal{C}$ is a $\Delta$-cofibration category such that $\mathbf{Ho}(\mathcal{C})$ is equivalent to $\mathbf{Ho}(f\mathcal{C})$. Moreover, if $\mathcal{C}$ is pointed, then so is $f\mathcal{C}$ and the category $f\mathcal{C}$ has functorial cones and functorial suspensions given by the smash product with the 'based interval' $I$ and the 'circle' $S^1$, respectively, cf. Example 4.4.

There is extra structure on a suspension, namely a certain *collapse morphism* $\kappa_A : \Sigma A \to \Sigma A \vee \Sigma A$ in $\mathbf{Ho}(\mathcal{C})$. To define it, we consider a pushout $CA \cup_A CA$ of two copies of the cone $CA$ along $i_A$. The gluing lemma guarantees that the map $0 \cup p : CA \cup_A CA \to CA/A = \Sigma A$ induced on horizontal pushouts of the left commutative diagram

$$\begin{tikzcd} CA \arrow[d, "\sim"'] & A \arrow[l, "i_A"'] \arrow[r, "i_A"] \arrow[d, equal] & CA \arrow[d, equal] \\ * & A \arrow[l] \arrow[r, "i_A"'] & CA \end{tikzcd} \qquad \begin{tikzcd} CA \arrow[d, "p"'] & A \arrow[l, "i_A"'] \arrow[r, "i_A"] \arrow[d] & CA \arrow[d, "p"] \\ \Sigma A & * \arrow[l] \arrow[r] & \Sigma A \end{tikzcd}$$

is a weak equivalence. We define the $\kappa_A$ as the composite

$$\Sigma A \xrightarrow{\gamma(0\cup p)^{-1}} CA \cup_A CA \xrightarrow{\gamma(p\cup p)} \Sigma A \vee \Sigma A,$$

where the second morphism is the image of the $\mathcal{C}$-morphism induced on horizontal pushouts of the right commutative diagram above.

Proposition A.6. *The morphism $\kappa_A : \Sigma A \to \Sigma A \vee \Sigma A$ satisfies the relations*

$$(0+\mathrm{Id})\kappa_A = \mathrm{Id} \quad \textit{and} \quad (\mathrm{Id}+\mathrm{Id})\kappa_A = 0$$

*as endomorphisms of $\Sigma A$ in $\mathbf{Ho}(\mathcal{C})$. The endomorphism*

$$m_A = (\mathrm{Id}+0)\kappa_A$$

*of $\Sigma A$ is an involution, that is, $m_A^2 = \mathrm{Id}$. The morphism $\kappa_A$ is natural, that is, for every morphism $a : A \to B$ we have $(\Sigma a \vee \Sigma a) \circ \kappa_A = \kappa_B \circ (\Sigma a)$.*

*Proof.* We observe that $(0+\mathrm{Id}) \circ (p \cup p) = 0 \cup p$ as $\mathcal{C}$-morphisms $CA \cup_A CA \to \Sigma A$, so

$$(0+\mathrm{Id}) \circ \kappa_A = (0+\mathrm{Id}) \circ \gamma(p \cup p) \circ \gamma(0 \cup p)^{-1} = \gamma(0 \cup p) \circ \gamma(0 \cup p)^{-1} = \mathrm{Id}\,.$$

The square

$$\begin{tikzcd} CA \cup_A CA \arrow[r, "p\cup p"] \arrow[d, "\mathrm{Id}\cup\mathrm{Id}"'] & \Sigma A \vee \Sigma A \arrow[d, "\mathrm{Id}+\mathrm{Id}"] \\ CA \arrow[r, "p"'] & \Sigma A \end{tikzcd}$$

commutes in $\mathcal{C}$, so the morphism $(\mathrm{Id}+\mathrm{Id}) \circ \kappa_A = (\mathrm{Id}+\mathrm{Id}) \circ \gamma(p \cup p) \circ \gamma(0 \cup p)^{-1}$ factors through the cone $CA$, which is a zero object in $\mathbf{Ho}(\mathcal{C})$. Thus $(\mathrm{Id}+\mathrm{Id})\kappa_A = 0$.

For the next relation, we denote by $\tau$ the involution of $CA \cup_A CA$ that interchanges the two cones. Then we have

$$\begin{aligned} m_A &= (\mathrm{Id}+0) \circ \gamma(p \cup p) \circ \gamma(0 \cup p)^{-1} = \gamma(p \cup 0) \circ \gamma(0 \cup p)^{-1} \\ &= \gamma(0 \cup p) \circ \gamma(\tau) \circ \gamma(0 \cup p)^{-1}. \end{aligned}$$

Since $\tau^2 = \mathrm{Id}$, this leads to $m_A^2 = \mathrm{Id}$.

Every morphism in $\mathbf{Ho}(\mathcal{C})$ is of the form $\gamma(s)^{-1}\gamma(\alpha)$ for $\mathcal{C}$-morphisms $\alpha$ and $s$; so it suffices to prove the naturality statement for morphisms of the form $a = \gamma(\alpha)$ for a $\mathcal{C}$-morphism $\alpha : A \to B$. We choose a cone extension $(\bar{\alpha}, s)$ of $\alpha$ and consider the commutative diagram:

$$\begin{tikzcd} \Sigma A \arrow[r, "\bar{\alpha}/\alpha"] & \bar{C}B/B & \Sigma B \arrow[l, "s/\mathrm{Id}_B"', "\sim"] \\ CA \cup_A CA \arrow[u, "0\cup p", "\sim"'] \arrow[r, "\bar{\alpha}\cup\bar{\alpha}"] \arrow[d, "p\cup p"] & \bar{C}B \cup_{si_B} \bar{C}B \arrow[u, "0\cup p", "\sim"'] \arrow[d, "p\cup p"] & CB \cup_B CB \arrow[l, "s\cup s"', "\sim"] \arrow[u, "0\cup p"', "\sim"] \arrow[d, "p\cup p"] \\ \Sigma A \vee \Sigma A \arrow[r, "\bar{\alpha}/\alpha\vee\bar{\alpha}/\alpha"'] & \bar{C}B/B \vee \bar{C}B/B & \Sigma B \vee \Sigma B. \arrow[l, "\sim", "s/\mathrm{Id}_B \vee s/\mathrm{Id}_B"'] \end{tikzcd}$$

The vertical maps going up and the horizontal maps going left are weak equivalences, so they become isomorphisms in the homotopy category. After inverting these weak equivalences in $\mathbf{Ho}(\mathcal{C})$, the composite through the upper right corner becomes $\kappa_B \circ \Sigma\gamma(\alpha)$ and the composite through the lower left corner becomes $(\Sigma\gamma(\alpha) \vee \Sigma\gamma(\alpha)) \circ \kappa_A$. $\square$

We call a pointed cofibration category *stable* if the suspension functor $\Sigma : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(\mathcal{C})$ is an auto-equivalence. We can now show that the homotopy category of a stable cofibration category is additive. By Theorem A.1(iii), the coproduct in any cofibration category $\mathcal{C}$ descends to a coproduct in the homotopy category $\mathcal{C}$. We will show that, for stable $\mathcal{C}$, the coproduct $X \vee Y$ is also a product of $X$ and $Y$ in $\mathbf{Ho}(\mathcal{C})$ with respect to the morphisms $p_X = \mathrm{Id}+0 : X \vee Y \to X$ and $p_Y = 0+\mathrm{Id} : X \vee Y \to Y$. So we have to show that, for every object $B$ of $\mathcal{C}$, the map

$$\mathbf{Ho}(\mathcal{C})(B, X \vee Y) \to \mathbf{Ho}(\mathcal{C})(B, X) \times \mathbf{Ho}(\mathcal{C})(B, Y), \quad \varphi \longmapsto (p_X\varphi,\, p_Y\varphi) \tag{A.7}$$

is bijective.

PROPOSITION A.8. *Let $\mathcal{C}$ be a pointed cofibration category.*

(i) *If the object $B$ is a suspension, then the map (A.7) is surjective.*

(ii) *Let $\varphi, \psi : B \to X \vee Y$ be morphisms in $\mathbf{Ho}(\mathcal{C})$ such that $p_X\varphi = p_X\psi$ and $p_Y\varphi = p_Y\psi$. Then $\Sigma\varphi = \Sigma\psi$.*

(iii) *If $\mathcal{C}$ is stable, then the homotopy category $\mathbf{Ho}(\mathcal{C})$ is additive and, for every object $A$ of $\mathcal{C}$, the morphism $m_A : \Sigma A \to \Sigma A$ is the negative of the identity of $\Sigma A$.*

*Proof.* (i) Given two morphisms $\alpha : \Sigma A \to X$ and $\beta : \Sigma A \to Y$ in $\mathbf{Ho}(C)$, we consider the morphism $((\alpha \circ m_A) \vee \beta) \circ \kappa_A : \Sigma A \to X \vee Y$. This morphism then satisfies

$$p_X \circ ((\alpha \circ m_A) \vee \beta) \circ \kappa_A = \alpha \circ m_A \circ (\mathrm{Id} + 0) \circ \kappa_A = \alpha \circ m_A^2 = \alpha$$

and similarly $p_Y \circ ((\alpha \circ m_A) \vee \beta) \circ \kappa_A = \beta$. So the map (A.7) is surjective for $B = \Sigma A$.

(ii) We first show that the composite

$$\Sigma(X \vee Y) \xrightarrow{\kappa_{X\vee Y}} \Sigma(X \vee Y) \vee \Sigma(X \vee Y) \xrightarrow{(\Sigma\iota_X)m_X(\Sigma p_X)+(\Sigma\iota_Y)(\Sigma p_Y)} \Sigma(X \vee Y) \tag{A.9}$$

is the identity of $\Sigma(X \vee Y)$. Here $\iota_X : X \to X \vee Y$ and $\iota_Y : Y \to X \vee Y$ are the canonical morphisms. Indeed, after precomposition with $\Sigma\iota_X : \Sigma X \to \Sigma(X \vee Y)$ we have

$$\begin{aligned}
&((\Sigma\iota_X)m_X(\Sigma p_X) + (\Sigma\iota_Y)(\Sigma p_Y)) \circ \kappa_{X\vee Y} \circ (\Sigma\iota_X) \\
&\quad = ((\Sigma\iota_X)m_X(\Sigma p_X) + (\Sigma\iota_Y)(\Sigma p_Y)) \circ (\Sigma\iota_X \vee \Sigma\iota_X) \circ \kappa_X \\
&\quad = ((\Sigma\iota_X)m_X + 0) \circ \kappa_X = ((\Sigma\iota_X)m_X) \circ (\mathrm{Id}_{\Sigma X} + 0) \circ \kappa_X \\
&\quad = (\Sigma\iota_X) \circ m_X^2 = \Sigma\iota_X.
\end{aligned}$$

Similarly, we have $((\Sigma\iota_X)m_X(\Sigma p_X) + (\Sigma\iota_Y)(\Sigma p_Y)) \circ \kappa_{X\vee Y} \circ (\Sigma\iota_Y) = \Sigma\iota_Y$. Since the suspension functor preserves coproducts, a morphism out of $\Sigma(X \vee Y)$ is determined by precomposition with $\Sigma\iota_X$ and $\Sigma\iota_Y$. This proves that the composite (A.9) is the identity. For $\varphi : B \to X \vee Y$ we then have

$$\begin{aligned}
\Sigma\varphi &= ((\Sigma\iota_X)m_X(\Sigma p_X) + (\Sigma\iota_Y)(\Sigma p_Y)) \circ \kappa_{X\vee Y} \circ (\Sigma\varphi) \\
&= ((\Sigma\iota_X)m_X(\Sigma p_X) + (\Sigma\iota_Y)(\Sigma p_Y)) \circ (\Sigma\varphi \vee \Sigma\varphi) \circ \kappa_B \\
&= ((\Sigma\iota_X)m_X\Sigma(p_X\varphi) + (\Sigma\iota_Y)\Sigma(p_Y\varphi)) \circ \kappa_B.
\end{aligned}$$

So $\Sigma\varphi$ is determined by the composites $p_X\varphi$ and $p_Y\varphi$, and this proves the claim.

(iii) Since $\mathcal{C}$ is stable, every object is isomorphic to a suspension, so the map (A.7) is always surjective by part (i). Moreover, suspension is faithful, so the map (A.7) is always injective by part (ii). Thus the map (A.7) is bijective for all objects $B, X$ and $Y$, and so coproducts in $\mathbf{Ho}(\mathcal{C})$ are also products.

It is well known that in any category with zero object that has coproducts that are also products, the morphism sets in $\mathcal{T}$ can then be endowed with a natural structure of an abelian monoid as follows; see, for example, [**17**, Theorem 8.2.14]. Given $f, g : B \to Z$, let $f \perp g : B \to Z \vee Z$ be the unique morphism such that $(\mathrm{Id} + 0)(f \perp g) = f$ and $(0 + \mathrm{Id})(f \perp g) = g$. Then the assignment $f + g = (\mathrm{Id} + \mathrm{Id})(f \perp g)$ is an associative, commutative and binatural operation on the set of morphisms from $B$ to $Z$ with neutral element given by the zero morphism.

The collapse map $\kappa_A : \Sigma A \to \Sigma A \vee \Sigma A$ satisfies $(\mathrm{Id} + 0)\kappa = m_A$ and $(0 + \mathrm{Id})\kappa_A = \mathrm{Id}$, and so $\kappa_A = m_A \perp \mathrm{Id}$. So we have $m_A + \mathrm{Id} = (\mathrm{Id} + \mathrm{Id})\kappa_A = 0$. This shows that the morphism $m_A$ is the additive inverse of the identity of $\Sigma A$. In particular, the abelian monoid $\mathbf{Ho}(\mathcal{C})(\Sigma A, Z)$ has inverses, and is thus an abelian group. Since every object is isomorphic to a suspension, the abelian monoid $\mathbf{Ho}(\mathcal{C})(B, Z)$ is a group for all objects $B$ and $Z$, and so $\mathbf{Ho}(\mathcal{C})$ is an additive category. □

Now we introduce the class of distinguished triangles. Given a cofibration $j : A \to B$ in a pointed cofibration category $\mathcal{C}$, we define the *connecting morphism* $\delta(j) : B/A \to \Sigma A$ in $\mathbf{Ho}(\mathcal{C})$ as

$$\delta(j) = \gamma(p \cup 0) \circ \gamma(0 \cup q)^{-1} \; : \; B/A \longrightarrow \Sigma A. \tag{A.10}$$

Here $q : B \to B/A$ is the quotient morphism, $p \cup 0 : CA \cup_j B \to \Sigma A$ is the morphism that collapses $B$ and $0 \cup q : CA \cup_j B \to B/A$ is the weak equivalence that collapses $CA$. The *elementary distinguished triangle* associated to the cofibration $j$ is the sequence

$$A \xrightarrow{\gamma(j)} B \xrightarrow{\gamma(q)} B/A \xrightarrow{\delta(j)} \Sigma A.$$

A *distinguished triangle* is any triangle in the homotopy category that is isomorphic to the elementary distinguished triangle of a cofibration in $\mathcal{C}$.

Proposition A.11. *The connecting morphism* (A.10) *is natural in the following sense: for every commutative square in* $\mathcal{C}$ *on the left*

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\ A' \arrow[r, "j'"'] & B' \end{tikzcd} \qquad\qquad \begin{tikzcd} B/A \arrow[r, "\delta(j)"] \arrow[d, "\gamma(\beta/\alpha)"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\ B'/A' \arrow[r, "\delta(j')"'] & \Sigma A' \end{tikzcd}$$

*such that $j$ and $j'$ are cofibrations, the square on the right commutes in* $\mathbf{Ho}(\mathcal{C})$.

*Proof.* We choose a cone extension $(\bar{\alpha}, s)$ of the morphism $\alpha$ as in the commutative diagram on the left:

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[d, "\alpha"'] & CA \arrow[d, "\bar{\alpha}"] \\ A' \arrow[r, "si_{A'}"] \arrow[d, equal] & \bar{C} \\ A' \arrow[r, "i_{A'}"'] & CA' \arrow[u, "\sim"', "s"] \end{tikzcd} \qquad \begin{tikzcd} B/A \arrow[d, "\beta/\alpha"'] & CA \cup_j B \arrow[l, "0\cup q"', "\sim"] \arrow[r, "p\cup 0"] \arrow[d] & \Sigma A \arrow[d, "\bar{\alpha}/\alpha"] \\ B'/A' \arrow[d, equal] & \bar{C} \cup_{j'} B' \arrow[l, "0\cup q'"', "\sim"] \arrow[r, "\bar{p}'\cup 0"] & \bar{C}/A' \\ B'/A' & CA' \cup_{j'} B' \arrow[l, "0\cup q'"] \arrow[r, "p'\cup 0"'] \arrow[u, "\sim"', "s\cup B'"] & \Sigma A'. \arrow[u, "\sim"', "s/A'"] \end{tikzcd}$$

From this we form the commutative diagram on the right. After passage to $\mathbf{Ho}(\mathcal{C})$ we can invert the weak equivalences that point to the left or upwards, and then the composite through the lower left corner becomes $\delta(j') \circ \gamma(\beta/\alpha)$ and the composite through the upper right corner becomes $\Sigma\gamma(\alpha) \circ \delta(j)$. □

Now we can prove that the homotopy category of a stable category is triangulated. For us, a triangulated category is an additive category $\mathcal{T}$ equipped with an auto-equivalence $\Sigma : \mathcal{T} \to \mathcal{T}$ and a class of distinguished triangles of the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A,$$

closed under isomorphism, that satisfies the following axioms.

(T1) For every object $X$ the triangle $0 \to X \xrightarrow{\mathrm{Id}} X \to 0$ is distinguished.
(T2) [Rotation] If a triangle $(f, g, h)$ is distinguished, then so is the triangle $(g, h, -\Sigma f)$.
(T3) [Completion of triangles] Given distinguished triangles $(f, g, h)$ and $(f', g', h')$ and morphisms $a$ and $b$ satisfying $bf = f'a$, there exists a morphism $c$ making the following diagram commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"'] & C \arrow[r, "h"] \arrow[d, dotted, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'.
\end{tikzcd}$$

(T4) [Octahedral axiom] For every pair of composable morphisms $f : A \to B$ and $f' : B \to D$ there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, "g"] \arrow[d, "f'"'] & C \arrow[r, "h"] \arrow[d, "x"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "f'f"'] & D \arrow[r, "g''"'] \arrow[d, "g'"'] & E \arrow[r, "h''"'] \arrow[d, "y"] & \Sigma A \arrow[d, "\Sigma f"] \\
& F \arrow[r, equal] \arrow[d, "h'"'] & F \arrow[r, "h'"'] \arrow[d, "(\Sigma g)\circ h'"] & \Sigma B \\
& \Sigma B \arrow[r, "\Sigma g"'] & \Sigma C &
\end{tikzcd}$$

such that the triangles $(f, g, h)$, $(f', g', h')$, $(f'f, g'', h'')$ and $(x, y, (\Sigma g) \circ h')$ are distinguished.

This version of the axioms appears to be weaker, at first sight, than the original formulation: Verdier [**34**, II.1] requires an 'if and only if' in (T2), and his formulation of (T4) appears stronger. However, the two formulations of the axioms are equivalent.

Theorem A.12. *The suspension functor and the class of distinguished triangles make the homotopy category* $\mathbf{Ho}(\mathcal{C})$ *of a stable cofibration category into a triangulated category.*

This result generalizes and provides a uniform approach to proofs of triangulations for homotopy categories of pretriangulated dg-categories, additive categories, derived categories of abelian categories, stable categories of Frobenius categories and for homotopy categories of stable model categories. Indeed, all these classes of categories have underlying stable cofibration categories. The only triangulated categories I am aware of that cannot be established via Theorem A.12 are the exotic examples of [**23**].

Remark A.13. Although I am not aware of a complete proof of Theorem A.12 in the present generality, I do not claim much originality; indeed, various parts of this result, assuming the same or closely related structure, are scattered throughout the literature. Nevertheless, I hope that a self-contained and complete account on the triangulation of the homotopy category, assuming only the axioms of a stable cofibration category, is useful.

Before embarking on the proof of Theorem A.12, I want to point out the relevant related sources that I am aware of. In Section I.2 of his monograph [**25**], Quillen shows that in any pointed closed model category, the homotopy category supports a functorial suspension functor with values in cogroup objects. In Section I.3, Quillen introduces the class of cofibration sequences and shows that they satisfy most of (the unstable versions of) the axioms of a triangulated category. In [**4**, Section I.4], Brown adapts Quillen's arguments to the more general context of 'categories of fibrant objects', which is strictly dual to the cofibration categories that we use. In [**12**, Section 3], Heller introduces the notion of 'h-c-category', an axiomatization closely related to (but slightly different from) that of cofibration categories; Heller indicates in his Theorem 9.2 how the stabilization of the homotopy category of an h-c-category is triangulated. We want to stress, though, that none of these three sources bothers to prove the octahedral axiom.

In the context of stable model categories, a complete proof of the triangulation of the homotopy category, including the octahedral axiom, is given by Hovey [**14**, Section 7.1]. However, Hovey's account is hiding the fact that the triangulation is available under much weaker hypotheses; for example, the fibrations, the existence of general colimits and functorial factorization are irrelevant for this particular purpose.

Yet another approach to triangulating the homotopy category of a cofibration category $\mathcal{C}$ uses the whole system of homotopy categories of suitable diagrams in $\mathcal{C}$. This idea has been made precise in slightly different forms by different people, for example by Grothendieck as a *derivator* [**11**, **21**], by Heller as a *homotopy theory* [**13**], by Keller as an *epivalent tower of suspended categories* [**18**] and by Franke as a *system of triangulated diagram categories* [**10**]. The respective 'stable' versions come with theorems showing that the underlying category of such a stable collection of homotopy categories is naturally triangulated; cf. [**11**, Theorem 4.16] or [**10**, Section 1.4, Theorem 1].

Cisinski shows in [**6**, Corollary 2.24] that the homotopy category of every cofibration category is underlying a right derivator (parameterized by finite, direct indexing categories). I am convinced that our definition of 'stable' (that is, the suspension functor is an auto-equivalence of the homotopy category) implies that the associated derivator is 'triangulated', that is, that the right derivator is automatically also a left derivator, hence a 'derivator' (without any adjective), and that homotopy cartesian square and homotopy co-cartesian square coincide. However, as far as I know, this link is not established anywhere in the literature.

*Proof of Theorem* A.12. We have seen in Proposition A.8(iii) that the homotopy category of a stable cofibration category is additive, and the suspension functor is an equivalence by assumption. So it remains to verify the axioms (T1)–(T4). We want to emphasize that the following proof of (T1)–(T4) works in any pointed cofibration category, without a stability assumption. The only place where stability is used is at the very end of the rotation axiom (T2), where the morphism $(\Sigma f) \circ \delta(i_A)$ is identified with $-\Sigma f$.

(T1) The unique morphism from any zero object to $X$ is a cofibration with quotient morphism the identity of $X$. The triangle $(0, \mathrm{Id}_X, 0)$ is the associated elementary distinguished triangle.

(T2) We start with a distinguished triangle $(f, g, h)$ and want to show that the triangle $(g, h, -\Sigma f)$ is also distinguished. It suffices to consider the elementary distinguished triangle $(\gamma(j), \gamma(q), \delta(j))$ associated to a cofibration $j : A \to B$. We choose pushouts for the left and the outer square as in the left diagram:

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, "i_A"'] & B \arrow[r] \arrow[d, "k"'] & * \arrow[d] \\
CA \arrow[r] & CA \cup_j B \arrow[r, "p\cup 0"'] & \Sigma A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, "\gamma(k)"] \arrow[d, equal] & CA \cup_j B \arrow[r, "\gamma(p\cup 0)"] \arrow[d, "\gamma(0\cup q)"', "\cong"] & \Sigma A \arrow[r, "\delta(k)"] \arrow[d, equal] & \Sigma B \arrow[d, equal] \\
B \arrow[r, "\gamma(q)"'] & B/A \arrow[r, "\delta(j)"'] & \Sigma A \arrow[r, "\Sigma\gamma(j)\circ\delta(i_A)"'] & \Sigma B.
\end{tikzcd}$$

The second square in the left diagram is then also a pushout and the morphism $p \cup 0 : CA \cup_j B \to \Sigma A$ is the quotient projection associated to the cofibration $k : B \to CA \cup_j B$. Moreover, both $i_A$ and $k$ are cofibrations, and so, by the naturality of the connecting morphisms, we get

$\delta(k) \circ \mathrm{Id}_{\Sigma A} = (\Sigma\gamma(j)) \circ \delta(i_A)$. Hence the diagram on the right commutes. The upper row is the elementary distinguished triangle of the cofibration $k$, and all vertical maps are isomorphisms, so the lower triangle is distinguished, as claimed. By definition, the connecting morphism $\delta(i_A)$ coincides with the involution $m_A$ of $\Sigma A$. In the stable context, $m_A$ is the negative of the identity (see Proposition A.8(iii)), so $(\Sigma f) \circ \delta(i_A) = -\Sigma f$.

(T3) We are given two distinguished triangles $(f, g, h)$ and $(f', g', h')$ and two morphisms $a$ and $b$ in $\mathbf{Ho}(\mathcal{C})$ satisfying $bf = f'a$ as in the diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[r, "g"] \arrow[d, "b"] & C \arrow[r, "h"] \arrow[d, dashed, "c"] & \Sigma A \arrow[d, "\Sigma a"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & \Sigma A'. \end{tikzcd}$$

We have to extend these data to a morphism of triangles, that is, find a morphism $c$ making the entire diagram commute. If we can solve the problem for isomorphic triangles, then we can also solve it for the original triangles. We can thus assume that the triangles $(f, g, h)$ and $(f', g', h')$ are the elementary distinguished triangles arising from two cofibrations $j : A \to B$ and $j' : A' \to B'$.

We start with the special case where $a = \gamma(\alpha)$ and $b = \gamma(\beta)$ for $\mathcal{C}$-morphisms $\alpha : A \to A'$ and $\beta : B \to B'$. Then $\gamma(j'\alpha) = \gamma(\beta j)$, so Theorem A.1(ii) provides an acyclic cofibration $s : B' \to \bar{B}$, a cylinder object $(I, i_0, i_1, p)$ for $A$ and a homotopy $H : I \to \bar{B}$ from $Hi_0 = sj'\alpha$ to $Hi_1 = s\beta j$. The following diagram of cofibrations on the left commutes in $\mathcal{C}$, so the diagram of elementary distinguished triangles on the right commutes in $\mathbf{Ho}(\mathcal{C})$ by the naturality of the connecting morphisms:

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, equal] & B \\ A \arrow[r, "i_0"] \arrow[d, "\alpha"'] & I \cup_{i_1} B \arrow[u, "\sim", "jp\cup B"'] \arrow[d, "H\cup s\beta"] \\ A' \arrow[r, "sj'"] \arrow[d, equal] & \bar{B} \\ A' \arrow[r, "j'"'] & B' \arrow[u, "\sim", "s"'] \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q)"] & B/A \arrow[r, "\delta(j)"] & \Sigma A \arrow[d, equal] \\ A \arrow[r, "\gamma(i_0)"] \arrow[d, "\gamma(\alpha)"'] & I \cup_{i_1} B \arrow[r, "\gamma(q)"] \arrow[u, "\gamma(jp\cup B)", "\cong"'] \arrow[d, "\gamma(H\cup s\beta)"'] & (I \cup_{i_1} B)/A \arrow[r, "\delta(i_0)"] \arrow[u, "\cong", "\gamma((jp\cup B)/A)"'] \arrow[d, "\gamma((H\cup s\beta)/\alpha)"] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\ A' \arrow[r, "\gamma(sj')"] \arrow[d, equal] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/A' \arrow[r, "\delta(sj')"] & \Sigma A' \arrow[d, equal] \\ A' \arrow[r, "\gamma(j')"'] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(s)", "\cong"'] & B'/A' \arrow[r, "\delta(j')"'] \arrow[u, "\cong", "\gamma(s/A')"'] & \Sigma A'. \end{tikzcd}$$

The morphism

$$c = \gamma(s/A')^{-1} \circ \gamma((H \cup s\beta)/\alpha) \circ \gamma((jp \cup B)/A)^{-1} \ : \ B/A \longrightarrow B'/A'$$

is the desired filler.

In the general case we write $a = \gamma(s)^{-1}\gamma(\alpha)$ where $\alpha : A \to \bar{A}$ is a $\mathcal{C}$-morphism and $s : A' \to \bar{A}$ is an acyclic cofibration. We choose a pushout

$$\begin{tikzcd} \bar{A} \arrow[r, "k"] & \bar{A} \cup_{A'} B' \\ A' \arrow[r, "j'"'] \arrow[u, "s", "\sim"'] & B'. \arrow[u, "\sim", "s'"'] \end{tikzcd}$$

We write $\gamma(s')b = \gamma(t)^{-1}\gamma(\beta) : B \to \bar{A} \cup_{A'} B'$ where $\beta : B \to \bar{B}$ is a $\mathcal{C}$-morphism and $t : \bar{A} \cup_{A'} B' \to \bar{B}$ is an acyclic cofibration. We then have

$$\gamma(tk)\gamma(\alpha) = \gamma(tk)\gamma(s)a = \gamma(ts')\gamma(j')a = \gamma(ts')b\gamma(j) = \gamma(\beta)\gamma(j),$$

and so, by the special case, applied to the cofibrations $j : A \to B$ and $tk : \bar{A} \to \bar{B}$ and the morphisms $\alpha : A \to \bar{A}$ and $\beta : B \to \bar{B}$, there exists a morphism $c : B/A \to \bar{B}/\bar{A}$ in $\mathbf{Ho}(\mathcal{C})$ making the diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, "\gamma(\alpha)"'] & B \arrow[r, "\gamma(q)"] \arrow[d, "\gamma(\beta)"'] & B/A \arrow[r, "\delta(j)"] \arrow[d, dotted, "c"'] & \Sigma A \arrow[d, "\Sigma\gamma(\alpha)"] \\
\bar{A} \arrow[r, "\gamma(tk)"] & \bar{B} \arrow[r, "\gamma(\bar{q})"] & \bar{B}/\bar{A} \arrow[r, "\delta(tk)"] & \Sigma\bar{A} \\
A' \arrow[r, "\gamma(j')"'] \arrow[u, "\gamma(s)"] & B' \arrow[r, "\gamma(q')"'] \arrow[u, "\gamma(ts')"'] & B'/A' \arrow[r, "\delta(j')"'] \arrow[u, "\gamma(ts'/s)"'] & \Sigma A' \arrow[u, "\Sigma\gamma(s)"']
\end{tikzcd}$$

commute (the lower part commutes by the naturality of connecting morphisms). Since $s$ is an acyclic cofibration, so is its cobase change $s'$. By the gluing lemma the weak equivalences $s : A' \to \bar{A}$ and $ts' : B' \to \bar{B}$ induce a weak equivalence $ts'/s : B'/A' \to \bar{B}/\bar{A}$ on quotients and the composite

$$B/A \xrightarrow{c} \bar{B}/\bar{A} \xrightarrow{\gamma(ts'/s)^{-1}} B'/A'$$

in $\mathbf{Ho}(\mathcal{C})$ thus solves the original problem.

(T4) We start with the special case where $f = \gamma(j)$ and $f' = \gamma(j')$ for cofibrations $j : A \to B$ and $j' : B \to D$. Then the composite $j'j : A \to D$ is a cofibration with $\gamma(j'j) = f'f$. The diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma(j)"] \arrow[d, equal] & B \arrow[r, "\gamma(q_j)"] \arrow[d, "\gamma(j')"'] & B/A \arrow[r, "\delta(j)"] \arrow[d, "\gamma(j'/A)"] & \Sigma A \arrow[d, equal] \\
A \arrow[r, "\gamma(j'j)"'] & D \arrow[r, "\gamma(q_{j'j})"'] \arrow[d, "\gamma(q_{j'})"'] & D/A \arrow[r, "\delta(j'j)"'] \arrow[d, "\gamma(D/j)"] & \Sigma A \arrow[d, "\Sigma\gamma(j)"] \\
 & D/B \arrow[r, equal] \arrow[d, "\delta(j')"'] & D/B \arrow[r, "\delta(j')"'] \arrow[d, "\delta(j'/A)=(\Sigma\gamma(q_j))\delta(j')"] & \Sigma B \\
 & \Sigma B \arrow[r, "\Sigma\gamma(q_j)"'] & \Sigma(B/A) &
\end{tikzcd}$$

then commutes by the naturality of connecting morphisms. Moreover, the four triangles in question are the elementary distinguished triangles of the cofibrations $j$, $j'$, $j'j$ and $j'/A : B/A \to D/A$.

In the general case we write $f = \gamma(s)^{-1}\gamma(a)$ for a $\mathcal{C}$-morphism $a : A \to B'$ and a weak equivalence $s : B \to B'$. Then $a$ can be factored as $a = pj$ for a cofibration $j : A \to \bar{B}$ and a weak equivalence $p : \bar{B} \to B'$. Altogether we then have $f = \varphi \circ \gamma(j)$ where $\varphi = \gamma(s)^{-1} \circ \gamma(p) : \bar{B} \to B$ is an isomorphism in $\mathbf{Ho}(\mathcal{C})$. We can apply the same reasoning to the morphism $f'\varphi : \bar{B} \to D$ and write it as $f' \circ \varphi = \psi \circ \gamma(j')$ for a cofibration $j' : \bar{B} \to \bar{D}$ in $\mathcal{C}$ and an isomorphism $\psi : \bar{D} \to D$ in $\mathbf{Ho}(\mathcal{C})$. The special case can then be applied to the cofibrations $j : A \to \bar{B}$ and $j' : \bar{B} \to \bar{D}$. The resulting commutative diagram that solves (T4) for $(\gamma(j), \gamma(j'))$ can then be translated back into a commutative diagram that solves (T4) for $(f, f')$ by conjugating with the isomorphisms $\varphi : \bar{B} \to B$ and $\psi : \bar{D} \to D$. This completes the proof of the octahedral axiom (T4), and hence the proof of Theorem A.12. $\square$

Now we discuss how exact functors between cofibration categories give rise to exact functors between the triangulated homotopy categories. A functor $F : \mathcal{C} \to \mathcal{D}$ between cofibration categories is *exact* if it preserves initial objects, cofibrations, weak equivalences and the particular pushouts (1.2) along cofibrations that are guaranteed by axiom (C3). Since $F$ preserves weak equivalences, the composite functor $\gamma^{\mathcal{D}} \circ F : \mathcal{C} \to \mathbf{Ho}(\mathcal{D})$ takes weak equivalences to isomorphisms and the universal property of the homotopy category provides a unique *derived functor* $\mathbf{Ho}(F) : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(\mathcal{D})$ such that $\mathbf{Ho}(F) \circ \gamma^{\mathcal{C}} = \gamma^{\mathcal{D}} \circ F$. An exact functor between cofibration categories in particular preserves coproducts. Since coproducts in $\mathcal{C}$ descend to coproducts in the homotopy category, the derived functor of any exact functor also preserves coproducts.

We will now explain that, for pointed cofibration categories $\mathcal{C}$ and $\mathcal{D}$, the derived functor $\mathbf{Ho}(F)$ commutes with suspension up to a preferred natural isomorphism

$$\tau_F \ : \ \mathbf{Ho}(F) \circ \Sigma \xrightarrow{\cong} \Sigma \circ \mathbf{Ho}(F)$$

of functors from $\mathbf{Ho}(\mathcal{C})$ to $\mathbf{Ho}(\mathcal{D})$. If $A$ is any object of $\mathcal{C}$, then the cofibration $F(i_A) : F(A) \to F(CA)$ is a cone since $F$ is exact. Lemma A.3 provides a cone extension of the identity of $F(A)$, that is, a morphism $\bar{\alpha} : F(CA) \to \bar{C}$, necessarily a weak equivalence, and an acyclic cofibration $s : C(F(A)) \to \bar{C}$ such that $si_{F(A)} = \bar{\alpha} F(i_A)$. The composite in $\mathbf{Ho}(\mathcal{D})$

$$\tau_{F,A} \ : \ F(\Sigma A) = F(CA)/F(A) \xrightarrow{\gamma(\bar{\alpha}/F(A))} \bar{C}/F(A) \xrightarrow{\gamma(s/F(A))^{-1}} \Sigma F(A)$$

is then an isomorphism, and independent (by Lemma A.3) of the cone extension $(\bar{\alpha}, s)$.

Proposition A.14. *Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between pointed cofibration categories. Then the isomorphism $\tau_{F,A} : F(\Sigma A) \to \Sigma(FA)$ is natural in $A$ and makes the derived functor $\mathbf{Ho}(F) : \mathbf{Ho}(\mathcal{C}) \to \mathbf{Ho}(\mathcal{D})$ into an exact functor.*

*Proof.* Let $j : A \to B$ be a cofibration in $\mathcal{C}$, with $q : B \to B/A$ a quotient morphism. Since $F$ is exact, $F(j)$ is a cofibration in $\mathcal{D}$ and $F(q) : F(B) \to F(B/A)$ is a quotient morphism for $F(j)$. We claim that the connecting morphism $\delta_{F(j)} : F(B/A) \to \Sigma F(A)$ of the cofibration $F(j)$ equals $\tau_{F,A} \circ \mathbf{Ho}(F)(\delta(j))$. To see this we choose a cone extension $(\bar{\alpha}, s)$ of the identity of $F(A)$ as in the construction of $\tau_{F,A}$. We can build the commutative diagram

$$\begin{tikzcd}
F(B/A) \arrow[d, equal] & F(CA) \cup_{F(j)} F(B) \arrow[l, "0\cup F(q)"', "\sim"] \arrow[r, "F(p)\cup 0"] \arrow[d, "\sim"', "\bar{\alpha}\cup \mathrm{Id}"] \arrow[rrr, dashed, bend left, "\mathbf{Ho}(F)(\delta(j))"] & F(CA)/F(A) \arrow[r, equal] \arrow[d, "\sim"', "\bar{\alpha}/F(A)"] & F(\Sigma A) \arrow[dd, dashed, "\tau_{F,A}"] \\
F(B/A) \arrow[d, equal] & \bar{C} \cup_{F(j)} F(B) \arrow[l, "0\cup F(q)", "\sim"'] \arrow[r, "\bar{p}\cup 0"] & \bar{C}/F(A) & \\
F(B/A) & C(F(A)) \cup_{F(j)} F(B) \arrow[l, "\sim", "0\cup F(q)"'] \arrow[r, "p\cup 0"'] \arrow[u, "\sim", "s\cup \mathrm{Id}"'] \arrow[rrr, dashed, bend right, "\delta(F(j))"'] & C(F(A))/F(A) \arrow[r, equal] \arrow[u, "\sim", "s/F(A)"'] & \Sigma F(A).
\end{tikzcd}$$

After passing to the homotopy category of $\mathcal{D}$, we can invert the weak equivalences and form the dashed morphisms. Since the exact functor $F$ preserves the pushout that defines $CA \cup_j B$, the composite of the top row then becomes $\mathbf{Ho}(F)(\delta(j))$. Since the vertical composite on the right is the isomorphism $\tau_{F,A}$, this proves the relation $\tau_{F,A} \circ \mathbf{Ho}(F)(\delta(j)) = \delta(F(j))$.

Now we can prove the proposition. We have to show that, for every distinguished triangle $(f, g, h)$ in $\mathbf{Ho}(\mathcal{C})$, the triangle

$$F(A) \xrightarrow{\mathbf{Ho}(F)(f)} F(B) \xrightarrow{\mathbf{Ho}(F)(g)} F(C) \xrightarrow{\tau_{F,A} \circ \mathbf{Ho}(F)(h)} \Sigma F(A)$$

is distinguished in $\mathbf{Ho}(\mathcal{D})$. If suffices to consider the elementary distinguished triangle of a cofibration $j : A \to B$ in $\mathcal{C}$; because $\mathbf{Ho}(F) \circ \gamma = \gamma \circ F$, we are then dealing with the triangle

$$F(A) \xrightarrow{\gamma(F(j))} F(B) \xrightarrow{\gamma(F(q))} F(B/A) \xrightarrow{\tau_{F,A} \circ \mathbf{Ho}(F)(\delta(j))} \Sigma F(A).$$

By the claim above, this triangle is the elementary distinguished triangle of the cofibration $F(j)$, and this concludes the proof. □

Remark A.15. Let us consider two different choices of cones $\{i_A : A \to CA\}_A$ and $\{i'_A : A \to C'A\}_A$ on a pointed cofibration category $\mathcal{C}$, where $A$ runs through all objects of $\mathcal{C}$. These different cones give rise to different suspension functors $\Sigma$ and $\Sigma'$ and different collections $\Delta$ and $\Delta'$ of distinguished triangles. We can apply the previous proposition with $\mathcal{C} = \mathcal{D}$ and $F = \mathrm{Id}$, the identity functor of $\mathcal{C}$, to the two triangulations. The proposition provides a natural isomorphism $\tau : \Sigma \to \Sigma'$ between the two suspension functors such that for every distinguished $\Delta$-triangle $(f, g, h)$, the triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\tau_A \circ h} \Sigma' A$$

is a distinguished $\Delta'$-triangle. So the triangulations arising from different choices of cones are canonically isomorphic.

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