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EQUIVARIANT PROPERTIES OF SYMMETRIC PRODUCTS

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Introduction

We let $Sp^\infty(X)$ denote the infinite symmetric product of a based space X. It comes with a filtration by finite symmetric products $Sp^n(X) = X^n/\Sigma_n$. We denote by

$$Sp^n = \{Sp^n(S^m)\}_{m\geq 0}$$

the spectrum whose terms are the nth symmetric products of spheres. A celebrated theorem of Dold and Thom [7] asserts that $Sp^\infty(S^m)$ is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, m)$ for $m \geq 1$; so Sp^∞ is an Eilenberg-Mac Lane spectrum for the group $\mathbb{Z}$. The resulting filtration

$$\mathbb{S} \;=\; Sp^1 \;\subseteq\; Sp^2 \;\subseteq\; \cdots \;\subseteq\; Sp^n \;\subseteq\; \cdots$$

of the Eilenberg-Mac Lane spectrum Sp^∞, starting with the sphere spectrum $\mathbb{S}$, has been much studied. The subquotient Sp^n/Sp^{n-1} is stably contractible unless n is a prime power. If $n = p^k$ for a prime p, then Sp^n/Sp^{n-1} is p-torsion, and its mod-p cohomology has been worked out by Nakaoka [19]. For $p = 2$ these subquotient spectra feature in the work of Mitchell and Priddy on stable splitting of classifying spaces [18], and in Kuhn's solution of the Whitehead conjecture [10]. Arone and Dwyer relate these spectra to the partition complex, the homology of the dual Lie representation, and the Tits building [1].

While the symmetric product filtration has been a major focus of research since the 1980s, essentially nothing was known when one adds group actions into the picture. This paper is about equivariant features of the symmetric product filtration, for actions of compact Lie groups G. If V is a finite dimensional orthogonal G-representation, then G acts continuously on the one-point compactification S^V, and hence on $Sp^n(S^V)$ and $Sp^\infty(S^V)$ by functoriality of symmetric products. As V varies over all such G-representations, the G-spaces $Sp^n(S^V)$ form a G-spectrum that represents a "genuine" G-equivariant stable homotopy type. For understanding these equivariant homotopy types it is extremely beneficial not to study one compact Lie group at a time, but to use the "global" perspective. Here "global" refers to simultaneous and compatible actions of all compact Lie groups. Various ways to formalize this idea have been explored (compare [12, Chapter II], [9, Section 5], [4]); we use a different approach via orthogonal spectra.

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In Definition 2.7 we introduce the notion of *global functor*, a useful language to describe the collection of equivariant homotopy groups of an orthogonal spectrum as a whole, i.e., when the compact Lie group is allowed to vary. The category of global functors is a symmetric monoidal abelian category that takes up the role in global homotopy theory played by abelian groups in ordinary homotopy theory, or by G-Mackey functors in G-equivariant homotopy theory. As a consequence of Theorem 2.12 we will see that a global functor is a certain kind of "global Mackey functor" that assigns abelian groups to all compact Lie groups and comes with restriction maps along continuous group homomorphisms and transfer maps along inclusions of closed subgroups.

In this language, we can then identify the global functor $\underline{\pi}_0(Sp^n)$ as a quotient of the Burnside ring global functor $\mathbb{A}$ by a single relation. We define an element t_n in the Burnside ring of the nth symmetric group by

$$t_n \;=\; n \cdot 1 \;-\; \mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}(1) \;\in\; \mathbb{A}(\Sigma_n).$$

As an element in the Grothendieck group of finite Σ_n-sets, the class t_n corresponds to the formal difference of a trivial Σ_n-set with n elements and the tautological Σ_n-set $\{1, \ldots, n\}$. Since t_n has zero augmentation, the global subfunctor $\langle t_n \rangle$ generated by t_n lies in the augmentation ideal global functor I. The restriction of t_n to the Burnside ring of Σ_{n-1} equals t_{n-1}, so we obtain a nested sequence of global functors

$$0 = \langle t_1 \rangle \;\subset\; \langle t_2 \rangle \;\subset\; \cdots \;\subset\; \langle t_n \rangle \;\subset\; \cdots \subset I \;\subset\; \mathbb{A}\;.$$

As part of our main result, Theorem 3.12, we prove the following theorem.

Theorem. *For every $n \geq 1$ the morphism of global functors $i_* : \mathbb{A} = \underline{\pi}_0(\mathbb{S}) \longrightarrow \underline{\pi}_0(Sp^n)$ induced by the embedding $i : \mathbb{S} = Sp^1 \longrightarrow Sp^n$ passes to an isomorphism of global functors*

$$\mathbb{A}/\langle t_n \rangle \;\cong\; \underline{\pi}_0(Sp^n)\;.$$

It is then a purely algebraic exercise to describe $\pi_0^G(Sp^n)$ as an explicit quotient of the Burnside ring $\mathbb{A}(G)$: one has to enumerate all relations in $\mathbb{A}(G)$ obtained by applying restrictions and transfers to the class t_n. We do this in Proposition 4.1 and then work out the examples of p-groups and some symmetric groups. The author thinks that the explicit answer for $\pi_0^G(Sp^n)$ is far less enlightening than the global description of $\underline{\pi}_0(Sp^n)$. Since all the inclusions $\langle t_{n-1} \rangle \subset \langle t_n \rangle$ are proper, the subquotients Sp^n/Sp^{n-1} are all globally non-trivial, in sharp contrast to the non-equivariant situation.

Our calculation of $\underline{\pi}_0(Sp^n)$ is a consequence of a global homotopy pushout square (see Theorem 3.8), showing that Sp^n is obtained from Sp^{n-1} by coning off a certain morphism from the suspension spectrum of the global classifying space of the family of non-transitive subgroups of Σ_n. This homotopy pushout square is a global equivariant refinement of a non-equivariant homotopy pushout established by Lesh [11].

Another consequence of our calculations is a possibly unexpected feature of the equivariant homotopy groups $\pi_0^G(Sp^\infty)$ when G has a positive dimension. We let I_∞ denote the global subfunctor of $\mathbb{A}$ generated by all the classes t_n for $n \geq 1$. Also in Theorem 3.12 we show that the embedding $\mathbb{S} \longrightarrow Sp^\infty$ induces an isomorphism of global functors

$$\mathbb{A}/I_\infty \;\cong\; \underline{\pi}_0(Sp^\infty)\;.$$

For every compact Lie group G the restriction map

$$\mathrm{res}^G_e \ : \ \pi_0^G(Sp^\infty) \ \longrightarrow \ \pi_0^e(Sp^\infty) \ \cong \ \mathbb{Z}$$

to the non-equivariant zeroth homotopy group is a split epimorphism onto a free abelian group of rank 1. When the group G is *finite*, then this restriction map is an isomorphism and all G-equivariant homotopy groups of Sp^∞ vanish in dimensions different from 0. So through the eyes of finite groups, Sp^∞ is an Eilenberg-Mac Lane spectrum for the constant global functor $\underline{\mathbb{Z}}$. This does *not*, however, generalize to compact Lie groups of positive dimension. In that generality, the restriction map res^G_e can have a non-trivial kernel; equivalently, the value of the global functor I_∞ at some compact Lie groups is strictly smaller than the augmentation ideal. We discuss these phenomena in more detail at the end of Section 4.

1. Orthogonal spaces

In this section we recall orthogonal spaces from a global equivariant perspective. We work in the category $\mathcal{T}$ of *compactly generated spaces* in the sense of [17], i.e., k-spaces (also called *Kelley spaces*) that satisfy the weak Hausdorff condition. An *inner product space* is a finite dimensional $\mathbb{R}$-vector space V equipped with a scalar product. We write $O(V)$ for the orthogonal group of V, i.e., the Lie group of linear isometries of V. We denote by $\mathbf{L}$ the category with objects the inner product spaces and morphisms the linear isometric embeddings. This is a topological category as follows: if $\varphi : V \longrightarrow W$ is one linear isometric embedding, then the action of the orthogonal group $O(W)$, by postcomposition, induces a bijection

$$O(W)/O(\varphi^\perp) \ \cong \ \mathbf{L}(V,W) \ , \quad A\cdot O(\varphi^\perp) \ \longmapsto A\circ\varphi \ ,$$

where $\varphi^\perp = W - \varphi(V)$ is the orthogonal complement of the image of φ. We topologize $\mathbf{L}(V,W)$ so that this bijection is a homeomorphism, and this topology is independent of φ. So if $n = \dim(V)$, then $\mathbf{L}(V,W)$ is homeomorphic to the Stiefel manifold of orthonormal n-frames in W.

Definition 1.1. An *orthogonal space* is a continuous functor $Y : \mathbf{L} \longrightarrow \mathcal{T}$ to the category of spaces. A morphism of orthogonal spaces is a natural transformation. We denote by *spc* the category of orthogonal spaces.

The systematic use of inner product spaces to index objects in homotopy theory seems to go back to Boardman's thesis [2]. The category $\mathbf{L}$ (or its extension that also contains countably infinite dimensional inner product spaces) is denoted $\mathscr{I}$ by Boardman and Vogt [3], and this notation is also used in [15]; other sources use the symbol $\mathcal{I}$. Accordingly, orthogonal spaces are sometimes referred to as $\mathscr{I}$-functors, $\mathscr{I}$-spaces, or $\mathcal{I}$-spaces. Our justification for using yet another name is twofold: on the one hand, we shift the emphasis away from a focus on non-equivariant homotopy types and toward viewing an orthogonal space as representing compatible equivariant homotopy types for all compact Lie groups. Second, we want to stress the analogy between orthogonal spaces and orthogonal spectra, the former being an unstable global world and the latter a corresponding stable global world.

We let G be a compact Lie group. By a *G-representation* we mean an orthogonal G-representation, i.e., an inner product space V equipped with a continuous G-action by linear isometries. For every orthogonal space Y and every

G-representation V, the value $Y(V)$ inherits a G-action from V through the functoriality of Y. For a G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the induced map $Y(\varphi) : Y(V) \longrightarrow Y(W)$ is G-equivariant.

Now we discuss the equivariant homotopy set $\pi_0^G(Y)$ of an orthogonal space Y; this is an unstable precursor of the zeroth equivariant stable homotopy group of an orthogonal spectrum.

Definition 1.2. Let G be a compact Lie group. A *G-universe* is an orthogonal G-representation $\mathcal{U}$ of countably infinite dimension with the following two properties:

- the representation $\mathcal{U}$ has non-zero G-fixed points,
- if a finite dimensional representation V embeds into $\mathcal{U}$, then a countable infinite sum of copies of V also embeds into $\mathcal{U}$.

A G-universe is *complete* if every finite dimensional G-representation embeds into it.

A G-universe is characterized, up to equivariant isometry, by the set of irreducible G-representations that embed into it. A universe is complete if and only if every irreducible G-representation embeds into it. In the following we fix, for every compact Lie group G, a complete G-universe $\mathcal{U}_G$. We let $s(\mathcal{U}_G)$ denote the poset, under inclusion, of finite dimensional G-subrepresentations of $\mathcal{U}_G$.

We let Y be an orthogonal space and define its G-equivariant path components as

$$\pi_0^G(Y) \ = \ \operatorname{colim}_{V\in s(\mathcal{U}_G)} \pi_0\left(Y(V)^G\right) \ . \tag{1.3}$$

As the group varies, the homotopy sets $\pi_0^G(Y)$ have contravariant functoriality in G: every continuous group homomorphism $\alpha : K \longrightarrow G$ between compact Lie groups induces a restriction map $\alpha^* : \pi_0^G(Y) \longrightarrow \pi_0^K(Y)$, as we shall now explain. We denote by α^* the restriction functor from G-spaces to K-spaces (or from G-representations to K-representations), i.e., α^*Z (respectively, α^*V) is the same topological space as Z (respectively, the same inner product space as V) endowed with K-action via

$$k \cdot z \ = \ \alpha(k) \cdot z \ .$$

Given an orthogonal space Y, we note that for every G-representation V, the K-spaces $\alpha^*(Y(V))$ and $Y(\alpha^*V)$ are equal (not just isomorphic).

The restriction $\alpha^*(\mathcal{U}_G)$ is a K-universe, but if α has a non-trivial kernel, then this K-universe is not complete. When α is injective, then $\alpha^*(\mathcal{U}_G)$ is a complete K-universe, but typically different from the chosen complete K-universe $\mathcal{U}_K$. To deal with this we explain how a G-fixed point $y \in Y(V)^G$, for an arbitrary G-representation V, gives rise to an unambiguously defined element $\langle y\rangle$ in $\pi_0^G(Y)$. The point here is that V need not be a subrepresentation of the chosen universe $\mathcal{U}_G$ and the resulting class does not depend on any additional choices. To construct $\langle y\rangle$ we choose a linear isometric G-embedding $j : V \longrightarrow \mathcal{U}_G$ and look at the image $Y(j)(y)$ under the G-map

$$Y(V) \xrightarrow{Y(j)} Y(j(V)) \ .$$

Here we have used the letter j to also denote the isometry $j : V \longrightarrow j(V)$ to the image of V; since $j(V)$ is a finite dimensional G-invariant subspace of $\mathcal{U}_G$, we obtain an element

$$\langle y\rangle \ = \ [Y(j)(y)] \ \in \ \pi_0^G(Y) \ .$$

It is crucial, although not particularly difficult, that $\langle y\rangle$ does not depend on the choice of embedding j.

Proposition 1.4. *Let Y be an orthogonal space, G a compact Lie group, V a G-representation, and $y \in Y(V)^G$ a G-fixed point.*

(i) *The class $\langle y \rangle$ in $\pi_0^G(Y)$ is independent of the choice of linear isometric embedding $j : V \longrightarrow \mathcal{U}_G$.*

(ii) *For every G-equivariant linear isometric embedding $\varphi : V \longrightarrow W$ the relation*

$$\langle Y(\varphi)(y) \rangle \;=\; \langle y \rangle \qquad \textit{holds in} \quad \pi_0^G(Y).$$

Proof.

(i) We let $j' : V \longrightarrow \mathcal{U}_G$ be another G-equivariant linear isometric embedding. If the images $j(V)$ and $j'(V)$ are orthogonal, then $H : V \times [0,1] \longrightarrow j(V) \oplus j'(V)$ defined by

$$H(v,t) \;=\; \sqrt{1-t^2}\; \cdot j(v) \;+\; t \cdot j'(v)$$

is a homotopy from j to j' through G-equivariant linear isometric embeddings. Thus

$$t \;\longmapsto\; Y(H(-,t))(y)$$

is a path in $Y(j(V) \oplus j'(V))^G$ from $Y(j)(y)$ to $Y(j')(y)$, so $[Y(j)(y)] = [Y(j')(y)]$ in $\pi_0^G(Y)$. In general we can choose a third G-equivariant linear isometric embedding $l : V \longrightarrow \mathcal{U}_G$ whose image is orthogonal to the images of j and j'. Then $[Y(j)(y)] = [Y(l)(y)] = [Y(j')(y)]$ by the previous paragraph.

(ii) If $j : W \longrightarrow \mathcal{U}_G$ is an equivariant linear isometric embedding, then so is $j\varphi : V \longrightarrow \mathcal{U}_G$. Since we can use any equivariant isometric embedding to define the class $\langle y \rangle$, we get

$$\langle Y(\varphi)(y) \rangle \;=\; [Y(j)(Y(\varphi)(y))] \;=\; [Y(j\varphi)(y)] \;=\; \langle y \rangle \;. \qquad \square$$

We can now define the *restriction map* associated to a continuous group homomorphism $\alpha : K \longrightarrow G$ by

$$\alpha^* \;:\; \pi_0^G(Y) \;\longrightarrow\; \pi_0^K(Y) \;, \quad [y] \;\longmapsto\; \langle y \rangle \;.$$

This makes sense because every G-fixed point of $Y(V)$ is also a K-fixed point of $\alpha^*(Y(V)) = Y(\alpha^* V)$. For a second continuous group homomorphism $\beta : L \longrightarrow K$ we have

$$\beta^* \circ \alpha^* \;=\; (\alpha\beta)^* \;:\; \pi_0^G(Y) \;\longrightarrow\; \pi_0^L(Y) \;.$$

Since restriction along the identity homomorphism is the identity, the collection of equivariant homotopy sets $\pi_0^G(Y)$ becomes a contravariant functor in the group variable. A key fact is that inner automorphisms act trivially.

Proposition 1.5. *For every orthogonal space Y, every compact Lie group G, and every $g \in G$, the restriction map $c_g^* : \pi_0^G(Y) \longrightarrow \pi_0^G(Y)$ along the inner automorphism*

$$c_g \;:\; G \;\longrightarrow\; G \;, \quad c_g(h) \;=\; g^{-1}hg$$

is the identity of $\pi_0^G(Y)$.

Proof. We consider a finite dimensional G-subrepresentation V of $\mathcal{U}_G$ and a G-fixed point $y \in Y(V)^G$ that represents an element in $\pi_0^G(Y)$. Then the map $l_g : c_g^*(V) \longrightarrow \mathcal{U}$ given by left multiplication by g is a G-equivariant linear isometric embedding. So

$$c_g^*[y] \;=\; [Y(l_g^V)(y)] \;=\; [g \cdot y] \;=\; [y] \;,$$

by the very definition of the restriction map, where $l_g^V : c_g^*(V) \longrightarrow V$. The second equation is the definition of the G-action on $Y(V)$ through the G-action on V. The third equation is the hypothesis that y is G-fixed. □

We denote by **Rep** the category whose objects are the compact Lie groups and whose morphisms are conjugacy classes of continuous group homomorphisms. We can summarize the discussion thus far by saying that for every orthogonal space Y the restriction maps make the equivariant homotopy sets $\{\pi_0^G(Y)\}$ into a contravariant functor

$$\underline{\pi}_0(Y) \ : \ \mathbf{Rep} \ \longrightarrow (\text{sets}) \ .$$

In fact, the restriction maps along continuous homomorphisms give *all* natural operations: As we show in [21], every natural transformation $\pi_0^G \longrightarrow \pi_0^K$ of set valued functors on the category of orthogonal spaces is of the form α^* for a unique conjugacy class of continuous group homomorphism $\alpha : K \longrightarrow G$.

If V is any inner product space, then the evaluation functor sending an orthogonal space Y to $Y(V)$ is represented by the hom functor $\mathbf{L}(V, -)$. Consequently, if V is a G-representation, then the functor

$$spc \ \longrightarrow \ \mathcal{T} \ , \quad Y \ \longmapsto \ Y(V)^G$$

that sends an orthogonal space Y to the space of G-fixed points of $Y(V)$ is represented by an orthogonal space $\mathbf{L}_{G,V}$, the *free orthogonal space* generated by (G, V). The value of $\mathbf{L}_{G,V}$ at an inner product space W is

$$\mathbf{L}_{G,V}(W) \ = \ \mathbf{L}(V, W)/G \ ,$$

the orbit space of the right G-action on $\mathbf{L}(V, W)$ by $(\varphi \cdot g)(v) = \varphi(g \cdot v)$. Every G-fixed point $y \in Y(V)^G$ gives rise to a morphism $\hat{y} : \mathbf{L}_{G,V} \longrightarrow Y$ of orthogonal spaces, defined at W as

$$\hat{y}(W) \ : \ \mathbf{L}(V, W)/G \ \longrightarrow \ Y(W) \ , \quad \varphi \cdot G \ \longmapsto \ Y(\varphi)(y) \ .$$

The morphism $\hat{y}$ is uniquely determined by the property $\hat{y}(V)(\mathrm{Id}_V \cdot G) = y$ in $Y(V)^G$.

We calculate the zeroth equivariant homotopy sets of a free orthogonal space. The *tautological class*

$$u_{G,V} \ \in \ \pi_0^G(\mathbf{L}_{G,V}) \tag{1.6}$$

is the path component of the G-fixed point

$$\mathrm{Id}_V \cdot G \ \in \ (\mathbf{L}(V, V)/G)^G = (\mathbf{L}_{G,V}(V))^G \ ,$$

the G-orbit of the identity of V.

Theorem 1.7. *Let K and G be compact Lie groups and V a faithful G-representation. Then the map*

$$\mathbf{Rep}(K, G) \ \longrightarrow \ \pi_0^K(\mathbf{L}_{G,V}) \ , \quad [\alpha : K \longrightarrow G] \ \longmapsto \ \alpha^*(u_{G,V})$$

is bijective.

Proof. We construct the inverse explicitly. We consider any element

$$[\varphi G] \ \in \ \pi_0^K(\mathbf{L}_{G,V}) \ ;$$

here $W \in s(\mathcal{U}_K)$, and $\varphi \in \mathbf{L}(V, W)$ is such that the orbit $\varphi G \in \mathbf{L}(V, W)/G$ is K-fixed. Thus $k\varphi G = \varphi G$ for every element $k \in K$. Since G acts faithfully on V, there is a unique $\alpha(k) \in G$ with $k\varphi = \varphi\alpha(k)$, and this defines a continuous homomorphism $\alpha : K \longrightarrow G$. If we replace φ by φg for some $g \in G$, then α changes into its g-conjugate. If we replace W by a larger K-representation in the poset $s(\mathcal{U}_K)$, then α does not change.

Now we consider a path

$$\omega \ : \ [0,1] \ \longrightarrow \ (\mathbf{L}(V,W)/G)^K$$

starting with φG. Since the projection $\mathbf{L}(V,W) \longrightarrow \mathbf{L}(V,W)/G$ is a locally trivial fiber bundle, we can choose a continuous lift

$$\tilde{\omega} \ : \ [0,1] \ \longrightarrow \ \mathbf{L}(V,W)$$

with $\tilde{\omega}(0) = \varphi$ and $\tilde{\omega}(t)G = \omega(t)$ for all $t \in [0,1]$. Then each t determines a continuous homomorphism $\alpha_t : K \longrightarrow G$ by $k\tilde{\omega}(t) = \tilde{\omega}(t)\alpha_t(k)$, and the assignment

$$[0,1] \ \longrightarrow \ \mathrm{Hom}(K,G) \ , \quad t \longmapsto \alpha_t$$

to the space of continuous group homomorphisms (with the topology of uniform convergence) is itself continuous. But that means that α_0 and α_1 are conjugate by an element of G, compare [6, VIII, Lemma 38.1]. In particular, the conjugacy class of α only depends on the path component of φG in the space $(\mathbf{L}(V,W)/G)^K$. Altogether this shows that the map

$$\pi_0^K(\mathbf{L}_{G,V}) \ \longrightarrow \ \mathbf{Rep}(K,G) \ , \quad [\varphi G] \ \longmapsto \ [\alpha]$$

is well-defined. It is straightforward from the definitions that this map is inverse to evaluation at $u_{G,V}$. □

We end this section by discussing certain orthogonal spaces that are closely related to the symmetric product filtration.

Construction 1.8. For an inner product space V we set

$$S(V,n) \ = \ \left\{ (v_1, \ldots, v_n) \in V^n \ : \ \sum_{i=1}^n v_i = 0 \ , \ \sum_{i=1}^n |v_i|^2 = 1 \right\} \ .$$

In other words, $S(V,n)$ is the unit sphere in the kernel of the summation map from V^n to V. The symmetric group Σ_n acts from the right on $S(V,n)$ by permuting the coordinates, i.e.,

$$(v_1, \ldots, v_n) \cdot \sigma \ = \ (v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \ .$$

We define

$$(B_{\mathrm{gl}}\mathcal{F}_n)(V) \ = \ S(V,n)/\Sigma_n \ ,$$

the orbit space of the Σ_n-action. A linear isometric embedding $\varphi : V \longrightarrow W$ induces the map

$$(B_{\mathrm{gl}}\mathcal{F}_n)(\varphi) \ = \ S(\varphi,n)/\Sigma_n \ , \quad (v_1, \ldots, v_n)\Sigma_n \ \longmapsto \ (\varphi(v_1), \ldots, \varphi(v_n))\Sigma_n \ .$$

We call $B_{\mathrm{gl}}\mathcal{F}_n$ the *global classifying space* of the family $\mathcal{F}_n$ of non-transitive subgroups of the symmetric group Σ_n. Proposition 1.11 below justifies this terminology.

Remark 1.9. The *reduced natural* Σ_n*-representation* (also called the *standard* Σ_n*-representation*) is the vector space

$$\nu_n \ = \ \{(x_1, \ldots, x_n) \in \mathbb{R}^n \ : \ x_1 + \ldots + x_n = 0\}$$

with the standard scalar product and left Σ_n-action by permutation of coordinates,

$$\sigma \cdot (x_1, \ldots, x_n) \ = \ (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \ .$$

In the proof of the following proposition we exploit that for every inner product space V (possibly infinite dimensional), the kernel of the summation map $V^n \longrightarrow V$ is isometrically and $(O(V)\times\Sigma_n)$-equivariantly isomorphic to $V\otimes\nu_n$. Hence $S(V,n)$ is $(O(V)\times\Sigma_n)$-equivariantly homeomorphic to $S(V\otimes\nu_n)$.

We show now that for every compact Lie group K the K-space $(B_{\mathrm{gl}}\mathcal{F}_n)(\mathcal{U}_K) = S(\mathcal{U}_K,n)/\Sigma_n$ is a certain classifying space, thereby justifying the term "global classifying space" for $B_{\mathrm{gl}}\mathcal{F}_n$. We denote by $\mathcal{F}_n(K)$ the family of those closed subgroups Γ of $K\times\Sigma_n$ whose trace $H=\{\sigma\in\Sigma_n \mid (1,\sigma)\in\Gamma\}$ is a non-transitive subgroup of Σ_n. For the purpose of the next proposition we combine the left K-action and the right Σ_n-action on $S(\mathcal{U}_K,n)$ into a left action of $K\times\Sigma$ by

$$(k,\sigma)\cdot(v_1,\dots,v_n) \;=\; (k\cdot v_{\sigma^{-1}(1)},\dots,k\cdot v_{\sigma^{-1}(n)})\ . \tag{1.10}$$

Proposition 1.11. *Let K be a compact Lie group and $n\geq 2$. Then the $(K\times\Sigma_n)$-space $S(\mathcal{U}_K,n)$ is a universal space for the family $\mathcal{F}_n(K)$ of subgroups of $K\times\Sigma_n$.*

Proof. We let Γ be a closed subgroup of $K\times\Sigma_n$. If the trace $H=\{\sigma\in\Sigma_n\mid(1,\sigma)\in\Gamma\}$ is a transitive subgroup of Σ_n, then all H-fixed points of $S(\mathcal{U}_K,n)$ are diagonal, i.e., of the form $(v,\dots,v)$ for some $v\in\mathcal{U}_K$. Since the components must add up to 0, this forces $v=0$, which cannot happen for tuples in the unit sphere. So if H is a transitive subgroup, then $S(\mathcal{U}_K,n)$ has no H-fixed points, and hence no Γ-fixed points either.

Now we suppose that the trace H is not transitive. We view the subgroup $\Gamma\leq K\times\Sigma_n$ as a generalized graph: we denote by L the image of Γ under the projection $K\times\Sigma_n\longrightarrow K$ and define a group homomorphism $\beta : L\longrightarrow W_{\Sigma_n}H$ to the Weyl group of the trace H by

$$\beta(l) \;=\; \{\sigma\in\Sigma_n\mid(l,\sigma)\in\Gamma\} \;\in\; W_{\Sigma_n}H\ .$$

We can recover Γ as the graph of β, i.e.,

$$\Gamma \;=\; \bigcup\nolimits_{l\in L}\{l\}\times\beta(l)\ .$$

We let L act on $(\nu_n)^H$ by restriction along β; then $\beta^*((\nu_n)^H)$ is a non-zero L-representation because H is non-transitive. Since $\mathcal{U}_K$ is a complete K-universe, the underlying L-universe is also complete; hence so is the L-universe $\mathcal{U}_K\otimes\beta^*((\nu_n)^H)$. So

$$(S(\mathcal{U}_K\otimes\nu_n))^\Gamma \;=\; S((\mathcal{U}_K\otimes\beta^*((\nu_n)^H))^L)$$

is an infinite dimensional unit sphere, and hence is contractible. □

We define a specific class in the equivariant homotopy set $\pi_0^{\Sigma_n}(B_{\mathrm{gl}}\mathcal{F}_n)$. We set

$$b \;=\; (1/n,\dots,1/n) \;\in\; \mathbb{R}^n$$

and let e_i be the ith vector of the canonical basis of $\mathbb{R}^n$. Then $b-e_i$ lies in the reduced natural Σ_n-representation ν_n. Because $|b-e_i|^2=\frac{n-1}{n}$, the vector

$$D_n \;=\; \frac{1}{\sqrt{n-1}}(b-e_1,\dots,b-e_n) \;\in\; (\nu_n)^n$$

lies in the unit sphere $S(\nu_n,n)$. The Σ_n-orbit of D_n (with respect to the right action permuting the "outer" coordinates) is Σ_n-fixed (with respect to the left action permuting the "inner" coordinates), i.e.,

$$D_n\cdot\Sigma_n \;\in\; (S(\nu_n,n)/\Sigma_n)^{\Sigma_n} \;=\; ((B_{\mathrm{gl}}\mathcal{F}_n)(\nu_n))^{\Sigma_n}\ .$$

We denote by

$$(1.12) \qquad u_n \;=\; \langle D_n \cdot \Sigma_n \rangle \;\in\; \pi_0^{\Sigma_n}(B_{\mathrm{gl}}\mathcal{F}_n)$$

the class represented by this Σ_n-fixed point. The next theorem says that the class u_n generates the homotopy set Rep-functor $\underline{\pi}_0(B_{\mathrm{gl}}\mathcal{F}_n)$.

Theorem 1.13. *For every $n \geq 2$, every compact Lie group K, and every element x of $\pi_0^K(B_{\mathrm{gl}}\mathcal{F}_n)$ there is a continuous group homomorphism $\alpha : K \longrightarrow \Sigma_n$ such that $\alpha^*(u_n) = x$.*

Proof. An element of $\pi_0^K(B_{\mathrm{gl}}\mathcal{F}_n)$ is represented by a K-representation V in $s(\mathcal{U}_K)$ and a K-fixed Σ_n-orbit

$$v \cdot \Sigma_n \;\in\; S(V,n)/\Sigma_n \;=\; (B_{\mathrm{gl}}\mathcal{F}_n)(V) \;.$$

We let H denote the Σ_n-stabilizer of v, a non-transitive subgroup of Σ_n. We define a continuous homomorphism $\beta : K \longrightarrow W_{\Sigma_n}H$ to the Weyl group of H by

$$\beta(k) \;=\; \{\sigma \in \Sigma_n \mid kv = v\sigma\} \;.$$

As the Σ_n-stabilizer of a point in V^n, the group H is a Young subgroup of Σ_n, i.e., the product of the symmetric groups of all the orbits of the tautological H-action on $\{1,\dots,n\}$. Thus the projection $q : N_{\Sigma_n}H \longrightarrow W_{\Sigma_n}H$ has a multiplicative section $s : W_{\Sigma_n}H \longrightarrow N_{\Sigma_n}H$. We define $\alpha : K \longrightarrow \Sigma_n$ as the composite homomorphism

$$K \xrightarrow{\ \beta\ } W_{\Sigma_n}H \xrightarrow{\ s\ } N_{\Sigma_n}H \xrightarrow{\ \mathrm{incl}\ } \Sigma_n$$

and claim that

$$\alpha^*(u_n) \;=\; [v \cdot \Sigma_n] \qquad \text{in} \qquad \pi_0^K(B_{\mathrm{gl}}\mathcal{F}_n) \;.$$

We turn $S(V,n)$ into a left $(K \times \Sigma_n)$-space as in (1.10) and let $\Gamma \leq K \times \Sigma_n$ denote the graph of α. Since $\alpha(k) \in \beta(k)$ for every $k \in K$, the vector v is fixed by Γ. Increasing the K-representation V does not change the stabilizer group of the vector v nor the class represented by the orbit $v \cdot \Sigma_n$ in $\pi_0^K(B_{\mathrm{gl}}\mathcal{F}_n)$; we can thus assume without loss of generality that there is a K-equivariant linear isometric embedding $\varphi : \alpha^*(\nu_n) \longrightarrow V$. As the K-representations V exhaust a complete K-universe, the $(K \times \Sigma_n)$-spaces $S(V,n)$ approximate a universal space for the family $\mathcal{F}_n(K)$, by Proposition 1.11. The graph Γ of α belongs to $\mathcal{F}_n(K)$, so after increasing the K-representation V, if necessary, we can assume that the dimension of the fixed point sphere $S(V,n)^\Gamma$ is at least 1, so that this fixed point space is path connected. The class $\alpha^*(u_n)$ is represented by the Σ_n-orbit of the point

$$S(\varphi,n)(D_n) \;\in\; S(V,n)^\Gamma,$$

and the original class in $\pi_0^K(B_{\mathrm{gl}}\mathcal{F}_n)$ is represented by the vector v. Any path between $S(\varphi,n)(D_n)$ and v in the fixed point space $S(V,n)^\Gamma$ projects to a path of K-fixed points between the orbits

$$S(\varphi,n)(D_n)\cdot\Sigma_n \;, \quad v \cdot \Sigma_n \quad \in\; (S(V,n)/\Sigma_n)^K \;=\; ((B_{\mathrm{gl}}\mathcal{F}_n)(V))^K \;.$$

This proves that $\alpha^*(u_n) = [v \cdot \Sigma_n]$, and it finishes the proof. □

Remark 1.14. The orthogonal space $B_{\mathrm{gl}}\mathcal{F}_2$ is isomorphic to the free orthogonal space generated by (Σ_2, σ), where σ is the one dimensional sign representation of Σ_2 on $\mathbb{R}$. An isomorphism of orthogonal spaces

$$\mathbf{L}_{\Sigma_2,\sigma} \;\cong\; B_{\mathrm{gl}}\mathcal{F}_2$$

is induced at an inner product space V by the Σ_2-equivariant natural homeomorphism

$$\mathbf{L}(\sigma, V) \cong S(V,2) , \quad (\varphi : \sigma \longrightarrow V) \longmapsto \left(\varphi(1/\sqrt{2}), -\varphi(1/\sqrt{2})\right) .$$

This isomorphism sends the tautological class $u_{\Sigma_2,\sigma}$ (see (1.6)) in $\pi_0^{\Sigma_2}(\mathbf{L}_{\Sigma_2,\sigma})$ to the class $u_2 \in \pi_0^{\Sigma_2}(B_{\mathrm{gl}}\mathcal{F}_2)$. So by Theorem 1.7 every element of $\pi_0^K(B_{\mathrm{gl}}\mathcal{F}_2)$ is of the form $\alpha^*(u_2)$ for *unique* conjugacy classes of continuous group homomorphism $\alpha : K \longrightarrow \Sigma_2$. For $n \geq 3$, however, α is typically not unique up to conjugacy, and $\underline{\pi}_0(B_{\mathrm{gl}}\mathcal{F}_n)$ is *not* a representable Rep-functor.

2. Orthogonal spectra

In this section we recall orthogonal spectra, the objects that represent *stable* global homotopy types. Orthogonal spectra are used, at least implicitly, in [15], and the term "orthogonal spectrum" was introduced in [13], where a non-equivariant stable model structure for orthogonal spectra was constructed. Before giving the formal definition of orthogonal spectra we try to motivate it. An orthogonal space Y assigns values to all finite dimensional inner product spaces. Informally speaking, the global homotopy type is encoded in the G-spaces obtained as the "homotopy colimit of $Y(V)$ over all G-representations V." So besides the values $Y(V)$, we use the $O(V)$-action (which is turned into a G-action when G acts on V) and the information about inclusions of inner product spaces. All this information is conveniently encoded as a continuous functor from the category $\mathbf{L}$.

An orthogonal spectrum X is a stable analog of this: it assigns a based space $X(V)$ to every inner product space, and it keeps track of an $O(V)$-action on $X(V)$ (to get G-homotopy types when G acts on V) and of a way to stabilize by suspensions. When doing this in a coordinate free way, the stabilization data assign to a linear isometric embedding $\varphi : V \longrightarrow W$ a continuous based map

$$\varphi_\star \ : \ X(V) \wedge S^{\varphi^\perp} \ \longrightarrow \ X(W)$$

that "varies continuously with φ." To make the continuous dependence rigorous one exploits that the orthogonal complements $\varphi^\perp$ vary in a locally trivial way; i.e., they are the fibers of an "orthogonal complement" vector bundle over the space $\mathbf{L}(V,W)$ of linear isometric embeddings. All the structure maps $\varphi_\star$ together define a map on the smash product of $X(V)$ with the Thom space of this complement bundle, and the continuity in φ is formalized by requiring continuity of that map. The Thom spaces together form the morphism spaces of a based topological category, and the data of an orthogonal spectrum can conveniently be packaged as a continuous based functor on this category.

Construction 2.1. We let V and W be inner product spaces. The orthogonal complement vector bundle over the space $\mathbf{L}(V,W)$ is the subbundle of the trivial vector bundle $W \times \mathbf{L}(V,W)$ with total space

$$\xi(V,W) \ = \ \{ (w,\varphi) \in W \times \mathbf{L}(V,W) \mid \langle w, \varphi(v)\rangle = 0 \text{ for all } v \in V \} .$$

The fiber over $\varphi : V \longrightarrow W$ is the orthogonal complement of the image of φ.

We let $\mathbf{O}(V,W)$ be the one-point compactification of the total space of $\xi(V,W)$; since the base space $\mathbf{L}(V,W)$ is compact, $\mathbf{O}(V,W)$ is also the Thom space of the bundle $\xi(V,W)$. Up to non-canonical homeomorphism, we can describe the space $\mathbf{O}(V,W)$ differently as follows: if $\dim V = n$ and $\dim W = n+m$, then $\mathbf{L}(V,W)$ is homeomorphic to the homogeneous space $O(n+m)/O(m)$ and $\mathbf{O}(V,W)$ is homeomorphic to $O(n+m)_+ \wedge_{O(m)} S^m$.

The spaces $\mathbf{O}(V,W)$ are the morphism spaces of a based topological category. Given a third inner product space U, the bundle map

$$\xi(V,W)\times\xi(U,V) \longrightarrow \xi(U,W)\ , \quad ((w,\varphi),\,(v,\psi)) \longmapsto (w+\varphi(v),\,\varphi\psi)$$

covers the composition in $\mathbf{L}$. Passage to one-point compactification gives a based map

$$\circ \ : \ \mathbf{O}(V,W)\wedge\mathbf{O}(U,V) \longrightarrow \mathbf{O}(U,W),$$

which is the composition in the category $\mathbf{O}$. The identity of V is $(0,\mathrm{Id}_V)$ in $\mathbf{O}(V,V)$.

Definition 2.2. An *orthogonal spectrum* is a based continuous functor from $\mathbf{O}$ to the category of based spaces. A *morphism* is a natural transformation of functors. We denote by $\mathcal{S}p$ the category of orthogonal spectra.

We denote by S^V the one-point compactification of an inner product space V, with the base point at infinity. If X is an orthogonal spectrum and V and W are inner product spaces, we define the *structure map*

$$\sigma_{V,W} \ : \ X(V)\wedge S^W \longrightarrow X(V\oplus W)$$

as the composite

$$X(V)\wedge S^W \xrightarrow{X(V)\wedge((0,-),i_V)} X(V)\wedge\mathbf{O}(V,V\oplus W) \xrightarrow{X} X(V\oplus W),$$

where $i_V : V \longrightarrow V\oplus W$ is the inclusion of the first summand. If a compact Lie group G acts on V and W by linear isometries, then $X(V)$ becomes a based G-space by restriction of the action of $\mathbf{O}(V,V) = O(V)_+$, and the structure map $\sigma_{V,W}$ is G-equivariant.

Remark 2.3. Given an orthogonal spectrum X and a compact Lie group G, the collection of G-spaces $X(V)$ and the structure maps $\sigma_{V,W}$ form an *orthogonal G-spectrum* in the sense of [14] that we denote by X_G. However, only very special orthogonal G-spectra arise in this way from an orthogonal spectrum. More precisely, an orthogonal G-spectrum Y is isomorphic to X_G for some orthogonal spectrum X if and only if for every *trivial* G-representation V, the G-action on $Y(V)$ is trivial. An orthogonal G-spectrum that does not satisfy this condition is the equivariant suspension spectrum of a based G-space with non-trivial G-action. In Remark 2.16 below we isolate some conditions on the Mackey functor homotopy groups of an orthogonal G-spectrum that hold for all G-spectra of the special form X_G.

As we just explained, an orthogonal spectrum X has an underlying orthogonal G-spectrum for every compact Lie group G. As such, it has equivariant stable homotopy groups, whose definition we now recall. As before, $s(\mathcal{U}_G)$ denotes the poset, under inclusion, of finite dimensional G-subrepresentations of the complete G-universe $\mathcal{U}_G$. For $k\geq 0$ we consider the functor from $s(\mathcal{U}_G)$ to sets that sends $V\in s(\mathcal{U}_G)$ to

$$[S^{k+V}, X(V)]^G\ ,$$

the set of G-equivariant homotopy classes of based G-maps from S^{k+V} to $X(V)$ (where $k+V$ is shorthand for $\mathbb{R}^k\oplus V$ with trivial G-action on $\mathbb{R}^k$). The map induced by an inclusion $V \subseteq W$ in $s(\mathcal{U}_G)$ sends the homotopy class of $f : S^{k+V} \longrightarrow X(V)$ to the class of the composite

$$\begin{aligned} S^{k+W} \cong \ S^{k+V} \wedge S^{W-V} \ &\xrightarrow{f \wedge S^{W-V}} X(V) \wedge S^{W-V} \\ &\xrightarrow{\sigma_{V,W-V}} X(V \oplus (W-V)) \ = \ X(W) \ , \end{aligned}$$

where $W - V$ is the orthogonal complement of V in W. The *kth equivariant homotopy group* $\pi_0^G(X)$ is then defined as

$$\pi_k^G(X) \ = \ \mathrm{colim}_{V \in s(\mathcal{U}_G)} \, [S^{k+V}, X(V)]^G \ , \tag{2.4}$$

the colimit of this functor over the poset $s(\mathcal{U}_G)$. For $k < 0$, the definition is essentially the same, but we take a colimit over $s(\mathcal{U}_G)$ of the sets $[S^V, X(\mathbb{R}^{-k} \oplus V)]^G$. When the fixed points V^G have dimension at least 2, then $[S^V, X(V)]^G$ comes with a commutative group structure, and the maps out of it are homomorphisms. The G-subrepresentations V with $\dim(V^G) \geq 2$ are cofinal in the poset $s(\mathcal{U}_G)$, so the abelian group structures on $[S^V, X(V)]^G$ for $\dim(V^G) \geq 2$ assemble into a well-defined and natural abelian group structure on the colimit $\pi_0^G(X)$. The argument for $\pi_k^G(X)$ is similar.

Definition 2.5. A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a *global equivalence* if the induced map $\pi_k^G(f) : \pi_k^G(X) \longrightarrow \pi_k^G(Y)$ is an isomorphism for all compact Lie groups G and all integers k.

The *global stable homotopy category* is the category obtained from the category of orthogonal spectra by formally inverting the global equivalences. The global equivalences are the weak equivalences of the *global model structure* on the category of orthogonal spectra; see [21]. So the methods of homotopical algebra are available for studying global equivalences and the associated global homotopy category.

Now we set up the formalism of *global functors*, the natural (in fact, the tautological) home of the collection of equivariant homotopy groups of an orthogonal spectrum. In this language we then describe the equivariant homotopy groups $\pi_0^G(Sp^n)$ of the symmetric product spectrum Sp^n as a whole, i.e., when the compact Lie group G is varying: the global functor $\underline{\pi}_0(Sp^n)$ is the quotient of the Burnside ring global functor by a single basic relation.

Definition 2.6 (Global Burnside category)**.** The *global Burnside category* $\mathbf{A}$ has all compact Lie groups as objects; the morphisms from a group G to K are defined as

$$\mathbf{A}(G, K) \ = \ \mathrm{Nat}(\pi_0^G, \pi_0^K) \ ,$$

the set of natural transformations of functors, from orthogonal spectra to sets, between the equivariant homotopy group functors π_0^G and π_0^K. The composition in $\mathbf{A}$ is a composition of natural transformations.

It is not a priori clear that the natural transformations from π_0^G to π_0^K form a set (as opposed to a proper class), but this follows from the representability result in Proposition 2.11 below. The functor π_0^K is abelian group valued, so the set $\mathbf{A}(G, K)$ is an abelian group under an objectwise addition of transformations. The composition is additive in each variable, so $\mathbf{A}(G, K)$ is a pre-additive category.

The Burnside category $\mathbf{A}$ is skeletally small: isomorphic compact Lie groups are also isomorphic in the category $\mathbf{A}$, and every compact Lie group is isomorphic to a closed subgroup of an orthogonal group $O(n)$.

Definition 2.7. A *global functor* is an additive functor from the global Burnside category $\mathbf{A}$ to the category of abelian groups. A morphism of global functors is a natural transformation.

As a category of additive functors out of a skeletally small pre-additive category, the category of global functors is abelian with enough injectives and projectives. The global Burnside category $\mathbf{A}$ is designed so that the collection of equivariant homotopy groups of an orthogonal spectrum is tautologically a global functor. Explicitly, the global homotopy group functor $\underline{\pi}_0(X)$ of an orthogonal spectrum X is defined on objects by

$$\underline{\pi}_0(X)(G) \;=\; \pi_0^G(X)$$

and on morphisms by evaluating natural transformations at X.

It is less obvious that conversely every global functor is isomorphic to the homotopy group global functor $\underline{\pi}_0(X)$ of some orthogonal spectrum X. We show this in [21] by constructing Eilenberg-Mac Lane spectra from global functors. In fact, the full subcategories of globally connective (respectively, globally coconnective) orthogonal spectra define a non-degenerate t-structure on the triangulated global stable homotopy category, and the heart of this t-structure is (equivalent to) the abelian category of global functors.

The abstract definition of the global Burnside category $\mathbf{A}$ is convenient for formal considerations and for defining the global functor $\underline{\pi}_0(X)$ associated to an orthogonal spectrum X, but to facilitate calculations we should describe the groups $\mathbf{A}(G,K)$ more explicitly. As we shall explain, the operations between the equivariant homotopy groups come from two different sources: restriction maps along continuous group homomorphisms and transfer maps along inclusions of closed subgroups. A quick way to define the restriction maps, and to deduce some of their properties, is to interpret $\pi_0^G(X)$ as the G-equivariant homotopy set, as defined in (1.3), of a certain orthogonal space.

Construction 2.8. We recall the functor

$$\Omega^\bullet \;:\; \mathcal{S}p \;\longrightarrow\; spc$$

from orthogonal spectra to orthogonal spaces. Given an orthogonal spectrum X, the value of $\Omega^\bullet X$ at an inner product space V is

$$(\Omega^\bullet X)(V) \;=\; \mathrm{map}(S^V, X(V)) \;.$$

If $\varphi : V \longrightarrow W$ is a linear isometric embedding, the induced map

$$\varphi_* \;:\; (\Omega^\bullet X)(V) \;=\; \mathrm{map}(S^V, X(V)) \;\longrightarrow\; \mathrm{map}(S^W, X(W)) \;=\; (\Omega^\bullet X)(W)$$

is by "conjugation and extension by the identity." In more detail: given a continuous based map $f : S^V \longrightarrow X(V)$ we define $\varphi_*(f) : S^W \longrightarrow X(W)$ as the composite

$$S^W \cong\; S^V \wedge S^{\varphi^\perp} \;\xrightarrow{f\wedge S^{\varphi^\perp}}\; X(V)\wedge S^{\varphi^\perp} \;\xrightarrow{\sigma_{V,\varphi^\perp}} X(V\oplus\varphi^\perp) \;\cong\; X(W) \;,$$

where each of the two unnamed homeomorphisms uses φ to identify $V\oplus\varphi^\perp$ with W. In particular, the orthogonal group $O(V)$ acts on $(\Omega^\bullet X)(V)$ by conjugation. The assignment $(\varphi, f)\mapsto \varphi_*(f)$ is continuous in both variables and functorial in φ. In other words, we have defined an orthogonal space $\Omega^\bullet X$.

The functor $\Omega^\bullet$ has a left adjoint, defined as follows. To every orthogonal space Y we can associate an unreduced *suspension spectrum* $\Sigma^\infty_+ Y$ whose value on an inner product space is given by

$$(\Sigma^\infty_+ Y)(V) \; = \; Y(V)_+ \wedge S^V \; ;$$

the structure map

$$\mathbf{O}(V,W) \wedge Y(V)_+ \wedge S^V \; \longrightarrow \; Y(W)_+ \wedge S^W$$

is given by

$$(w,\varphi) \wedge y \wedge v \; \longmapsto \; Y(\varphi)(y) \wedge (w + \varphi(v)) \; .$$

If Y is the constant orthogonal space with value A, then $\Sigma^\infty_+ Y$ specializes to the usual suspension spectrum of A with a disjoint base point added.

If G acts on V by linear isometries, then the G-fixed subspace of $(\Omega^\bullet X)(V)$ is the space of G-equivariant based maps from S^V to $X(V)$. The path components of this space are precisely the equivariant homotopy classes of based G-maps, i.e.,

$$\pi_0 \left(((\Omega^\bullet X)(V))^G \right) \; = \; \pi_0 \left(\mathrm{map}^G(S^V, X(V)) \right) \; = \; [S^V, X(V)]^G \; .$$

Passing to the colimit over the poset $s(\mathcal{U}_G)$ gives

$$\pi_0^G(\Omega^\bullet X) \; = \; \pi_0^G(X) \; ,$$

i.e., the G-equivariant homotopy group of the orthogonal spectrum X equals the G-equivariant homotopy set (as previously defined in (1.3)) of the orthogonal space $\Omega^\bullet X$. So by specializing the restriction maps for orthogonal spaces we obtain restriction maps

$$\alpha^* \; : \; \pi_0^G(X) \; \longrightarrow \; \pi_0^K(X)$$

for every continuous group homomorphism $\alpha : K \longrightarrow G$. These restriction maps are again contravariantly functorial and depend only on the conjugacy class of α (by Proposition 1.5). Moreover, α^* is additive, i.e., a group homomorphism.

The transfer maps

$$\mathrm{tr}_H^G \; : \; \pi_0^H(X) \; \longrightarrow \; \pi_0^G(X)$$

are the classical ones that arise from the orthogonal G-spectrum underlying X; they are defined whenever H is a closed subgroup of G and constructed by an equivariant Thom-Pontryagin construction [16, Section IX.3], [20]. Transfers are additive and natural for homomorphisms of orthogonal spectra; since we only consider degree 0 transfers (as opposed to more general "dimension shifting transfers"), the transfer tr_H^G is trivial whenever H has infinite index in its normalizer in G.

As we shall now explain, the suspension spectrum functor "freely builds in" the extra structure that is available at the level of π_0^G for orthogonal *spectra* (as opposed to orthogonal *spaces*), namely the abelian group structure and transfers. We let Y be an orthogonal space and G a compact Lie group. We define a stabilization map

$$\sigma^G \; : \; \pi_0^G(Y) \; \longrightarrow \; \pi_0^G(\Sigma^\infty_+ Y) \tag{2.9}$$

as the effect of the adjunction unit $Y \longrightarrow \Omega^\bullet(\Sigma^\infty_+ Y)$ on the G-equivariant homotopy set π_0^G, using the identification $\pi_0^G(\Omega^\bullet(\Sigma^\infty_+ Y)) = \pi_0^G(\Sigma^\infty_+ Y)$. More explicitly: if V is a finite dimensional G-subrepresentation of the complete G-universe $\mathcal{U}_G$ and $y \in Y(V)^G$ a G-fixed point, then $\sigma^G[y]$ is represented by the G-map

$$S^V \xrightarrow{y \wedge -} Y(V)_+ \wedge S^V \; = \; (\Sigma^\infty_+ Y)(V) \; .$$

The stabilization maps (2.9) commute with restriction, since they arise from a morphism of orthogonal spaces.

For a closed subgroup L of a compact Lie group K, the normalizer $N_K L$ acts on L by conjugation, and hence on $\pi_0^L(Y)$ by restriction along the conjugation maps. Restriction along an inner automorphism is the identity, so the action of $N_K L$ factors over an action of the Weyl group $W_K L = N_K L/L$ on $\pi_0^L(Y)$. After passing to the stable classes along the map $\sigma^L : \pi_0^L(Y) \longrightarrow \pi_0^L(\Sigma_+^\infty Y)$, we can then transfer from L to K. For an element $k \in N_K L$ and a class $x \in \pi_0^L(Y)$ we have

$$\mathrm{tr}_L^K(\sigma^L(c_k^*(x))) \;=\; \mathrm{tr}_L^K(c_k^*(\sigma^L(x))) \;=\; c_k^*(\mathrm{tr}_L^K(\sigma^L(x))) \;=\; \mathrm{tr}_L^K(\sigma^L(x))$$

because transfer commutes with restriction along the conjugation maps

$$c_k \;:\; L \;\longrightarrow\; L, \qquad \text{respectively,} \qquad c_k \;:\; K \;\longrightarrow\; K,$$

defined by $c_k(h) = k^{-1}hk$. So transferring from L to K in the global functor $\underline{\pi}_0(\Sigma_+^\infty Y)$ equalizes the action of the Weyl group $W_K L$ on $\pi_0^L(Y)$.

Proposition 2.10. *Let Y be an orthogonal space. Then for every compact Lie group K the equivariant homotopy group $\pi_0^K(\Sigma_+^\infty Y)$ of the suspension spectrum of Y is a free abelian group with a basis given by the elements*

$$\mathrm{tr}_L^K(\sigma^L(x)),$$

where L runs through all conjugacy classes of closed subgroups of K with the finite Weyl group and x runs through a set of representatives of the $W_K L$-orbits of the set $\pi_0^L(Y)$.

Proof. We consider the functor on the product poset $s(\mathcal{U}_K)^2$ sending (V, U) to the set $[S^V, Y(U)_+ \wedge S^V]^K$. The diagonal is cofinal in $s(\mathcal{U}_K)^2$, and thus the induced map

$$\begin{aligned}\pi_0^K(\Sigma_+^\infty Y) = \;& \mathrm{colim}_{V\in s(\mathcal{U}_G)}[S^V, Y(V)_+ \wedge S^V]^K \\ & \longrightarrow \; \mathrm{colim}_{(V,U)\in s(\mathcal{U}_K)^2}[S^V, Y(U)_+ \wedge S^V]^K\end{aligned}$$

is an isomorphism. The target can be calculated in two steps, so the group we are after is isomorphic to

$$\mathrm{colim}_{U\in s(\mathcal{U}_K)}\left(\mathrm{colim}_{V\in s(\mathcal{U}_K)}[S^V, Y(U)_+ \wedge S^V]^K\right) \;=\; \mathrm{colim}_{U\in s(\mathcal{U}_K)}\, \pi_0^K\left(\Sigma_+^\infty Y(U)\right) .$$

We may thus show that the latter group is free abelian with the specified basis.

The rest of the argument is well known, and a version of it can be found in [12, V Corollary 9.3]. The tom Dieck splitting [26, Satz 2] provides an isomorphism

$$\bigoplus_{(L)} \pi_0^{WL}(\Sigma_+^\infty(EWL \times Y(U)^L)) \;\cong\; \pi_0^K(\Sigma_+^\infty Y(U)) \;,$$

where the sum is indexed over all conjugacy classes of closed subgroups L and $WL = W_K L$ is the Weyl group of L in K. By [26, Section 4] the group $\pi_0^{WL}(\Sigma_+^\infty(EWL \times Y(U)^L))$ vanishes if the Weyl group WL is infinite; so only the summands with finite Weyl group contribute to π_0^K. On the other hand, if the Weyl group WL is finite, then the group $\pi_0^{WL}(\Sigma_+^\infty(EWL \times Y(U)^L))$ is free abelian with a basis given by the set $WL\backslash\pi_0(Y(U)^L)$, the WL-orbit set of the path components of $Y(U)^L$.

Since colimits commute among themselves, we conclude that

$$\begin{aligned}\pi_0^K\left(\Sigma_+^\infty Y\right) \;&\cong\; \mathrm{colim}\; \pi_0^K\left(\Sigma_+^\infty Y(U)\right) \;\cong\; \mathrm{colim} \bigoplus_{(L)} \mathbb{Z}\{WL\backslash\pi_0(Y(U)^L)\} \\ &\cong\; \bigoplus_{(L)} \mathbb{Z}\{WL\backslash(\mathrm{colim}\, \pi_0(Y(U)^L))\} \;=\; \bigoplus_{(L)} \mathbb{Z}\{WL\backslash\pi_0^L(Y)\} \;,\end{aligned}$$

where all colimits are over the poset $s(\mathcal{U}_G)$ and the sums are indexed by conjugacy classes with finite Weyl groups. To verify that this composite isomorphism takes the class $\mathrm{tr}_L^K(\sigma^L(x))$ in $\pi_0^K\left(\Sigma_+^\infty Y\right)$ to the basis element corresponding to the orbit of $x \in \pi_0^L(Y)$ in the summand indexed by L, one needs to recall the definition of the isomorphism in tom Dieck's splitting from [26]; we omit this. □

Now we prove a representability result. The *stable tautological class*

$$e_{G,V} \;=\; \sigma^G(u_{G,V}) \;\in\; \pi_0^G(\Sigma_+^\infty \mathbf{L}_{G,V})$$

arises from the unstable tautological class $u_{G,V}$ defined in (1.6) by applying the stabilization map (2.9); so it is represented by the G-map

$$S^V \;\longrightarrow\; (\mathbf{L}(V,V)/G)_+ \wedge S^V \;=\; (\Sigma_+^\infty \mathbf{L}_{G,V})(V)\;, \quad v \;\longmapsto\; (\mathrm{Id}_V \cdot G) \wedge v\;.$$

Proposition 2.11. *Let G and K be compact Lie groups and V a faithful G-representation. Then evaluation at the stable tautological class is an isomorphism*

$$\mathbf{A}(G,K) \;\xrightarrow{\cong}\; \pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})\;, \quad \tau \;\longmapsto\; \tau(e_{G,V}),$$

to the zeroth K-equivariant homotopy group of the orthogonal spectrum $\Sigma_+^\infty \mathbf{L}_{G,V}$. Hence the morphism

$$\mathbf{A}(G,-) \;\longrightarrow\; \underline{\pi}_0(\Sigma_+^\infty \mathbf{L}_{G,V})$$

classified by the stable tautological class $e_{G,V}$ is an isomorphism of global functors.

Proof. We show first that every natural transformation $\tau : \pi_0^G \longrightarrow \pi_0^K$ is determined by the element $\tau(e_{G,V})$. We let X be any orthogonal spectrum and $x \in \pi_0^G(X)$ a G-equivariant homotopy class. Without loss of generality the class x is represented by a continuous based G-map

$$f \;:\; S^{V\oplus W} \;\longrightarrow\; X(V\oplus W)$$

for some G-representation W. This G-map is adjoint to a morphism of orthogonal spectra

$$\hat{f} \;:\; \Sigma_+^\infty \mathbf{L}_{G,V\oplus W} \;\longrightarrow\; X \qquad \text{that satisfies} \qquad \hat{f}_*(e_{G,V\oplus W}) \;=\; x \quad \text{in} \quad \pi_0^G(X)\;.$$

We consider the morphism of orthogonal spaces $r : \mathbf{L}_{G,V\oplus W} \longrightarrow \mathbf{L}_{G,V}$ that restricts a linear isometry from $V \oplus W$ to V. The relation

$$\pi_0^G(r)(u_{G,V\oplus W}) \;=\; u_{G,V}$$

shows that the composite

$$\mathbf{Rep}(K,G) \xrightarrow{[\alpha]\mapsto\alpha^*(u_{G,V\oplus W})} \pi_0^K(\mathbf{L}_{G,V\oplus W}) \xrightarrow{\pi_0^K(r)} \pi_0^K(\mathbf{L}_{G,V})$$

is evaluation at the class $u_{G,V}$. Evaluation at $u_{G,V\oplus W}$ and at $u_{G,V}$ are both bijective by Theorem 1.7, so $\pi_0^K(r)$ is bijective for all compact Lie groups K. By Proposition 2.10, the induced morphism of suspension spectra

$$\Sigma_+^\infty r \;:\; \Sigma_+^\infty \mathbf{L}_{G,V\oplus W} \;\longrightarrow\; \Sigma_+^\infty \mathbf{L}_{G,V}$$

thus induces an isomorphism on $\pi_0^K(-)$ for all compact Lie groups K, and it sends $e_{G,W\oplus V}$ to $e_{G,V}$. The diagram

$$\begin{tikzcd}
\pi_0^G(\Sigma^\infty_+ \mathbf{L}_{G,V}) \arrow[d, "\tau"'] & \pi_0^G(\Sigma^\infty_+ \mathbf{L}_{G,V\oplus W}) \arrow[l, "(\Sigma^\infty_+ r)_*"', "\cong"] \arrow[r, "\hat{f}_*"] \arrow[d, "\tau"'] & \pi_0^G(X) \arrow[d, "\tau"] \\
\pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V}) & \pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,W\oplus V}) \arrow[l, "\cong"', "(\Sigma^\infty_+ r)_*"] \arrow[r, "\hat{f}_*"'] & \pi_0^K(X)
\end{tikzcd}$$

commutes and the two left horizontal maps are isomorphisms. Since

$$x \;=\; \hat{f}_*((\Sigma^\infty_+ r)_*^{-1}(e_{G,V}))\ ,$$

naturality yields that

$$\tau(x) \;=\; \tau(\hat{f}_*((\Sigma^\infty_+ r)_*^{-1}(e_{G,V}))) \;=\; \hat{f}_*((\Sigma^\infty_+ r)_*^{-1}(\tau(e_{G,V})))\ .$$

So the transformation τ is determined by the value $\tau(e_{G,V})$.

It remains to construct, for every element $y \in \pi_0^K(\Sigma^\infty_+ \mathbf{L}_{G,V})$, a natural transformation $\tau : \pi_0^G \longrightarrow \pi_0^K$ with $\tau(e_{G,V}) = y$. The previous paragraph dictates what to do: we represent a given class $x \in \pi_0^G(X)$ by a continuous based G-map $f : S^{V\oplus W} \longrightarrow X(V\oplus W)$ as above and set

$$\tau(x) \;=\; \hat{f}_*((\Sigma^\infty_+ r)_*^{-1}(y)))\ .$$

We omit the verification that the element $\tau(x)$ only depends on the class x and that τ is indeed natural. $\square$

We show now that restriction and transfer maps generate all natural operations between the zero dimensional equivariant homotopy group functors for orthogonal spectra. Given compact Lie groups K and G, we consider pairs (L,α) consisting of

- a closed subgroup $L \leq K$ whose Weyl group $W_K L$ is finite, and
- a continuous group homomorphism $\alpha : L \longrightarrow G$.

The *conjugate* of (L,α) by a pair of group elements $(k,g) \in K\times G$ is the pair $({}^kL,\, c_g\circ\alpha\circ c_k)$ consisting of the conjugate subgroup ${}^kL$ and the composite homomorphism

$${}^kL \xrightarrow{\;c_k\;} L \xrightarrow{\;\alpha\;} G \xrightarrow{\;c_g\;} G\ .$$

Since inner automorphisms induce the identity on equivariant homotopy groups,

$$\mathrm{tr}^K_{{}^kL}\circ(c_g\circ\alpha\circ c_k)^* \;=\; \mathrm{tr}^K_L\circ\alpha^* \;:\; \pi_0^G(X) \;\longrightarrow\; \pi_0^K(X)\ ;$$

i.e., conjugate pairs define the same operation on equivariant homotopy groups.

Theorem 2.12. *Let G and K be compact Lie groups.*

(i) *Let V be a faithful G-representation. Then the homotopy group $\pi_0^K(\Sigma^\infty_+\mathbf{L}_{G,V})$ is a free abelian group with basis given by the classes*

$$\mathrm{tr}^K_L(\alpha^*(e_{G,V}))$$

as (L,α) runs over a set of representatives of all $(K\times G)$-conjugacy classes of pairs consisting of a closed subgroup L of K with finite Weyl group and a continuous homomorphism $\alpha : L \longrightarrow G$.

(ii) *The morphism group $\mathbf{A}(G,K)$ in the global Burnside category is a free abelian group with basis the operations $\mathrm{tr}^K_L\circ\alpha^*$, where (L,α) runs over all conjugacy classes of pairs consisting of a closed subgroup L of K with finite Weyl group and a continuous homomorphism $\alpha : L \longrightarrow G$.*

Proof.

(i) The map

$$\mathbf{Rep}(K,G) \longrightarrow \pi_0^K(\mathbf{L}_{G,V}) , \quad [\alpha : K \longrightarrow G] \longmapsto \alpha^*(u_{G,V}),$$

is bijective according to Theorem 1.7. Proposition 2.10 thus says that $\pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})$ is a free abelian group with a basis given by the elements

$$\mathrm{tr}_L^K(\sigma^L(\alpha^*(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(\sigma^G(u_{G,V}))) = \mathrm{tr}_L^K(\alpha^*(e_{G,V})),$$

where L runs through all conjugacy classes of closed subgroups of K with finite Weyl group and α runs through a set of representatives of the $W_K L$-orbits of the set $\mathbf{Rep}(L,G)$. The claim follows because $(K \times G)$-conjugacy classes of such pairs (L,α) biject with pairs consisting of a conjugacy class of subgroups (L) and a $W_K L$-equivalence class in $\mathbf{Rep}(L,G)$.

(ii) We let V be any faithful G-representation. By part (i) the composite

$$\mathbb{Z}\{[L,\alpha] \mid \ |W_K L| < \infty, \alpha : L \longrightarrow G\} \longrightarrow \mathrm{Nat}(\pi_0^G, \pi_0^K) \xrightarrow{\mathrm{ev}} \pi_0^K(\Sigma_+^\infty \mathbf{L}_{G,V})$$

is an isomorphism, where the first map takes a conjugacy class $[L,\alpha]$ to $\mathrm{tr}_L^K \circ \alpha^*$, and the second map is evaluation at the stable tautological class $e_{G,V}$. The evaluation map is an isomorphism by Proposition 2.11, so the first map is an isomorphism, as claimed. □

Theorem 2.12 (ii) is almost a complete calculation of the global Burnside category, but one important piece of information is still missing: how does one express the composite of two operations, each given in the basis of Theorem 2.12, as a sum of basis elements? Restrictions are contravariantly functorial and transfers are transitive, i.e., for every closed subgroup K of H we have

$$\mathrm{tr}_H^G \circ \mathrm{tr}_K^H = \mathrm{tr}_K^G : \pi_0^K(X) \longrightarrow \pi_0^G(X) .$$

So the key question is how to express a transfer followed by a restriction in terms of the specified basis.

Every group homomorphism is the composite of an epimorphism and a subgroup inclusion. Transfers commute with inflation (i.e., restriction along epimorphisms): for every surjective continuous group homomorphism $\alpha : K \longrightarrow G$ and every subgroup H of G the relation

$$\alpha^* \circ \mathrm{tr}_H^G = \mathrm{tr}_L^K \circ (\alpha|_L)^*$$

holds as maps $\pi_0^H(X) \longrightarrow \pi_0^K(X)$, where $L = \alpha^{-1}(H)$ and $\alpha|_L : L \longrightarrow H$ is the restriction of α. So the remaining issue is to rewrite the composite

$$\pi_0^H(X) \xrightarrow{\mathrm{tr}_H^G} \pi_0^G(X) \xrightarrow{\mathrm{res}_K^G} \pi_0^K(X)$$

of a transfer map and a restriction map, where H and K are two closed subgroups of a compact Lie group G. The answer is given by the *double coset formula*,

$$\mathrm{res}_K^G \circ \mathrm{tr}_H^G = \sum_{[M]} \chi^\sharp(M) \cdot \mathrm{tr}_{K \cap {}^g H}^K \circ c_g^* \circ \mathrm{res}_{K^g \cap H}^H . \tag{2.13}$$

The double coset formula was proved by Feshbach for Borel cohomology theories [8, Theorem II.11] and later generalized to equivariant cohomology theories by Lewis and May [12, IV Section 6]. The sum in the double coset formula (2.13) runs over all connected components M of orbit type manifolds, the group element $g \in G$ that occurs is such that $KgH \in M$, and $\chi^\sharp(M)$ is the internal Euler characteristic of M.

Only finitely many of the orbit type manifolds are non-empty, so the double coset formula is a finite sum.

In this paper we only need the double coset formula when H has a finite index in G, and then (2.13) simplifies. For any other subgroup K of G the intersection $K \cap {}^gH$ then has a finite index in K, so only finite index transfers are involved in the double coset formula. Since G/H is finite, so is the set $K\backslash G/H$ of double cosets, and all orbit type manifold components are points. So all internal Euler characteristics that occur are 1 and the double coset formula specializes to

$$\mathrm{res}^G_K \circ \mathrm{tr}^G_H \ = \sum_{[g]\in K\backslash G/H} \mathrm{tr}^K_{K\cap {}^gH} \circ c_g^* \circ \mathrm{res}^H_{K^g\cap H} \ ;$$

the sum runs over a set of representatives of the K-H-double cosets.

The explicit description of the groups $\mathbf{A}(G,K)$ allows us to relate our notion of global functor to other "global" versions of Mackey functors, which are typically introduced by specifying generating operations and relations between them. For example, our category of global functors is equivalent to the category of *functors with regular Mackey structure* in the sense of Symonds [25, Section 3].

Example 2.14. We list some explicit examples of global functors; for more details we refer to [21].

(i) For every compact Lie group G, the represented global functor $\mathbf{A}(G,-)$ is realized by the suspension spectrum of a free orthogonal space $\mathbf{L}_{G,V}$, by Proposition 2.11. In the special case $G=e$ of the trivial group we refer to this represented global functor as the *Burnside ring global functor* and denote it by $\mathbb{A} = \mathbf{A}(e,-)$. The value $\mathbb{A}(K)$ at a compact Lie group K is a free abelian group with the basis indexed by conjugacy classes of closed subgroups of K with the finite Weyl group. When K is finite, then the Weyl group condition is vacuous and $\mathbb{A}(K)$ is naturally isomorphic to the Burnside ring of K; compare Remark 2.15 below.

The Burnside ring global functor is realized by the sphere spectrum $\mathbb{S}$, given by $\mathbb{S}(V) = S^V$ with the canonical homeomorphisms $S^V \wedge S^W \cong S^{V\oplus W}$ as structure maps. The equivariant homotopy groups of the sphere spectrum are thus the equivariant stable stems. The action on the unit $1 \in \pi_0(\mathbb{S})$ is an isomorphism of global functors

$$\mathbb{A} \xrightarrow{\cong} \underline{\pi}_0(\mathbb{S})$$

from the Burnside ring global functor to the zeroth homotopy global functor of the sphere spectrum. For finite groups, this is originally due to Segal [23], and for general compact Lie groups to tom Dieck, as a corollary to his splitting theorem (see Satz 2 and Satz 3 of [26]).

(ii) Given an abelian group M, the *constant global functor* is given by $\underline{M}(G) = M$ and all restriction maps are identity maps. The transfer $\mathrm{tr}^G_H : \underline{M}(H) \longrightarrow \underline{M}(G)$ is multiplication by the Euler characteristic of the homogeneous space G/H. In particular, if H is a subgroup of the finite index of G, then tr^G_H is multiplication by the index $[G:H]$.

(iii) The *representation ring global functor* $\mathbf{RU}$ assigns to a compact Lie group G the representation ring $\mathbf{RU}(G)$, the Grothendieck group of finite dimensional complex G-representations. The fact that the representation rings form a global functor is classical in the restricted realm of finite groups, but somewhat less familiar for compact Lie groups in general. The restriction maps $\alpha^* : \mathbf{RU}(G) \longrightarrow \mathbf{RU}(K)$ are induced by the restriction of representations along a homomorphism $\alpha : K \longrightarrow$

G. The transfer $\mathrm{tr}_H^G : \mathbf{RU}(H) \longrightarrow \mathbf{RU}(G)$ along a closed subgroup inclusion $H \leq G$ is the *smooth induction* of Segal [22, Section 2]. If H has finite index in G, then this induction sends the class of an H-representation V to the induced G-representation $\mathrm{map}^G(H, V)$; in general, induction may send actual representations to virtual representations. In the generality of compact Lie groups, the double coset formula for $\mathbf{RU}$ was proved by Snaith [24, Theorem 2.4]. We show in [21] that the representation ring global functor $\mathbf{RU}$ is realized by the periodic global K-theory spectrum.

(iv) Given any generalized cohomology theory E (in the non-equivariant sense), we can define a global functor $\underline{E}$ by setting

$$\underline{E}(G) \ = \ E^0(BG) \ ,$$

the zeroth E-cohomology of a classifying space of the group G. The contravariant functoriality in group homomorphisms comes from the covariant functoriality of classifying spaces. The transfer maps for a subgroup inclusion $H \leq G$ comes from the stable transfer map

$$\Sigma^\infty_+ BG \ \longrightarrow \ \Sigma^\infty_+ BH \ .$$

The double coset formula was proved in this context by Feshbach [8, Theorem II.11]. The global functor $G \mapsto E^0(BG)$ is realized by a preferred global homotopy type: in [21] we introduce a "global Borel theory" functor b from the non-equivariant stable homotopy category to the global stable homotopy category such that the global functor $\underline{\pi}_0(bE)$ is isomorphic to $\underline{E}$. The functor b is in fact right adjoint to the forget functor from the global stable homotopy to the non-equivariant stable homotopy category.

Remark 2.15. The full subcategory $\mathbf{A}^{\mathrm{fin}}$ of the global Burnside category $\mathbf{A}$ spanned by *finite* groups has a different, more algebraic description, as we shall now recall. This alternative description is often taken as the definition in algebraic treatments of global functors. We define an algebraic Burnside category $\mathbf{B}$ whose objects are all finite groups. The abelian group $\mathbf{B}(G, K)$ of morphisms from a group G to K is the Grothendieck group of finite K-G-bisets where the right G-action is free. The composition

$$\circ \ : \ \mathbf{B}(K, L) \times \mathbf{B}(G, K) \ \longrightarrow \ \mathbf{B}(G, L)$$

is induced by the balanced product over K; i.e., it is the biadditive extension of

$$(S, T) \ \longmapsto \ S \times_K T \ .$$

Here S has a left L-action and a commuting free right K-action, whereas T has a left K-action and a commuting free right G-action. The balanced product $S \times_K T$ then inherits a left L-action from S and a free right G-action from T. Since the balanced product is associative up to isomorphism, this defines a pre-additive category $\mathbf{B}$.

An isomorphism of pre-additive categories $\mathbf{A}^{\mathrm{fin}} \cong \mathbf{B}$ is given by the identity on objects and by the group isomorphisms $\mathbf{A}^{\mathrm{fin}}(G, K) \longrightarrow \mathbf{B}(G, K)$ sending a basis element $\mathrm{tr}_H^G \circ \alpha^*$ to the class of the right free K-G-biset

$$K \times_{(L,\alpha)} G \ = \ (K \times G) \,/\, (kl, g) \sim (k, \alpha(l)g) \ .$$

The category of "global functors on finite groups," i.e., additive functors from $\mathbf{A}^{\mathrm{fin}}$ to abelian groups, is thus equivalent to the category of *inflation functors* in the sense of [27, page 271]. In the context of finite groups, these inflation functors and several other variations of the concept "global Mackey functor" have been much studied in algebra and representation theory.

Remark 2.16. In Remark 2.3 we observed that only very special kinds of orthogonal G-spectra are part of a "global family," i.e., isomorphic to an orthogonal G-spectrum of the form X_G for some orthogonal spectrum X. The previous obstructions were in terms of point set level conditions, and now we can also isolate obstructions to "being global" in terms of the Mackey functor homotopy groups of an orthogonal G-spectrum.

If we fix a compact Lie group G and let H run through all closed subgroups of G, then the collection of H-equivariant homotopy groups $\pi_0^H(X)$ of an orthogonal spectrum X forms a *Mackey functor* for the group G, with respect to the restriction, conjugation, and transfer maps. One obstruction for a general orthogonal G-spectrum Y to be global, i.e., equivariantly stably equivalent to X_G for some orthogonal spectrum X, is that the G-Mackey functor $H \mapsto \pi_0^H(Y)$ can be extended to a global functor.

An extension of a G-Mackey functor to a global functor requires us to specify values for groups that are not subgroups of G, but it also imposes restrictions on the existing data. In particular, the G-Mackey functor homotopy groups can be complemented by restriction maps along arbitrary group homomorphisms between the subgroups of G. As the extreme case this includes a restriction map $p^* : \pi_*^e(X) \longrightarrow \pi_*^G(X)$ associated to the unique homomorphism $p : G \longrightarrow e$, splitting the restriction map $\mathrm{res}_e^G : \pi_*^G(X) \longrightarrow \pi_*^e(X)$. So one obstruction to being global is that res_e^G must be a splittable epimorphism.

Another point is that for an orthogonal spectrum X (as opposed to a general orthogonal G-spectrum), the action of the Weyl group $W_G H$ on $\pi_0^H(X)$ factors through the outer automorphism group of H. In other words, if g centralizes H, then c_g^* is the identity of $\pi_0^H(X)$. The most extreme case of this is when $H = e$ is the trivial subgroup of G. Every element of G centralizes e, so for G-spectra of the form X_G, the conjugation maps on the value at the trivial subgroup are all identity maps.

3. The global homotopy type of symmetric products

Now we start the equivariant analysis of the symmetric product filtration. The main result is a global homotopy pushout square of orthogonal spectra (3.9), showing that Sp^n can be obtained from Sp^{n-1} by coning off, in the global stable homotopy category, a certain morphism from the suspension spectrum of $B_{\mathrm{gl}}\mathcal{F}_n$. Non-equivariantly, such a homotopy pushout square was exhibited by Lesh; see Theorem 1.1 and Proposition 7.4 of [11]. In Theorem 3.12 we then exploit the Mayer-Vietoris sequence of the global homotopy pushout square to calculate the global functors $\underline{\pi}_0(Sp^n)$ inductively.

We define an orthogonal space $C(B_{\mathrm{gl}}\mathcal{F}_n)$ by

$$(C(B_{\mathrm{gl}}\mathcal{F}_n))(V) \;=\; D(V,n)/\Sigma_n \;,$$

where

$$D(V,n) \;=\; \left\{ (v_1,\ldots,v_n) \in V^n \;:\; \sum_{i=1}^n v_i = 0 \;,\; \sum_{i=1}^n |v_i|^2 \le 1 \right\}$$

is the unit disc in the kernel of summation map. Since a unit disc is the cone on the unit sphere, $C(B_{\mathrm{gl}}\mathcal{F}_n)$ is the unreduced cone of the global classifying space $B_{\mathrm{gl}}\mathcal{F}_n$, whence the notation. Next we define a certain morphism of orthogonal spectra

$$\Phi \; : \; \Sigma^\infty_+ C(B_{\mathrm{gl}}\mathcal{F}_n) \longrightarrow Sp^n$$

that takes the orthogonal subspectrum $\Sigma^\infty_+ B_{\mathrm{gl}}\mathcal{F}_n$ to Sp^{n-1}, and that can be thought of as a highly structured, parametrized Thom-Pontryagin collapse map. I owe this construction to Markus Hausmann. Before giving the details we try to explain the main idea. For every inner product space V, the map $\Phi(V)$ has to assign to each tuple $(y_1, \dots, y_n) \in D(V, n)$ a based map $\Phi(V)(y_1, \dots, y_n) : S^V \longrightarrow Sp^n(V)$ that does not depend on the order of $y_1, \dots, y_n$. We would like to take $\Phi(V)(y_1, \dots, y_n)$ as the product of the Thom-Pontryagin collapse maps in balls of a sufficiently small radius centered at the points $y_1, \dots, y_n$. This would work fine for an individual inner product space V, but such maps would not form a morphism of orthogonal spectra as V increases.

The fix to this problem is to combine the collapse maps with orthogonal projection onto the subspace spanned by $y_1, \dots, y_n$. However, this orthogonal projection does *not* depend continuously on the tuple y at those points where the dimension of the span of $y_1, \dots, y_n$ jumps. So instead of the orthogonal projection to the span we use a certain positive semidefinite self-adjoint endomorphism $P(y)$ of V that has similar features and varies continuously with y.

Construction 3.1 (Collapse maps)**.** We let V be an inner product space and denote by $\mathrm{sa}^+(V)$ the space of positive semidefinite, self-adjoint endomorphisms of V, i.e., $\mathbb{R}$-linear maps $F : V \longrightarrow V$ that satisfy

- $\langle F(v), v\rangle \geq 0$ for all $v \in V$, and
- $\langle F(v), w\rangle = \langle v, F(w)\rangle$ for all $v, w \in V$.

We note that $\mathrm{sa}^+(V)$ is a convex subset of $\mathrm{End}(V)$, and hence is contractible. We fix the natural number $n \geq 2$ and set the radius for the collapse maps to

$$\rho \; = \; \frac{1}{2 \cdot n^{3/2}} \; .$$

We define a scaling function

$$s \; : \; [0, \rho) \; \longrightarrow \; [0, \infty) \qquad \text{by} \qquad s(x) \; = \; x/(\rho - x) \; .$$

What matters is not the precise form of the function s, but only that it is a homeomorphism from $[0, \rho)$ to $[0, \infty)$. We define a *parametrized collapse map*

$$c \; : \; \mathrm{sa}^+(V) \times S^V \; \longrightarrow \; S^V$$

by

$$c(F, v) \; = \; \begin{cases} v \; + \; s(|F(v)|) \cdot F(v) & \text{if } v \neq \infty \text{ and } |F(v)| < \rho, \text{ and} \\ \infty & \text{else.} \end{cases}$$

Lemma 3.2.

(i) *For all $(F, v) \in \mathrm{sa}^+(V) \times S^V$ the relation $|c(F, v)| \geq |v|$ holds.*
(ii) *The map c is continuous.*

Proof.

(i) There is nothing to show if $c(F, v) = \infty$, so we may assume that $v \neq \infty$ and $|F(v)| < \rho$. Since F is self-adjoint, V is the orthogonal direct sum of the image and kernel of F. So we can write

$$v \; = \; a \; + \; b \; ,$$

where $a \in \operatorname{im}(F)$, $b \in \ker(F)$, and b is orthogonal to $\operatorname{im}(F)$. The orthogonal decomposition

$$c(F, v) \;=\; v \;+\; s(|F(v)|) \cdot F(v) \;=\; \big(a + s(|F(a)|) \cdot F(a)\big) \;+\; b$$

allows us to conclude that

$$\begin{aligned} |c(F,v)|^2 \;&=\; |a + s(|F(a)|) \cdot F(a)|^2 + |b|^2 \\ &=\; |a|^2 \;+\; 2s(|F(a)|) \cdot \langle a, F(a) \rangle \\ &\quad + s(|F(a)|)^2 \cdot |F(a)|^2 \;+\; |b|^2 \;\geq\; |a|^2 \;+\; |b|^2 \;=\; |v|^2 \;. \end{aligned}$$

The inequality uses that F is positive semidefinite. Taking square roots proves the claim.

(ii) We consider a sequence (F_k, v_k) that converges in $\mathrm{sa}^+(V) \times S^V$ to a point (F, v). We need to show that the sequence $c(F_k, v_k)$ converges to $c(F, v)$ in S^V. If $v = \infty$, then $|v_k|$ converges to ∞; hence so does $c(F_k, v_k)$ by part (i). So we suppose that $v \neq \infty$ for the rest of the proof. Then we can assume without loss of generality that $v_k \neq \infty$ for all k. We distinguish three cases.

Case 1. $|F(v)| < \rho$. Then $|F_k(v_k)| < \rho$ for almost all k. So $c(F_k, v_k)$ converges to $c(F, v)$ because the formula in the definition of c is continuous in both parameters F and v.

Case 2. $|F(v)| = \rho$. If $|F_k(v_k)| \geq \rho$, then $c(F_k, v_k) = \infty$. Otherwise

$$|c(F_k, v_k)| \;=\; |v_k \;+\; s(|F_k(v_k)|) \cdot F_k(v_k)| \;\geq\; s(|F_k(v_k)|) \cdot |F_k(v_k)| \;-\; |v_k| \;.$$

The sequences $|v_k|$ and $|F_k(v_k)|$ converge to the finite numbers $|v|$, respectively, ρ; on the other hand, $s(|F_k(v_k)|)$ converges to ∞. So the sequence $|c(F_k, v_k)|$ also converges to ∞, which means that the sequence $c(F_k, v_k)$ converges to $\infty = c(F, v)$.

Case 3. $|F(v)| > \rho$. Then $|F_k(v_k)| > \rho$ for almost all k. So $c(F_k, v_k) = \infty$ for almost all k, and this sequence converges to $c(F, v) = \infty$. $\square$

Construction 3.3. We define a continuous map

$$P \;:\; V^n \;\longrightarrow\; \mathrm{sa}^+(V) \qquad \text{by} \qquad P(y)(v) \;=\; P(y_1, \ldots, y_n)(v) \;=\; \sum_{j=1}^{n} \langle v, y_j \rangle \cdot y_j \;.$$

Each of the summands $\langle -, y_j \rangle \cdot y_j$ is self-adjoint, and hence so is $P(y)$. Because

$$\langle P(y)(v), v \rangle \;=\; \sum_{j=1}^{n} \langle v, y_j \rangle^2 \;\geq\; 0,$$

the map $P(y)$ is also positive semidefinite. If the family $(y_1, \ldots, y_n)$ happens to be orthonormal, then $P(y)$ is the orthogonal projection onto the span of $y_1, \ldots, y_n$. In general, $P(y)$ need not be idempotent, but its image is always the span of the vectors $y_1, \ldots, y_n$, and hence its kernel is the orthogonal complement of that span.

For every linear isometric embedding $\varphi : V \longrightarrow W$ and an endomorphism $F \in \mathrm{sa}^+(V)$, we define ${}^{\varphi}F \in \mathrm{sa}^+(W)$ by "conjugation and extension by 0"; i.e., we set

$$({}^{\varphi}F)(\varphi(v) + w) \;=\; \varphi(F(v))$$

for all $(v,w) \in V \times \varphi^\perp$. Then

$$\begin{aligned} {}^{\varphi}(P(y))(\varphi(v)+w) = \varphi(P(y)(v)) &= \sum_{j=1}^{n} \langle \varphi(v), \varphi(y_j)\rangle \cdot \varphi(y_j) \\ &= P(\varphi(y))(\varphi(v)+w) \ , \end{aligned} \tag{3.4}$$

i.e., ${}^{\varphi}(P(y)) = P(\varphi(y))$ as endomorphisms of W.

It will be convenient to extend the meaning of the minus symbol and allow one to subtract a vector from infinity. We define a continuous map

$$\ominus \ : \ S^V \times V \ \longrightarrow \ S^V \qquad \text{by} \qquad v \ominus z \ = \ \begin{cases} v - z & \text{for } v \neq \infty, \text{ and} \\ \infty & \text{for } v = \infty. \end{cases}$$

We emphasize that only the first argument of the operator $\ominus$ is allowed to be infinity; in particular, we cannot subtract ∞ from itself. We define a continuous map

$$\begin{aligned} \tilde\Phi(V) \ &: \ D(V,n) \times S^V \ \longrightarrow \ Sp^n(S^V) \qquad \text{by} \\ \tilde\Phi(V)(y,v) \ &= \ [c(P(y), v\ominus y_1), \dots, c(P(y), v \ominus y_n)] \ . \end{aligned}$$

The map $\tilde\Phi(V)$ sends $D(V,n)\times\{\infty\}$ to the base point. For every permutation $\sigma \in \Sigma_n$ we have $P(y\cdot\sigma) = P(y)$ and hence

$$\tilde\Phi(V)(y\cdot\sigma, v) \ = \ \tilde\Phi(V)(y,v) \ .$$

So $\tilde\Phi(V)$ factors over a continuous map

$$\Phi(V) \ : \ (C(B_{\mathrm{gl}}\mathcal{F}_n)(V))_+ \wedge S^V \ = \ (D(V,n)/\Sigma_n)_+ \wedge S^V \ \longrightarrow \ Sp^n(S^V) \ .$$

Lemma 3.5. *As V varies over all inner product spaces, the maps $\Phi(V)$ form a morphism of orthogonal spectra*

$$\Phi \ : \ \Sigma^\infty_+ C(B_{\mathrm{gl}}\mathcal{F}_n) \ \longrightarrow \ Sp^n \ .$$

Proof. For every linear isometric embedding $\varphi : V \longrightarrow W$, every $F \in \mathrm{sa}^+(V)$, and all $(v,w)\in V\times\varphi^\perp$ with $|F(v)| < \rho$ we have

$$\begin{aligned} c({}^{\varphi}F, \varphi(v)+w) \ &= \ (\varphi(v)+w) \ + \ s(|({}^{\varphi}F)(\varphi(v)+w)|)\cdot({}^{\varphi}F)(\varphi(v)+w) \\ &= \ \varphi(v) + w \ + \ s(|\varphi(F(v))|)\cdot\varphi(F(v)) \ = \ \varphi(c(F,v)) + w \end{aligned} \tag{3.6}$$

in S^W. Hence for all $y \in D(V,n)$,

$$\begin{aligned} c(P(\varphi(y)), (\varphi(v)+w)\ominus\varphi(y_i)) \ &=_{(3.4)} \ c({}^{\varphi}(P(y)), (\varphi(v)+w)\ominus\varphi(y_i)) \\ &= \quad c({}^{\varphi}(P(y)), \varphi(v\ominus y_i)+w) \\ &=_{(3.6)} \ \varphi(c(P(y), v\ominus y_i)) + w \ . \end{aligned}$$

This shows that the square

$$\begin{tikzcd} \mathbf{O}(V,W)\times D(V,n)\times S^V \arrow[r, "\mathbf{O}(V,W)\times\tilde\Phi(V)"] \arrow[d] & \mathbf{O}(V,W)\times Sp^n(S^V) \arrow[d] & ((w,\varphi),[v_1,\dots,v_n]) \arrow[d, mapsto] \\ D(W,n)\times S^W \arrow[r, "\tilde\Phi(W)"'] & Sp^n(S^W) & {[w+\varphi(v_1),\dots,w+\varphi(v_n)]} \end{tikzcd}$$

commutes, where the vertical maps are the structure maps. □

We claim that the morphism Φ takes the orthogonal subspectrum $\Sigma_+^\infty(B_{\mathrm{gl}}\mathcal{F}_n)$ into the subspectrum Sp^{n-1}. We will need that kind of argument again later, so we formulate it more generally.

Lemma 3.7. *Let V be an inner product space and $y \in S(V,n)$. For $t \in [0,1]$ we define $F_t \in \mathrm{sa}^+(V)$ by $F_t = (1-t)\cdot P(y) + t \cdot \mathrm{Id}_V$. Then for every $v \in S^V$ the point*

$$[c(F_t, v \ominus y_1), \ldots, c(F_t, v \ominus y_n)] \;\in\; Sp^n(S^V)$$

belongs to the subspace $Sp^{n-1}(S^V)$.

Proof. Since $\sum_{i=1}^n |y_i|^2 = 1$ there is at least one $i \in \{1,\ldots,n\}$ with $|y_i|^2 \geq 1/n$. The Cauchy-Schwarz inequality gives

$$\begin{aligned} |y_i|\cdot|F_t(y_i)| \;&\geq\; |\langle y_i, F_t(y_i)\rangle| \\ &=\; t\langle y_i, y_i\rangle \;+\; (1-t)\sum_{j=1}^n \langle y_i, y_j\rangle^2 \;\geq\; t|y_i|^2 \;+\; (1-t)|y_i|^4 \;. \end{aligned}$$

Dividing by $|y_i|$ yields

$$|F_t(y_i)| \;\geq\; t|y_i| \;+\; (1-t)|y_i|^3 \;\geq\; \frac{t}{n^{1/2}} \;+\; \frac{1-t}{n^{3/2}} \;\geq\; \frac{1}{n^{3/2}} \;=\; 2\rho \;.$$

The relation

$$\begin{aligned} \sum_{j=1}^n |F_t(y_i - y_j)| \;&\geq \left|\sum_{j=1}^n F_t(y_i - y_j)\right| \\ &= |n\cdot F_t(y_i) - F_t(y_1 + \cdots + y_n)| \;=\; n|F_t(y_i)| \end{aligned}$$

shows that there is a $j \in \{1,\ldots,n\}$ such that

$$|F_t(y_i) - F_t(y_j)| \;=\; |F_t(y_i - y_j)| \;\geq\; |F_t(y_i)| \;\geq\; 2\rho \;.$$

So every $v \in V$ has distance at least ρ from $F_t(y_i)$ or from $F_t(y_j)$. Hence $c(F_t, v \ominus y_i)$ or $c(F_t, v \ominus y_j)$ is the base point at infinity of S^V. □

For $t = 0$, Lemma 3.7 shows that for every $v \in S^V$, the point

$$\Phi(V)(y\cdot\Sigma_n, v) \;=\; [c(P(y), v \ominus y_1), \ldots, c(P(y), v \ominus y_n)]$$

belongs to the subspace $Sp^{n-1}(S^V)$. So the map $\Phi(V)$ takes the subspace $(\Sigma_+^\infty B_{\mathrm{gl}}\mathcal{F}_n)(V)$ to $Sp^{n-1}(S^V)$. We denote by

$$\Psi \;:\; \Sigma_+^\infty B_{\mathrm{gl}}\mathcal{F}_n \;\longrightarrow\; Sp^{n-1}$$

the restriction of the morphism $\Phi : \Sigma_+^\infty C(B_{\mathrm{gl}}\mathcal{F}_n) \longrightarrow Sp^n$ to the suspension spectrum of $B_{\mathrm{gl}}\mathcal{F}_n$. The two vertical maps in the following commutative square (3.9) are levelwise equivariant h-cofibrations. So the following theorem effectively says that the square is a global homotopy pushout.

Theorem 3.8. *The morphism induced on vertical quotients by the commutative square of orthogonal spectra*

$$\begin{tikzcd} \Sigma^\infty_+ B_{\mathrm{gl}}\mathcal{F}_n \arrow[r, "\Psi"] \arrow[d] & Sp^{n-1} \arrow[d] \\ \Sigma^\infty_+ C(B_{\mathrm{gl}}\mathcal{F}_n) \arrow[r, "\Phi"'] & Sp^n \end{tikzcd} \tag{3.9}$$

is a global equivalence.

Proof. We show that for every inner product space V the map

$$\Phi(V)/\Psi(V) \ : \ \bar{D}(V,n)/\Sigma_n \wedge S^V \ \longrightarrow \ Sp^n(S^V)/Sp^{n-1}(S^V)$$

is $O(V)$-equivariantly based homotopic to an equivariant homeomorphism, where $\bar{D}(V,n) = D(V,n)/S(V,n)$. We define continuous maps

$$G_i \ : \ \big([0,1] \times D(V,n) \setminus \{1\} \times S(V,n)\big) \times S^V \ \longrightarrow \ S^V$$

for $1 \leq i \leq n$ by

$$G_i(t,y,v) \ = \ c\left((1-t)\cdot P(y),\ v \ominus \frac{y_i}{1-t|y|}\right) .$$

Here the domain of the definition of G_i is the space of those tuples $(t,y,v) \in [0,1] \times D(V,n) \times S^V$ such that $t < 1$ or $|y| < 1$.

We claim that the map

$$(G_1,\dots,G_n) \ : \ [0,1) \times D(V,n) \times S^V \ \longrightarrow \ (S^V)^n$$

takes the subspace $[0,1) \times S(V,n) \times S^V$ of the source into the wedge inside of the product $(S^V)^n$. Indeed, because $\Phi(V)$ takes $(S(V,n)/\Sigma_n)_+ \wedge S^V$ into $Sp^{n-1}(S^V)$, for every $v \in V$ there is an $i \in \{1,\dots,n\}$ with

$$c(P(y),((1-t)v) \ominus y_i) \ = \ \infty \ ,$$

i.e., $|P(y)((1-t)v - y_i)| \geq \rho$. Because

$$\left|(1-t)\cdot P(y)\left(v - \frac{y_i}{1-t}\right)\right| \ = \ |P(y)((1-t)v-y_i)| \ \geq \ \rho \ ,$$

this implies that $G_i(t,y,v) = \infty$.

We warn the reader that the maps G_i do *not* extend continuously to $[0,1] \times D(V,n) \times S^V$! However, smashing all G_i together remedies this. In other words, we claim that the map

$$\big([0,1] \times D(V,n) \setminus \{1\} \times S(V,n)\big) \times S^V \ \xrightarrow{G_1 \wedge \cdots \wedge G_n} \ (S^V)^{\wedge n}$$

has a continuous extension (necessarily unique)

$$\bar{G} \ : \ [0,1] \times D(V,n) \times S^V \ \longrightarrow \ (S^V)^{\wedge n}$$

that sends $\{1\} \times S(V,n) \times S^V$ to the base point. To prove the claim, we consider any sequence $(t_m, y^m, v^m)_{m \geq 1}$ in $\big([0,1] \times D(V,n) \setminus \{1\} \times S(V,n)\big) \times S^V$ that converges to a point $(1, y, v)$ with $|y| = 1$. We claim that there are $i, j \in \{1, \dots, n\}$ such that $|y_i - y_j| \geq 4\rho$. Indeed, if that were not the case, then we would have

$$\begin{aligned} 2n \;&=\; \left(n \sum_{i=1}^{n} |y_i|^2 \right) - 2 \left\langle \sum_{i=1}^{n} y_i, \sum_{j=1}^{n} y_j \right\rangle + \left(n \sum_{j=1}^{n} |y_j|^2 \right) \\ &=\; \sum_{i,j=1}^{n} \left(|y_i|^2 - 2\langle y_i, y_j \rangle + |y_j|^2 \right) \;=\; \sum_{i,j=1}^{n} |y_i - y_j|^2 \;<\; (4\rho)^2 n^2 \;=\; 4/n \;, \end{aligned}$$

a contradiction.

Since $\lim_{m \longrightarrow \infty} y^m = y$, we deduce that $|y_i^m - y_j^m| \geq 2\rho$ for all sufficiently large m. For these m there is then a $k \in \{i, j\}$ such that

$$|(1 - t_m |y^m|) v^m - y_k^m| \;\geq\; \rho \;,$$

and hence

$$\begin{aligned} |G_k(t_m, y^m, v^m)| \;&=\; \left| c\left((1 - t_m) \cdot P(y^m),\; v^m \ominus \frac{y_k^m}{1 - t_m |y^m|} \right) \right| \\ &\geq\; \left| v^m - \frac{y_k^m}{1 - t_m |y^m|} \right| \;\geq\; \frac{\rho}{1 - t_m |y^m|} \;. \end{aligned}$$

The first inequality is Lemma 3.2 (i). Since the sequences (t_m) and $|y^m|$ converge to 1, the length of the vector

$$(G_1(t_m, y^m, v^m), \dots, G_n(t_m, y^m, v^m))$$

tends to infinity with m, so it converges to the base point at infinity of $S^{V^n} = (S^V)^{\wedge n}$.

We have now completed the verification that the map $\bar{G} : [0,1] \times D(V, n) \times S^V \longrightarrow (S^V)^{\wedge n}$ is continuous. Because the endomorphism $P(y)$ does not depend on the order of the components of the tuple y, the maps G_i satisfy

$$G_i(t, y \cdot \sigma, v) \;=\; G_{\sigma(i)}(t, y, v) \;,$$

so the map $\bar{G}$ descends to a well-defined continuous and $O(V)$-equivariant map

$$[0,1] \times \left(\bar{D}(V, n)/\Sigma_n \wedge S^V \right) \;\longrightarrow\; (S^V)^{\wedge n}/\Sigma_n \;=\; Sp^n(S^V)/Sp^{n-1}(S^V) \;,$$

which is the desired equivariant homotopy. This homotopy starts with the map $\Phi(V)/\Psi(V)$ and ends with the map

$$\begin{aligned} \bar{D}(V, n)/\Sigma_n \wedge S^V \;&\longrightarrow\; Sp^n(S^V)/Sp^{n-1}(S^V), \\ (y \cdot \Sigma_n, v) \;&\longmapsto\; \left(v - \frac{y_1}{1 - |y|} \right) \wedge \dots \wedge \left(v - \frac{y_n}{1 - |y|} \right) \;; \end{aligned}$$

this map is a continuous bijection from a compact space to a Hausdorff space, hence a homeomorphism. $\square$

We recall from (1.12) the definition of the unstable homotopy class $u_n \in \pi_0^{\Sigma_n}(B_{\text{gl}}\mathcal{F}_n)$. The stabilization map (2.9) lets us define a Σ_n-equivariant stable homotopy class

$$w_n \;=\; \sigma^{\Sigma_n}(u_n) \;\in\; \pi_0^{\Sigma_n}(\Sigma_+^\infty B_{\text{gl}}\mathcal{F}_n) \;.$$

The last ingredient for our main calculation is to determine the image of w_n under the morphism of orthogonal spectra

$$\Psi \;:\; \Sigma_+^\infty B_{\text{gl}}\mathcal{F}_n \;\longrightarrow\; Sp^{n-1} \;.$$

Proposition 3.10. *The relation*

$$\Psi_*(w_n) \;=\; i_*\left(\mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}(1)\right)$$

holds in the group $\pi_0^{\Sigma_n}(Sp^{n-1})$, *where* $i : \mathbb{S} \longrightarrow Sp^{n-1}$ *is the inclusion.*

Proof. The class $\Psi_*(w_n)$ is represented by the composite Σ_n-map

$$(3.11)\qquad S^{\nu_n} \xrightarrow{(d_1,\dots,d_n)\cdot\Sigma_n\wedge -} (S(\nu_n,n)/\Sigma_n)_+ \wedge S^{\nu_n} \xrightarrow{\Psi(\nu_n)} Sp^{n-1}(S^{\nu_n}) \ ,$$

where ν_n is the reduced natural Σ_n-representation and

$$(d_1,\dots,d_n) \;=\; \frac{1}{\sqrt{n-1}}(b-e_1,\dots,b-e_n) \;\in\; S(\nu_n,n) \ .$$

We define an equivariant homotopy to a different map that is easier to understand.

The space $\mathrm{sa}^+(\nu_n)$ of positive semidefinite self-adjoint endomorphisms of ν_n is convex, so we can interpolate between $P(d_1,\dots,d_n)$ and the identity of ν_n in $\mathrm{sa}^+(\nu_n)$ by the linear homotopy

$$t \;\longmapsto\; F_t \;=\; (1-t)\cdot P(d_1,\dots,d_n) \;+\; t\cdot \mathrm{Id}_{\nu_n} \ .$$

This induces a homotopy

$$K \;:\; [0,1]\times S^{\nu_n} \;\longrightarrow\; Sp^n(S^{\nu_n}) \ , \quad K(t,v) \;=\; [c(F_t, v\ominus d_1),\dots,c(F_t,v\ominus d_n)] \ .$$

For every permutation $\sigma\in\Sigma_n$ we have $\sigma\cdot d_i = d_{\sigma(i)}$, and hence

$${}^{\sigma}(P(d_1,\dots,d_n)) =_{(3.4)} P(\sigma\cdot d_1,\dots,\sigma\cdot d_n) \;=\; P(d_{\sigma(1)},\dots,d_{\sigma(n)}) \;=\; P(d_1,\dots,d_n)$$

as endomorphisms of ν_n. Thus also ${}^{\sigma}(F_t) = F_t$ and hence

$$\begin{aligned}\sigma\cdot c(F_t, v\ominus d_i) &=_{(3.6)} c({}^{\sigma}(F_t), \sigma\cdot(v\ominus d_i)) \;=\; c(F_t, (\sigma\cdot v)\ominus(\sigma\cdot d_i))\\ &=\; c(F_t,(\sigma\cdot v)\ominus d_{\sigma(i)}) \ ,\end{aligned}$$

so $\sigma\cdot K(t,v) = K(t,\sigma\cdot v)$, i.e., the homotopy K is Σ_n-equivariant. A priori, the homotopy takes values in the nth symmetric product; however, Lemma 3.7 applied to $V=\nu_n$ and $y=(d_1,\dots,d_n)$ shows that $K(t,v)$ belongs to $Sp^{n-1}(S^{\nu_n})$.

The homotopy K starts with the composite (3.11), so the map

$$\begin{aligned} K(1,-) \;&:\; S^{\nu_n} \;\longrightarrow\; Sp^{n-1}(S^{\nu_n}) \qquad \text{given by}\\ K(1,v) \;&=\; \left[c(\mathrm{Id}_V, v\ominus d_1),\dots,c(\mathrm{Id}_V,v\ominus d_n)\right]\end{aligned}$$

is another representative of the class $\Psi_*(w_n)$. Because

$$|d_i-d_j| \;=\; \sqrt{\frac{2}{n-1}} \;\geq\; \frac{1}{n^{3/2}} \;=\; 2\rho$$

for all $i\neq j$, the interiors of the ρ-balls around the points $d_1,\dots,d_n$ are disjoint. So for every $v\in S^{\nu_n}$ at most one of the points $c(\mathrm{Id}_V,v\ominus d_1),\dots,c(\mathrm{Id}_V,v\ominus d_n)$ is different from the base point of S^{ν_n} at infinity. The map $K(1,-)$ thus equals the composite

$$S^{\nu_n} \xrightarrow{\;J\;} S^{\nu_n} \xrightarrow{\;i\;} Sp^{n-1}(S^{\nu_n}),$$

where the first map is defined by

$$J(v) \;=\; \begin{cases} c(\mathrm{Id}_V, v\ominus d_i) = \frac{v-d_i}{1-|v-d_i|/\rho} & \text{if } v\neq\infty \text{ and } |v-d_i|<\rho, \text{ and}\\ \infty & \text{else.}\end{cases}$$

The map J is a Σ_n-equivariant Thom-Pontryagin collapse map around the image of the equivariant embedding

$$\Sigma_n/\Sigma_{n-1} \longrightarrow \nu_n \ , \quad \sigma\Sigma_{n-1} \longmapsto d_{\sigma(n)} \ .$$

So J represents the class $\mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}(1)$ in the equivariant 0-stem $\pi_0^{\Sigma_n}(\mathbb{S})$. $\square$

Now we can put the pieces together and prove our main calculation. We let I_n denote the global subfunctor of the Burnside ring global functor $\mathbb{A}$ generated by the element $t_n = n\cdot 1 - \mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}(1)$ in $\mathbb{A}(\Sigma_n)$, and we let I_∞ denote the union of the global functors I_n for all $n \geq 1$.

Theorem 3.12. *The inclusion of orthogonal spectra $Sp^{n-1} \longrightarrow Sp^n$ induces an epimorphism*

$$\underline{\pi}_0(Sp^{n-1}) \longrightarrow \underline{\pi}_0(Sp^n)$$

of the zeroth homotopy global functors whose kernel is generated, as a global functor, by the class

$$i_*\Big(n\cdot 1 \ - \ \mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}(1)\Big) \ \in \ \pi_0^{\Sigma_n}(Sp^{n-1}) \ ,$$

where $i : \mathbb{S} \longrightarrow Sp^{n-1}$ is the inclusion. For every $n \geq 1$ and $n = \infty$ the action of the Burnside ring global functor on the class $i_*(1) \in \pi_0^e(Sp^n)$ passes to an isomorphism of global functors

$$\mathbb{A}/I_n \ \cong \ \underline{\pi}_0(Sp^n) \ .$$

Proof. For every inner product space V the embedding $Sp^{n-1}(S^V) \longrightarrow Sp^n(S^V)$ has the $O(V)$-equivariant homotopy extension property. The cone inclusion $q : B_{\mathrm{gl}}\mathcal{F}_n \longrightarrow C(B_{\mathrm{gl}}\mathcal{F}_n)$ also has the levelwise homotopy extension property, and hence so does the induced morphism $j = \Sigma_+^\infty q : \Sigma_+^\infty B_{\mathrm{gl}}\mathcal{F}_n \longrightarrow \Sigma_+^\infty C(B_{\mathrm{gl}}\mathcal{F}_n)$ of the suspension spectra. So Theorem 3.8 says that the commutative square of orthogonal spectra (3.9) is a global homotopy pushout square. Taking equivariant stable homotopy groups thus results in an exact Mayer-Vietoris sequence that ends in the exact sequence of global functors

(3.13)

$$\underline{\pi}_0(\Sigma_+^\infty B_{\mathrm{gl}}\mathcal{F}_n) \xrightarrow{\binom{j_*}{\Psi_*}} \underline{\pi}_0(\Sigma_+^\infty C(B_{\mathrm{gl}}\mathcal{F}_n)) \oplus \underline{\pi}_0(Sp^{n-1}) \xrightarrow{(-\Phi_*, \mathrm{incl}_*)} \underline{\pi}_0(Sp^n) \ \longrightarrow \ 0 \ .$$

By Proposition 1.13 the Rep-functor $\underline{\pi}_0(B_{\mathrm{gl}}\mathcal{F}_n)$ is generated by the element u_n in $\pi_0^{\Sigma_n}(B_{\mathrm{gl}}\mathcal{F}_n)$. So by Proposition 2.10 the global functor $\underline{\pi}_0(\Sigma_+^\infty B_{\mathrm{gl}}\mathcal{F}_n)$ is generated by the element $w_n = \sigma^{\Sigma_n}(u_n)$ in $\pi_0^{\Sigma_n}(\Sigma_+^\infty B_{\mathrm{gl}}\mathcal{F}_n)$. The orthogonal space $C(B_{\mathrm{gl}}\mathcal{F}_n)$ is contractible; so its suspension spectrum $\Sigma_+^\infty C(B_{\mathrm{gl}}\mathcal{F}_n)$ is globally equivalent to the sphere spectrum. Thus $\underline{\pi}_0(\Sigma_+^\infty C(B_{\mathrm{gl}}\mathcal{F}_n))$ is isomorphic to the Burnside ring global functor $\mathbb{A}$, and it is freely generated by the class $1 = \sigma^e(u)$ in $\pi_0^e(\Sigma_+^\infty C(B_{\mathrm{gl}}\mathcal{F}_n))$, where $u \in \pi_0^e(C(B_{\mathrm{gl}}\mathcal{F}_n))$ is the unique element. We record that

$$\begin{aligned} j_*(w_n) \ &= \ (\Sigma_+^\infty q)_*(\sigma^{\Sigma_n}(u_n)) \ = \ \sigma^{\Sigma_n}(q_*(u_n)) \ = \ \sigma^{\Sigma_n}(p^*(u)) \\ &= \ p^*(\sigma^e(u)) \ = \ p^*(1) \ , \end{aligned}$$

where $p : \Sigma_n \longrightarrow e$ is the unique homomorphism. The relation $q_*(u_n) = p^*(u)$ holds because the set $\pi_0^{\Sigma_n}(C(B_{\mathrm{gl}}\mathcal{F}_n))$ has only one element.

Since the global functor $\underline{\pi}_0(\Sigma^\infty_+ C(B_{\mathrm{gl}}\mathcal{F}_n))$ is freely generated by the class 1, there is a unique morphism $s : \underline{\pi}_0(\Sigma^\infty_+ C(B_{\mathrm{gl}}\mathcal{F}_n)) \longrightarrow \underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}\mathcal{F}_n)$ such that $s(1) = \mathrm{res}^{\Sigma_n}_e(w_n)$. Then

$$j_*(s(1)) \ = \ \mathrm{res}^{\Sigma_n}_e(j_*(w_n)) \ = \ \mathrm{res}^{\Sigma_n}_e(p^*(1)) \ = \ 1 \ .$$

Morphisms out of the global functor $\underline{\pi}_0(\Sigma^\infty_+ C(B_{\mathrm{gl}}\mathcal{F}_n))$ are determined by their effect on the universal class, so we conclude that $j_* \circ s = \mathrm{Id}$. In particular, j_* is an epimorphism and the Mayer-Vietoris sequence (3.13) restricts to an exact sequence of global functors

$$\ker(j_*) \ \xrightarrow{\Psi_*} \ \underline{\pi}_0(Sp^{n-1}) \ \xrightarrow{\mathrm{incl}_*} \ \underline{\pi}_0(Sp^n) \ \longrightarrow \ 0 \ .$$

The other composite $s \circ j_*$ is an idempotent endomorphism of $\underline{\pi}_0(\Sigma^\infty_+ B_{\mathrm{gl}}\mathcal{F}_n)$, and it satisfies

$$s(j_*(w_n)) \ = \ s(p^*(1)) \ = \ p^*(s(1)) \ = \ p^*(\mathrm{res}^{\Sigma_n}_e(w_n)) \ .$$

The kernel of j_* is thus generated as a global functor by

$$(s \circ j_* - \mathrm{Id})(w_n) \ = \ p^*(\mathrm{res}^{\Sigma_n}_e(w_n)) \ - \ w_n \ .$$

The global functor $\Psi_*(\ker(j_*))$ is then generated by the class

$$\begin{aligned} \Psi_*(p^*(\mathrm{res}^{\Sigma_n}_e(w_n)) - w_n) \ &= \ p^*\left(\mathrm{res}^{\Sigma_n}_e\left(i_*\left(\mathrm{tr}^{\Sigma_n}_{\Sigma_{n-1}}(1)\right)\right)\right) \ - \ i_*\left(\mathrm{tr}^{\Sigma_n}_{\Sigma_{n-1}}(1)\right) \\ &= \ i_*\left(p^*\left(\mathrm{res}^{\Sigma_n}_e\left(\mathrm{tr}^{\Sigma_n}_{\Sigma_{n-1}}(1)\right)\right) - \mathrm{tr}^{\Sigma_n}_{\Sigma_{n-1}}(1)\right) \\ &= \ i_*\left(n \cdot 1 - \mathrm{tr}^{\Sigma_n}_{\Sigma_{n-1}}(1)\right) \ . \end{aligned}$$

The first equality uses Proposition 3.10. This proves the first claim.

The second claim is then obtained by induction over n, using that $I_{n-1} \subset I_n$. For $n = \infty$ we use that the canonical map

$$\mathrm{colim}_n \ \underline{\pi}_0(Sp^n) \ \longrightarrow \ \underline{\pi}_0(Sp^\infty)$$

is an isomorphism because each embedding $Sp^{n-1} \longrightarrow Sp^n$ is levelwise an equivariant h-cofibration. □

4. Examples

In this last section we make the description of the global functor $\underline{\pi}_0(Sp^n)$ of Theorem 3.12 more explicit by exhibiting a generating set for the group $I_n(G)$, the kernel of the map $\mathbb{A}(G) \cong \pi_0^G(\mathbb{S}) \longrightarrow \pi_0^G(Sp^n)$, in terms of the subgroup structure of G. We use this to determine $\pi_0^G(Sp^n)$, for all n, when G is a p-group, a symmetric group Σ_k for $k \leq 4$, and the alternating group A_5. The purpose of these calculations is twofold: we want to illustrate that $\pi_0^G(Sp^n)$ can be worked out explicitly in terms of the poset of conjugacy classes of subgroups of G and their relative indices; and we want to convince the reader that the explicit answer for the group $\pi_0^G(Sp^n)$ is much less natural than the global description of $\underline{\pi}_0(Sp^n)$ given by Theorem 3.12.

For a pair of closed subgroups $K \leq H$ of a compact Lie group G such that K has the finite index in H we denote by $t^H_K \in \mathbb{A}(G)$ the class

$$t^H_K \ = \ [H : K] \cdot \mathrm{tr}^G_H(1) \ - \ \mathrm{tr}^G_K(1) \ .$$

For example, $t_n = t^{\Sigma_n}_{\Sigma_{n-1}}$. The notation is somewhat imprecise because it does not record the ambient group G, but that should always be clear from the context. In the next proposition these classes feature under the hypothesis that the Weyl group of H in G is finite (so that $\mathrm{tr}_H^G(1)$ is a non-trivial class in the Burnside ring of G). However, the group K may have the infinite Weyl group, in which case $\mathrm{tr}_K^G(1) = 0$ and t_K^H simplifies to $[H : K] \cdot \mathrm{tr}_H^G(1)$.

Proposition 4.1. *For every $n \geq 2$ and every compact Lie group G, the abelian group $I_n(G)$ is generated by the classes t_K^H as (H, K) runs through a set of representatives of all G-conjugacy classes of nested pairs $K \leq H$ of closed subgroups of G such that*

- *$[H : K] \leq n$ and*
- *the Weyl group $W_G H$ is finite.*

Proof. By definition I_n is the image of the morphism of global functors

$$\mathbf{A}(\Sigma_n, -) \longrightarrow \mathbb{A}$$

represented by $t_n \in \mathbb{A}(\Sigma_n)$. By Theorem 2.12 (ii) the group $\mathbf{A}(\Sigma_n, G)$ is generated by the operations $\mathrm{tr}_H^G \circ \alpha^*$ where (H, α) runs through the $(G \times \Sigma_n)$-conjugacy classes of pairs consisting of a closed subgroup $H \leq G$ with finite Weyl group and a continuous homomorphism $\alpha : H \longrightarrow \Sigma_n$. So $I_n(G)$ is generated, as an abelian group, by the classes

$$\mathrm{tr}_H^G(\beta^*(\mathrm{res}_\Gamma^{\Sigma_n}(t_n))) \in \mathbb{A}(G) ,$$

where Γ is a subgroup of Σ_n and $\beta : H \longrightarrow \Gamma$ a continuous epimorphism. The double coset formula (in the finite index case) gives

$$\begin{aligned} \mathrm{res}_\Gamma^{\Sigma_n}(t_n) &= n \cdot 1 - \mathrm{res}_\Gamma^{\Sigma_n}(\mathrm{tr}_{\Sigma_{n-1}}^{\Sigma_n}(1)) \\ &= n \cdot 1 - \sum_{[\sigma] \in \Gamma \backslash \Sigma_n / \Sigma_{n-1}} \mathrm{tr}_{\Gamma \cap {}^\sigma \Sigma_{n-1}}^\Gamma(1) = \sum_{[\sigma] \in \Gamma \backslash \Sigma_n / \Sigma_{n-1}} t_{\Gamma \cap {}^\sigma \Sigma_{n-1}}^\Gamma \end{aligned}$$

in $\mathbb{A}(\Gamma)$, where we used that

$$\sum_{[\sigma] \in \Gamma \backslash \Sigma_n / \Sigma_{n-1}} [\Gamma : \Gamma \cap {}^\sigma \Sigma_{n-1}] = n .$$

Thus

$$\beta^*(\mathrm{res}_\Gamma^{\Sigma_n}(t_n)) = \sum_{[\sigma] \in \Gamma \backslash \Sigma_n / \Sigma_{n-1}} \beta^*\left(t_{\Gamma \cap {}^\sigma \Sigma_{n-1}}^\Gamma\right) = \sum_{[\sigma] \in \Gamma \backslash \Sigma_n / \Sigma_{n-1}} t_{\beta^{-1}({}^\sigma \Sigma_{n-1})}^H$$

in $\mathbb{A}(H)$. Transferring from H to G gives

$$\mathrm{tr}_H^G(\beta^*(\mathrm{res}_\Gamma^{\Sigma_n}(t_n))) = \sum_{[\sigma] \in \Gamma \backslash \Sigma_n / \Sigma_{n-1}} t_{\beta^{-1}({}^\sigma \Sigma_{n-1})}^H \quad \text{in } \mathbb{A}(G).$$

Since ${}^\sigma \Sigma_{n-1}$ has index n in Σ_n, the group $\Gamma \cap {}^\sigma \Sigma_{n-1}$ has index at most n in Γ, and hence the group $\beta^{-1}({}^\sigma \Sigma_{n-1}) = \beta^{-1}(\Gamma \cap {}^\sigma \Sigma_{n-1})$ has index at most n in H. So $I_n(G)$ is indeed contained in the group described in the statement of the proposition.

For the other inclusion we consider a pair of closed subgroups $K \leq H$ in the ambient group G with $m = [H : K] \leq n$. A choice of bijection between H/K and $\{1, \ldots, m\}$ turns the left translation action of H on H/K into a homomorphism $\beta : H \longrightarrow \Sigma_m$ such that H/K is isomorphic, as an H-set, to $\beta^*(\{1, \ldots, m\})$. Since $t_m \in I_m(\Sigma_m) \subset I_n(\Sigma_m)$ and I_n is a global functor, we conclude that

$$t_K^H = \mathrm{tr}_H^G\left([H : K] \cdot 1 - \mathrm{tr}_K^H(1)\right) = \mathrm{tr}_H^G(\beta^*(t_m)) \in I_n(G) . \qquad \square$$

For every finite group G the augmentation ideal $I(G)$ is generated by the classes t^G_H where H runs through all subgroups of G. So Proposition 4.1 shows that the filtration by the subfunctors I_n exhausts the augmentation ideal at the $|G|$th stage.

Corollary 4.2. *Let G be a finite group. Then $I_n(G) = I(G)$ for $n \geq |G|$.*

However, often the filtration stops earlier, for example for p-groups.

Example 4.3 (Finite p-groups)**.** Let p be a prime and P a finite p-group. Proposition 4.1 shows that $I_n(P) = \{0\}$ for $n < p$. On the other hand, every proper subgroup H of P admits a sequence of intermediate subgroups

$$H \ = \ H_0 \ \subset \ H_1 \ \subset \ \cdots \ \subset \ H_k \ = \ P$$

such that $[H_i : H_{i-1}] = p$ for all $i = 1, \ldots, k$. Then the class

$$t^P_H \ = \ p^k \cdot 1 - \mathrm{tr}^P_H(1) \ = \ \sum_{i=1}^{k} p^{i-1} \cdot t^{H_i}_{H_{i-1}}$$

belongs to $I_p(P)$ by Proposition 4.1. Since the classes t^P_H generate the augmentation ideal, we conclude that $I_p(P) = I(P)$. Hence the group $\pi^P_0(Sp^n)$ is isomorphic to the Burnside ring $\mathbb{A}(P)$ for $1 \leq n < p$ and free of rank 1 for $n \geq p$.

We work out the symmetric product filtration on equivariant homotopy groups for the symmetric groups Σ_k for $k \leq 4$ and for the alternating group A_5. The groups $G = \Sigma_4$ and $G = A_5$ provide explicit examples of homotopy groups $\pi^G_0(Sp^n)$ with non-trivial torsion.

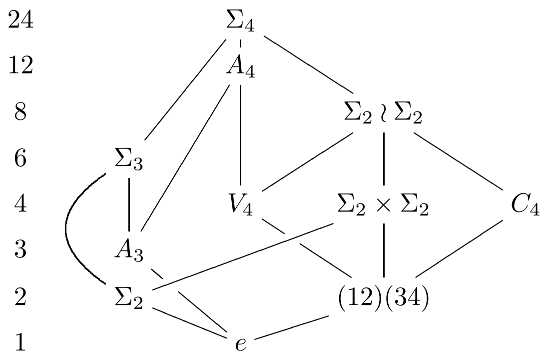
Example 4.4 (Symmetric group Σ_2)**.** For the group Σ_2 we have $I_2(\Sigma_2) = I(\Sigma_2)$, freely generated by the class t_2, i.e., the filtration terminates at the second step. Hence the group $\pi^{\Sigma_2}_0(\mathbb{S})$ is free of rank 2, while the groups $\pi^{\Sigma_2}_0(Sp^n)$ are free of rank 1 for all $n \geq 2$.

Example 4.5 (Symmetric group Σ_3)**.** The group Σ_3 has four conjugacy classes of subgroups with representatives e, Σ_2, A_3, and Σ_3. So the augmentation ideal $I(\Sigma_3)$ is free of rank 3, and a basis is given by the classes

$$t_3 \ = \ t^{\Sigma_3}_{\Sigma_2} \ , \quad p^*(t_2) \ = \ t^{\Sigma_3}_{A_3}, \qquad \text{and} \qquad \mathrm{tr}^{\Sigma_3}_{\Sigma_2}(t_2) \ = \ 2 \cdot t^{\Sigma_3}_{\Sigma_2} \ - \ t^{\Sigma_3}_{e} \ ,$$

where $p : \Sigma_3 \longrightarrow \Sigma_2$ is the unique epimorphism. Hence $I_2(\Sigma_3)$ is freely generated by the classes $p^*(t_2)$ and $\mathrm{tr}^{\Sigma_3}_{\Sigma_2}(t_2)$, and $I_3(\Sigma_3) = I(\Sigma_3)$, i.e., the filtration stabilizes at the third step. Theorem 3.12 lets us conclude that the homotopy group $\pi^{\Sigma_3}_0(Sp^n)$ is free for every $n \geq 1$ and has rank 4 for $n = 1$, rank 2 for $n = 2$, and rank 1 for $n \geq 3$.

Example 4.6 (Symmetric group Σ_4)**.** The group Σ_4 has 11 conjugacy classes of subgroups, displayed below; the left column lists the order of a subgroup, and lines denote subconjugacy,



The augmentation ideal $I(\Sigma_4)$ is free of rank 10, and the classes

$$(4.7) \qquad t_e^{\Sigma_2}\ ,\quad t_{\Sigma_2}^{\Sigma_2\times\Sigma_2}\ ,\quad t_{(12)(34)}^{V_4}\ ,\quad t_{A_3}^{\Sigma_3}\ ,\quad t_{V_4}^{\Sigma_2\wr\Sigma_2}\ ,\quad t_{\Sigma_2\times\Sigma_2}^{\Sigma_2\wr\Sigma_2}\ ,\quad t_{C_4}^{\Sigma_2\wr\Sigma_2}\ ,\quad t_{A_4}^{\Sigma_4}$$

together with the two classes

$$t_{\Sigma_2\wr\Sigma_2}^{\Sigma_4} \qquad \text{and} \qquad t_{\Sigma_3}^{\Sigma_4} \;=\; t_4$$

form a basis of $I(\Sigma_4)$.

The group $I_2(\Sigma_4)$ is generated by the classes t_K^H as (H,K) runs over all pairs of nested subgroups with $[H:K]=2$. All classes of this particular form are linear combinations of the eight classes (4.7),

$$\begin{aligned}
t_{(12)(34)}^{\Sigma_2\times\Sigma_2} &= t_{(12)(34)}^{V_4} + 2\cdot t_{V_4}^{\Sigma_2\wr\Sigma_2} - 2\cdot t_{\Sigma_2\times\Sigma_2}^{\Sigma_2\wr\Sigma_2},\\
t_{(12)(34)}^{C_4} &= t_{(12)(34)}^{V_4} + 2\cdot t_{V_4}^{\Sigma_2\wr\Sigma_2} - 2\cdot t_{C_4}^{\Sigma_2\wr\Sigma_2},\\
t_e^{(12)(34)} &= t_e^{\Sigma_2} + 2\cdot t_{\Sigma_2}^{\Sigma_2\times\Sigma_2} - 2\cdot t_{(12)(34)}^{\Sigma_2\times\Sigma_2}.
\end{aligned}$$

So the eight classes (4.7) form a basis of $I_2(\Sigma_4)$.

The group $I_3(\Sigma_4)$ is generated by the classes t_K^H for all nested subgroup pairs with $[H:K]\leq 3$. We observe that

$$(4.8) \qquad 3\cdot t_4 \;=\; 3\cdot t_{\Sigma_3}^{\Sigma_4} \;=\; t_{\Sigma_2}^{\Sigma_2\times\Sigma_2} + 2\cdot t_{\Sigma_2\times\Sigma_2}^{\Sigma_2\wr\Sigma_2} + 4\cdot t_{\Sigma_2\wr\Sigma_2}^{\Sigma_4} - t_{\Sigma_2}^{\Sigma_3} \;\in\; I_3(\Sigma_4)\ ;$$

Proposition 4.15 below explains in which way this relation is an exceptional feature for $n=4$. All classes t_K^H with $[H:K]\leq 3$ are linear combinations of the classes (4.7) and the two classes

$$(4.9) \qquad t_{\Sigma_2\wr\Sigma_2}^{\Sigma_4} \qquad \text{and} \qquad 3\cdot t_{\Sigma_3}^{\Sigma_4}\ .$$

Indeed:

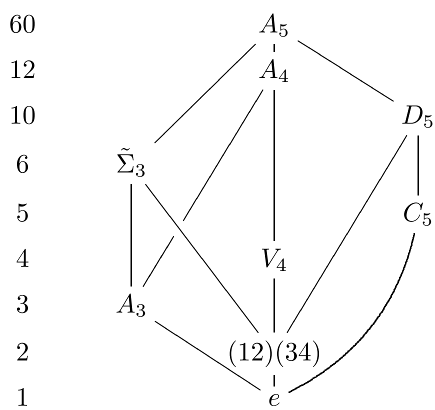
$$\begin{aligned}
t_{V_4}^{A_4} &= t_{V_4}^{\Sigma_2\wr\Sigma_2} + 2\cdot t_{\Sigma_2\wr\Sigma_2}^{\Sigma_4} - 3\cdot t_{A_4}^{\Sigma_4},\\
t_{\Sigma_2}^{\Sigma_3} &= t_{\Sigma_2}^{\Sigma_2\times\Sigma_2} + 2\cdot t_{\Sigma_2\times\Sigma_2}^{\Sigma_2\wr\Sigma_2} + 4\cdot t_{\Sigma_2\wr\Sigma_2}^{\Sigma_4} - 3\cdot t_{\Sigma_3}^{\Sigma_4},\\
t_e^{A_3} &= t_e^{\Sigma_2} + 2\cdot t_{\Sigma_2}^{\Sigma_3} - 3\cdot t_{A_3}^{\Sigma_3}.
\end{aligned}$$

Since the eight elements (4.7) and the two elements (4.9) together are linearly independent, they form a basis of $I_3(\Sigma_4)$.

Since $[\Sigma_4 : \Sigma_3] = 4$, the last basis element $t_4 = t^{\Sigma_4}_{\Sigma_3}$ belongs to $I_4(\Sigma_4)$, and thus $I_4(\Sigma_4) = I(\Sigma_4)$. The relation (4.8) shows that $I_3(\Sigma_4)$ has index 3 in $I_4(\Sigma_4) = I(\Sigma_4)$. Altogether, Theorem 3.12 lets us conclude the following:

- the group $\pi_0^{\Sigma_4}(\mathbb{S}) = \pi_0^{\Sigma_4}(Sp^1)$ is free of rank 11,
- the group $\pi_0^{\Sigma_4}(Sp^2)$ is free of rank 3,
- the group $\pi_0^{\Sigma_4}(Sp^3)$ has rank 1 and its torsion subgroup has order 3, and
- for all $n \geq 4$, the group $\pi_0^{\Sigma_4}(Sp^n)$ is free of rank 1.

Example 4.10 (Alternating group A_5)**.** The last example that we treat in detail is the alternating group A_5. The point is not just to have another explicit example, but we also need the calculation of $I_5(A_5)$ in Example 4.14 for identifying when the filtration of Σ_5 stabilizes. The group A_5 has nine conjugacy classes of subgroups:



The group $\tilde{\Sigma}_3$ is generated by the elements (123) and (12)(45) and is isomorphic to Σ_3 (but not conjugate in Σ_5 to the "standard" copy of Σ_3 generated by (123) and (12)). The dihedral group D_5 is generated by the elements (12345) and (25)(34).

The augmentation ideal $I(A_5)$ is free of rank 8, and a convenient basis for our purposes is given by the classes

(4.11)
$$t_e^{(12)(34)}\ ,\quad t^{V_4}_{(12)(34)}\ ,\quad t^{\tilde{\Sigma}_3}_{A_3}\ ,\quad t^{D_5}_{C_5}\ ,\quad t^{A_4}_{V_4}\ ,\quad t^{\tilde{\Sigma}_3}_{(12)(34)} + t^{A_4}_{A_3}\ ,\quad t^{A_5}_{A_4} \quad \text{and} \quad t^{A_5}_{D_5}\ .$$

Proposition 4.1 says that the group $I_2(A_5)$ is generated by the classes t^H_K as (H, K) runs over all pairs of nested subgroups with $[H : K] = 2$, i.e., by the first four classes of the basis (4.11). So these four classes form a basis of $I_2(A_5)$.

We observe that

(4.12)
$$3 \cdot (t^{\tilde{\Sigma}_3}_{(12)(34)} + t^{A_4}_{A_3}) \;=\; 2 \cdot t^{V_4}_{(12)(34)} \;+\; 4 \cdot t^{A_4}_{V_4} \;+\; 3 \cdot t^{\tilde{\Sigma}_3}_{A_3} \;+\; t^{\tilde{\Sigma}_3}_{(12)(34)} \;\in\; I_3(A_5)\ .$$

The group $I_3(A_5)$ is generated by the classes t^H_K for all nested subgroup pairs with $[H : K] \leq 3$. Because

$$\begin{aligned} t^{\tilde{\Sigma}_3}_{(12)(34)} \;&=\; 3 \cdot (t^{\tilde{\Sigma}_3}_{(12)(34)} + t^{A_4}_{A_3}) \;-\; 2 \cdot t^{V_4}_{(12)(34)} \;-\; 3 \cdot t^{\tilde{\Sigma}_3}_{A_3} \;-\; 4 \cdot t^{A_4}_{V_4}, \\ t^{A_3}_e \;&=\; t_e^{(12)(34)} \;-\; 3 \cdot t^{\tilde{\Sigma}_3}_{A_3} \;+\; 2 \cdot t^{\tilde{\Sigma}_3}_{(12)(34)}, \end{aligned}$$

all such classes are linear combinations of the six classes

$$t_e^{(12)(34)}\ ,\quad t^{V_4}_{(12)(34)}\ ,\quad t^{\tilde{\Sigma}_3}_{A_3}\ ,\quad t^{D_5}_{C_5}\ ,\quad t^{A_4}_{V_4}, \qquad \text{and} \qquad 3 \cdot (t^{\tilde{\Sigma}_3}_{(12)(34)} + t^{A_4}_{A_3})\ .$$

Since these classes are linearly independent, they form a basis of the group $I_3(A_5)$.

The group $I_4(A_5)$ is generated by the classes t_K^H for all nested subgroups with $[H:K]\leq 4$. There is only one new generator, the class $t_{A_3}^{A_4}$; since $t_{(12)(34)}^{\tilde\Sigma_3}\in I_3(A_5)\subset I_4(A_5)$, the group $I_4(A_5)$ is freely generated by the first six elements of the basis (4.11). Because $3\cdot t_{A_3}^{A_4}\in I_3(A_5)$, the group $I_3(A_5)$ has index 3 in $I_4(A_5)$.

The group $I_5(A_5)$ is generated by the classes t_K^H for all nested subgroup pairs with $[H:K]\leq 5$. In particular, $I_5(A_5)$ contains the seventh element $t_{A_4}^{A_5}$ of the basis (4.11). We observe that

$$5\cdot t_{D_5}^{A_5} \;=\; t_{(12)(34)}^{V_4} \;+\; 2\cdot t_{V_4}^{A_4} \;+\; 6\cdot t_{A_4}^{A_5} \;-\; t_{(12)(34)}^{D_5} \;\in\; I_5(A_5)\ . \tag{4.13}$$

All classes of the form t_K^H with $[H:K]\leq 5$ are linear combinations of the first seven classes of the basis (4.11) and the class (4.13),

$$\begin{aligned} t_{(12)(34)}^{D_5} \;&=\; t_{(12)(34)}^{V_4} \;+\; 2\cdot t_{V_4}^{A_4} \;+\; 6\cdot t_{A_4}^{A_5} \;-\; 5\cdot t_{D_5}^{A_5},\\ t_e^{C_5} \;&=\; t_e^{(12)(34)} \;+\; 2\cdot t_{(12)(34)}^{V_4} \;-\; 5\cdot t_{C_5}^{D_5} \;+\; 4\cdot t_{V_4}^{A_4} \;+12\cdot t_{A_4}^{A_5} \;-\; 2\cdot(5\cdot t_{D_5}^{A_5})\ . \end{aligned}$$

The group $I_5(A_5)$ is thus generated by the linearly independent classes

$$t_e^{(12)(34)}\ ,\quad t_{(12)(34)}^{V_4}\ ,\quad t_{A_3}^{\tilde\Sigma_3}\ ,\quad t_{C_5}^{D_5}\ ,\quad t_{V_4}^{A_4}\ ,\quad t_{(12)(34)}^{\tilde\Sigma_3}+t_{A_3}^{A_4}\ ,\quad t_{A_4}^{A_5},\quad \text{and}\quad 5\cdot t_{D_5}^{A_5}\ .$$

So $I_5(A_5)$ has full rank 8, but index 5 in the augmentation ideal $I(A_5)$. Since $[A_5:D_5]=6$, the last basis element $t_{D_5}^{A_5}$ belongs to $I_6(A_5)$, and we conclude that $I_6(A_5)=I(A_5)$ is the full augmentation ideal. Altogether, Theorem 3.12 lets us conclude that

- the group $\pi_0^{A_5}(\mathbb{S})=\pi_0^{A_5}(Sp^1)$ is free of rank 9,
- the group $\pi_0^{A_5}(Sp^2)$ is free of rank 5,
- the group $\pi_0^{A_5}(Sp^3)$ has rank 3 and its torsion subgroup has order 3,
- the group $\pi_0^{A_5}(Sp^4)$ is free of rank 3,
- the group $\pi_0^{A_5}(Sp^5)$ has rank 1 and its torsion subgroup has order 5, and
- for all $n\geq 6$, the group $\pi_0^{A_5}(Sp^n)$ is free of rank 1.

Example 4.14 (Symmetric group Σ_5)**.** We refrain from a complete calculation of the groups $\pi_0^{\Sigma_5}(Sp^n)$, but we work out where the filtration for Σ_5 stabilizes. The previous examples could be mistaken as evidence that the group $I_n(\Sigma_n)$ coincides with the full augmentation ideal $I(\Sigma_n)$ for every n; equivalently, one could get the false impression that the group $\pi_0^{\Sigma_n}(Sp^n)$ is always free of rank 1. While this is true for $n\leq 4$, we will now see that it fails for $n=5$, i.e., that $I_5(\Sigma_5)$ is strictly smaller than $I(\Sigma_5)$.

We let B denote the subgroup of Σ_5 generated by the elements (12345) and (2354); this group has order 20 and is isomorphic to the semidirect product $\mathbb{F}_5\rtimes(\mathbb{F}_5)^\times$, the affine linear group of the field $\mathbb{F}_5$. The intersection of B with the alternating group A_5 is the dihedral group D_5. The double coset formula thus gives

$$\mathrm{res}_{A_5}^{\Sigma_5}(t_B^{\Sigma_5}) \;=\; 6 \;-\; \mathrm{res}_{A_5}^{\Sigma_5}(\mathrm{tr}_B^{\Sigma_5}(1)) \;=\; 6 \;-\; \mathrm{tr}_{D_5}^{A_5}(\mathrm{res}_{D_5}^{B}(1)) \;=\; t_{D_5}^{A_5}\ .$$

We showed in the previous Example 4.10 that the class $t_{D_5}^{A_5}$ does *not* belong to $I_5(A_5)$. Since I_5 is closed under restriction maps, the class $t_B^{\Sigma_5}$ does *not* belong to $I_5(\Sigma_5)$, and hence $I_5(\Sigma_5)\neq I(\Sigma_5)$.

Every subgroup H of Σ_5 admits a nested sequence of subgroups

$$H \;=\; H_0 \;\subset\; H_1 \;\subset\; \cdots \;\subset\; H_k$$

with $[H_i : H_{i-1}] \leq 6$ for all $i = 1, \dots, k$ and such that the last group H_k is either the full group Σ_5 or conjugate to the maximal subgroup $\Sigma_3 \times \Sigma_2$ of index 10. The relation

$$t^{\Sigma_5}_{\Sigma_3\times\Sigma_2} \;=\; t^{\Sigma_3\times\Sigma_2}_{\Sigma_3} \;-\; t^{\Sigma_3\times\Sigma_2}_{\Sigma_2\times\Sigma_2} \;-\; t^{\Sigma_4}_{\Sigma_3} \;+\; t^{\Sigma_2\wr\Sigma_2}_{\Sigma_2\times\Sigma_2} \;+\; 2\cdot t^{\Sigma_4}_{\Sigma_2\wr\Sigma_2} \;+\; 2\cdot t^{\Sigma_5}_{\Sigma_4}$$

shows that the class $t^{\Sigma_5}_{\Sigma_3\times\Sigma_2}$ lies in $I_5(\Sigma_5)$. So $t^{\Sigma_n}_H$ belongs to $I_6(\Sigma_5)$ for every subgroup H of Σ_n, and hence $I_6(\Sigma_5) = I(\Sigma_5)$.

We pause to point out a curious phenomenon that happens only for $n = 4$. This exceptional behavior can be traced back to the fact that the alternating group A_4 has a subgroup of "unusually small index" (the Klein group V_4 of index 3); compare the proof of the next proposition. We have seen in (4.8) that $3 \cdot t_4$ lies in $I_3(\Sigma_4)$. Since the class t_4 generates the global functor I_4, this implies that

$$3 \cdot I_4 \;\subset\; I_3 \;\subset\; I_4\ .$$

So after inverting 3, the inclusion $I_3 \longrightarrow I_4$ and the epimorphism of global functors

$$\underline{\pi}_0(Sp^3) \;\longrightarrow\; \underline{\pi}_0(Sp^4)$$

induced by the inclusion $Sp^3 \longrightarrow Sp^4$ both become isomorphisms. However:

Proposition 4.15. *For every $n \geq 2$ with $n \neq 4$, the inclusion $I_{n-1} \longrightarrow I_n$ is not a rational isomorphism.*

Proof. Example 4.4 shows that no non-zero multiple of the class t_2 belongs to $I_1(\Sigma_2)$. Example 4.5 shows that no non-zero multiple of t_3 belongs to $I_2(\Sigma_3)$. So we assume $n \geq 5$ for the rest of the argument. We recall that the alternating group A_n has no proper subgroup H of index less than n. Indeed, the left translation action on A_n/H provides a homomorphism $\rho : A_n \longrightarrow \Sigma(A_n/H)$ to the symmetric group of the underlying set of A_n/H. For $[A_n : H] < n$, the order of $\Sigma(A_n/H)$ is strictly less than the order of A_n. So the homomorphism ρ has a non-trivial kernel. Since the group A_n is simple, ρ must be trivial, which forces $H = A_n$.

Now we prove the proposition. The class $t^{A_n}_{A_{n-1}} = \mathrm{res}^{\Sigma_n}_{A_n}(t_n)$ belongs to $I_n(A_n)$, but for $n > 4$ no non-zero multiple of it belongs to $I_{n-1}(A_n)$. Indeed, otherwise Proposition 4.1 would allow us to write

$$k \cdot t^{A_n}_{A_{n-1}} \;=\; \lambda_1 \cdot t^{H_1}_{K_1} + \cdots + \lambda_m \cdot t^{H_m}_{K_m}$$

in $\mathbb{A}(A_n)$, for certain integers $k, \lambda_1, \dots, \lambda_m$ and nested subgroup pairs with $1 < [H_i : K_i] < n$. Since A_n has no proper subgroup of index less than n, the groups $H_1, \dots, H_m$ must all be different from the full group A_n. We expand both sides in terms of the basis of $\mathbb{A}(A_n)$ given by the classes $\mathrm{tr}^{A_n}_H(1)$ (for H running through the conjugacy classes of subgroups). On the right-hand side the coefficient of the basis element $\mathrm{tr}^{A_n}_{A_n}(1) = 1$ is zero, whereas the coefficient on the left-hand side is kn. So we must have $k = 0$. □

We conclude by looking more closely at the limit case, the infinite symmetric product spectrum. We remark without proof that, generalizing the non-equivariant situation, the orthogonal spectrum Sp^∞ is globally equivalent to the orthogonal spectrum $H\mathbb{Z}$ defined by

$$(H\mathbb{Z})(V) \;=\; \mathbb{Z}[S^V]\ ,$$

the reduced free abelian group generated by the n-sphere. Theorem 3.12 shows that

$$\mathbb{A}/I_\infty \;\cong\; \underline{\pi}_0(Sp^\infty)\;,$$

induced by the action of $\mathbb{A}$ on the class $i_*(1)$. For every compact Lie group G, the map

$$\mathrm{res}^G_e \;:\; \pi_0^G(Sp^\infty) \;\longrightarrow\; \pi_0^e(Sp^\infty) \cong \mathbb{Z}$$

is a split epimorphism, so the group $\pi_0^G(Sp^\infty)$ is free of rank 1 if and only if $I_\infty(G) = I(G)$.

We can split the group $\mathbb{A}(G)/I_\infty(G)$, and hence the group $\pi_0^G(Sp^\infty)$, into summands indexed by conjugacy classes of connected subgroups of G. If C is such a connected subgroup, we denote by $\mathbb{A}(G;C)$ the subgroup of the Burnside ring $\mathbb{A}(G)$ that is generated by the transfers $\mathrm{tr}_H^G(1)$ for all subgroups H with $C = H^\circ$, the path component of the identity of H (or equivalently, H contains C as a finite index subgroup). Then

$$\mathbb{A}(G) \;=\; \bigoplus_{(C)} \mathbb{A}(G;C),$$

where the sum runs over conjugacy classes of connected subgroups of G. Proposition 4.1 shows that $I_\infty(G)$ is generated as an abelian group by the classes

$$t_K^H \;=\; [H:K]\cdot \mathrm{tr}_H^G(1) - \mathrm{tr}_K^G(1) \;\in\; \mathbb{A}(G)$$

as (H,K) runs through all pairs of nested closed subgroups such that K has a finite index in H, and H has the finite Weyl group in G. Then K and H have the same connected component of the identity, i.e., $K^\circ = H^\circ$, so the relation t_K^H belongs to the direct summand $\mathbb{A}(G;K^\circ)$. Hence

$$\pi_0^G(Sp^\infty) \;\cong\; \mathbb{A}(G)/I_\infty(G) \;=\; \bigoplus_{(C)} \left(\mathbb{A}(G;C)/I_\infty(G;C)\right)\;,$$

where $I_\infty(G;C)$ is the subgroup of $\mathbb{A}(G;C)$ generated by the classes t_K^H with $H^\circ = K^\circ = C$. The summands behave quite differently according to whether C has an infinite or finite Weyl group:

- If C has an infinite Weyl group, then for every subgroup $H \le G$ with $H^\circ = C$ the class $[H:C]\cdot\mathrm{tr}_H^G(1)$ belongs to $I_\infty(G;C)$. So the class $\mathrm{tr}_H^G(1)$ becomes torsion in the quotient group $\mathbb{A}(G;C)/I_\infty(G;C)$, which is thus a torsion group.
- If C has the finite Weyl group, and $H \le G$ satisfies $H^\circ = C$, then the relations
$$C = H^\circ \;\le\; H \;\le\; N_G H \;\le\; N_G C$$
show that H has the finite Weyl group and finite index in $N_G C$. So
$$t_H^{N_G C} \;=\; [N_G C : H]\cdot \mathrm{tr}_{N_G C}^G(1) \;-\; \mathrm{tr}_H^G(1) \;\in\; I_\infty(G;C)$$
and in the quotient group $\mathbb{A}(G;C)/I_\infty(G;C)$, the class $\mathrm{tr}_H^G(1)$ becomes a multiple of the class $\mathrm{tr}_{N_G C}^G(1)$. Hence the group $\mathbb{A}(G;C)/I_\infty(G;C)$ is free of rank 1, generated by $\mathrm{tr}_{N_G C}^G(1)$.

  In the situation at hand, the subgroup C can be recovered as the identity component of its normalizer. A compact Lie group has only finitely many conjugacy classes of subgroups that are normalizers of connected subgroups, see [5, VII Lemma 3.2]. So there are only finitely many conjugacy classes of connected subgroups with finite Weyl group.

So altogether we conclude that the group $\pi_0^G(Sp^\infty)$ is a direct sum of a torsion group and a free abelian group of finite rank. In particular, the rationalization $\mathbb{Q}\otimes\pi_0^G(Sp^\infty)$ is a finite dimensional $\mathbb{Q}$-vector space with the basis consisting of the classes $\mathrm{tr}_C^G(1)$ as C runs through the conjugacy classes of connected subgroups of G with the finite Weyl group. Unfortunately, the author does not know an example when the torsion subgroup of $\pi_0^G(Sp^\infty)$ is non-trivial.

Example 4.16. If every subgroup H with the finite Weyl group also has a finite index in G, then $I_\infty(G) = I(G)$ and $\pi_0^G(Sp^\infty)$ is free of rank 1. This holds, for example, when G is finite or a torus.

An example for which $\pi_0^G(Sp^\infty)$ has rank bigger than 1 is $G = SU(2)$. Here there are three conjugacy classes of connected subgroups: the trivial subgroup, the conjugacy class of the maximal tori, and the full group $SU(2)$. Among these, the maximal tori and $SU(2)$ have finite Weyl groups, so the classes 1 and $\mathrm{tr}_N^{SU(2)}(1)$ are a $\mathbb{Z}$-basis for $\pi_0^{SU(2)}(Sp^\infty)$ modulo torsion, where N is a maximal torus normalizer.

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