# An untitled book project about symmetric spectra

This document is a preliminary and incomplete version of what may one day become a book about symmetric spectra. It probably contains an embarrassing number of typos and inconsistencies, and hopefully not too many actual mistakes. I intend to post updates regularly, so you may want to check my homepage for newer versions. I am interested in feedback.

Stefan Schwede
Mathematisches Institut
Universität Bonn, Germany
schwede@math.uni-bonn.de
www.math.uni-bonn.de/people/schwede

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# Contents

## Introduction

This textbook is an introduction to the modern foundations of stable homotopy theory and 'algebra' over structured ring spectra, based on symmetric spectra. We begin with a quick historical review and attempt at motivation.

A crucial prerequisite for spectral algebra is an associative and commutative smash product on a good point-set level category of spectra, which lifts the well-known smash product pairing on the *homotopy category*. The first construction of what is now called 'the stable homotopy category', including its symmetric monoidal smash product, is due to Boardman [**4**] (unpublished); accounts of Boardman's construction appear in [**65**], [**62**] and [**2**, Part III] (Adams has to devote more than 30 pages to the construction and formal properties of the smash product).

To illustrate the drastic simplification that occurred in the foundations in the mid-90s, let us draw an analogy with the algebraic context. Let $R$ be a commutative ring and imagine for a moment that the notion of a chain complex (of $R$-modules) has not been discovered, but nevertheless various complicated constructions of the unbounded derived category $\mathcal{D}(R)$ of the ring $R$ exist. Moreover, constructions of the *derived* tensor product on the *derived* category exist, but they are complicated and the proof that the derived tensor product is associative and commutative occupies 30 pages. In this situation, you could talk about objects $A$ in the derived category together with morphisms $A \otimes_R^L A \longrightarrow A$, in the derived category, which are associative and unital, and possibly commutative, again in the derived category. This notion may be useful for some purposes, but it suffers from many defects – as one example, the category of modules (under derived tensor product in the derived category), does not in general form a triangulated category.

Now imagine that someone proposes the definition of a chain complex of $R$-modules and shows that by formally inverting the quasi-isomorphisms, one can construct the derived category. She also defines the tensor product of chain complexes and proves that tensoring with suitably nice (i.e., *homotopically projective*) complexes preserves quasi-isomorphisms. It immediately follows that the tensor product descends to an associative and commutative product on the derived category. What is even better, now one can suddenly consider differential graded algebras, a 'rigidified' version of the crude multiplication 'up-to-chain homotopy'. We would quickly discover that this notion is much more powerful and that differential graded algebras arise all over the place (while chain complexes with a multiplication which is merely associative up to chain homotopy seldom come up in nature).

Fortunately, this is not the historical course of development in homological algebra, but the development in stable homotopy theory was, in several aspects, as indicated above. In the stable homotopy category people could consider ring spectra 'up to homotopy', which are closely related to multiplicative cohomology theories. However, the need and usefulness of ring spectra with rigidified multiplications soon became apparent, and topologists developed different ways of dealing with them. One line of approach uses operads for the bookkeeping of the homotopies which encode all higher forms of associativity and commutativity, and this led to the notions of $A_\infty$- respectively $E_\infty$-ring spectra. Various notions of point-set level ring spectra had been used (which were only later recognized as the monoids in a symmetric monoidal model category). For example, the orthogonal ring spectra had appeared as *$\mathscr{I}_*$-prefunctors* in [**42**], the *functors with smash product* were introduced in [**6**] and symmetric ring spectra appeared as *strictly associative ring spectra* in [**22**, Def. 6.1] or as *FSPs defined on spheres* in [**23**, 2.7].

At this point it had become clear that many technicalities could be avoided if one had a smash product on a good point-set category of spectra which was associative and unital *before* passage to the homotopy category. For a long time no such category was known, and there was even evidence that it might not exist [**32**]. In retrospect, the modern spectra categories could maybe have been found earlier if Quillen's formalism of *model categories* [**46**] had been taken more seriously; from the model category perspective, one should not expect an intrinsically 'left adjoint' construction like a smash product to have a good homotopical behavior in general, and along with the search for a smash product, one should look for a compatible notion of cofibrations.

In the mid-90s, several categories of spectra with nice smash products were discovered, and simultaneously, model categories experienced a major renaissance. Around 1993, Elmendorf, Kriz, Mandell and May introduced the *$S$-modules* [**19**] and Jeff Smith gave the first talks about *symmetric spectra*; the details of the model structure were later worked out and written up by Hovey, Shipley and Smith [**25**]. In 1995, Lydakis [**34**] independently discovered and studied the smash product for $\Gamma$-spaces (in the sense of Segal [**55**]), and a little later he developed model structures and smash product for *simplicial functors* [**35**]. Except for the *$S$-modules* of Elmendorf, Kriz, Mandell and May, all other known models for spectra with nice smash product have a very similar flavor; they all arise as categories of continuous (or simplicial), space-valued functors from a symmetric monoidal indexing category, and the smash product is a convolution product (defined as a left Kan extension), which had much earlier been studied by the category theorist Day [**15**]. This unifying context was made explicit by Mandell, May, Schwede and Shipley in [**39**], where another example, the *orthogonal spectra* were first worked out in detail. The different approaches to spectra categories with smash product have been generalized and adapted to equivariant homotopy theory [**16, 37, 38**] and motivic homotopy theory [**17, 26, 27**].

**Why symmetric spectra?** The author is a big fan of symmetric spectra; two important reasons are that symmetric spectra are easy to define and require the least amount of symmetry among the models of the stable homotopy category with smash product. A consequence of the second point is that many interesting homotopy types can be written down explicitly and in closed form. We give examples of this in Section I.2, right after the basic definitions, among these are the sphere spectrum, suspension spectra, Eilenberg-Mac Lane spectra, Thom spectra such as $MO, MSO$ and $MU$, topological $K$-theory and algebraic $K$-theory spectra.

Another consequence of the second point is that whenever someone writes down or constructs a model for a homotopy type in one of the other worlds of spectra, then we immediately get a model as a symmetric spectrum by applying one of the 'forgetful' functors from spectra with more symmetries which we recall in Section I.2.3. In fact, symmetric spectra have a certain universal property (see Shipley's paper [**57**]), making them 'initial' among stable model categories with a compatible smash product.

There are already good sources available which explain the stable homotopy category, and there are many research papers and at least one book devoted to structured ring spectra. However, my experience is that for students learning the subject it is hard to reconcile the treatment of the stable homotopy category as given, for example, in Adams' notes [**2**], with the more recent model category approaches to, say, $S$-modules or symmetric spectra. So one aim of this book is to provide a source where one can learn about the triangulated stable homotopy category and about stable model categories and a good point-set level smash product with just one notion of what a spectrum is.

The monograph [**19**] by Elmendorf, Kriz, Mandell and May develops the theory of one of the competing frameworks, the $S$-modules, in detail. It has had a big impact and is widely used, for example because many standard tools can simply be quoted from that book. The theory of symmetric spectra is by now highly developed, but the results are spread over many research papers. The aim of this book is to collect basic facts in one place, thus providing an alternative to [**19**].

**Prerequisites.** As a general principle, I assume knowledge of basic algebraic topology and unstable homotopy theory. I will develop in parallel the theory of symmetric spectra based on topological spaces (compactly generated and weak Hausdorff) and simplicial sets. Whenever simplicial sets are used, I assume basic knowledge of simplicial homotopy theory, as found for example in [**21**] or [**41**]. However, the use of simplicial sets is often convenient but hardly ever essential, so not much understanding is lost by just thinking about topological spaces throughout.

On the other hand, no prior knowledge of *stable* homotopy theory is assumed. In particular, we define the stable homotopy category using symmetric spectra and develop its basic properties from scratch.

From Chapter III on I will freely use the language of Quillen's model categories and basic results of homotopical algebra. The original source is [**46**], a good introduction is [**18**], and [**24**] is a thorough, extensive treatment.

**Organization.** We organize the book into chapters, each chapter into sections and some sections into subsections. The numbering scheme for referring to definitions, theorems, examples etc. is as follows. If we refer to something in the same chapter, then the reference number consists only of the arabic section number and then a running number for all kind of environments. If the reference is to another chapter, then we add the roman number of the chapter in front. So 'Lemma 3.14' refers to a Lemma in Section 3 of the same chapter, with running number 14, while 'Example I.2.21' is an example from the second section of the first chapter, with running number 21.

In the first chapter we introduce the basic concepts of a symmetric spectrum and symmetric ring spectrum and then, before developing any extensive theory, discuss lots of examples. There is a section on the smash product where we concentrate on its formal properties, leaving the homotopical properties of the smash product to a later chapter. One of the few points where symmetric spectra are more complicated than other frameworks is that the usual homotopy groups can be somewhat pathological. So we spend the last section of the first chapter on the structure of homotopy groups and the notion of semistable symmetric spectra.

The second chapter is devoted to the stable homotopy category. We define it as the homotopy category of injective Ω-spectra, based on simplicial sets. Model categories will not be discussed explicitly until Chapter III, but the justification behind this definition is that there is a stable model structure (the *injective stable* model structure) in which every symmetric spectrum is cofibrant and where the fibrant objects are precisely the injective Ω-spectra. We develop some basic theory around the stable homotopy category, such as the triangulated structure, derived smash product, homotopy (co-)limits, Postnikov sections, localization and completion, and discuss the relationship to the Spanier-Whitehead category and generalized cohomology theories.

In Chapter III model structures enter the scene. We start by establishing the various level model structures (projective, flat, injective, and their positive versions) for symmetric spectra, and then discuss the associated, more important, stable model structures. We also develop the model structures for modules over a fixed symmetric ring spectrum and for algebras over an operad of simplicial sets. The latter includes the stable model structures for symmetric ring spectra and for commutative symmetric ring spectra.

Each chapter has a section containing exercises.

As a general rule, I do not attribute credit for definitions and theorems in the body of the text. Instead, there is a section 'History and credits' at the end of each chapter, where I summarize, to the best of my knowledge, who contributed what. Additions and corrections are welcome.

**Two philosophical points.** (i) We make isomorphisms explicit. Thus we avoid phrases like 'the canonical isomorphism' unless the isomorphism we have in mind has previously been defined. The main reason for this is the author's experience that what seems canonical to the expert may often not be clear to a newcomer. Another reason is that here and there, one can get sign trouble if one is not careful about choices of isomorphisms. A disadvantage is that we have to introduce lots of symbols or numbers to refer to the isomorphisms.

(ii) We sometimes give two (and occasionally even three) proofs or constructions when they are sufficiently different and shed light on the theorem or notion under consideration. While this is logically redundant, we think that to understand a mathematical phenomenon it is good to see it from as many different angles as possible.

**Some conventions.** Let us fix some terminology and enact several useful conventions. We think that some slight abuse of language and notation can often make statements more transparent, but when we allow ourselves such imprecision we feel obliged to state them clearly here, at the risk of being picky.

We denote by $\mathcal{T}$ the category of pointed, compactly generated, weak Hausdorff topological spaces. A *map* between topological spaces always refers to a *continuous* map, unless explicitly stated otherwise. Similarly, an *action* of a group on a space refers to a *continuous action*.

It will be convenient to define the $n$-sphere $S^n$ as the one-point compactification of $n$-dimensional euclidian space $\mathbb{R}^n$, with the point at infinity as the basepoint.

For $n \geq 0$, the symmetric group $\Sigma_n$ is the group of bijections of the set $\{1, 2, \ldots, n\}$; in particular, $\Sigma_0$ consists only of the identity of the empty set. It will often be convenient to identify the product group $\Sigma_n \times \Sigma_m$ with the subgroup of $\Sigma_{n+m}$ of those permutations which take the sets $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$ to themselves. Whenever we do so, we implicitly use the monomorphism

$$\Sigma_n \times \Sigma_m \longrightarrow \Sigma_{n+m} , \quad (\tau, \kappa) \mapsto \tau \times \kappa$$

given by

$$(\tau \times \kappa)(i) \;=\; \begin{cases} \tau(i) & \text{for } 1 \le i \le n, \\ \kappa(i-n)+n & \text{for } n+1 \le i \le n+m. \end{cases}$$

We let the symmetric group $\Sigma_n$ act from the left on $\mathbb{R}^n$ by permuting the coordinates, i.e., $\gamma(x_1,\dots,x_n) = (x_{\gamma^{-1}(1)},\dots,x_{\gamma^{-1}(n)})$. This action compactifies to an action on $S^n$ which fixes the basepoint. The canonical linear isomorphism

$$\mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^{n+m} \;, \quad ((x_1,\dots,x_n),(y_1,\dots,x_m)) \;\mapsto\; (x_1,\dots,x_n,y_1,\dots,x_m)$$

induces a homeomorphism $S^n \wedge S^m \longrightarrow S^{n+m}$ which is equivariant with respect to the action of the group $\Sigma_n \times \Sigma_m$, acting on the target by restriction from $\Sigma_{n+m}$.

The topological spaces we consider are usually pointed, and we use the notation $\pi_n X$ for the $n$-th homotopy group with respect to the distinguished basepoint, which we do not record in the notation.

**Limits and colimits.** Limits and colimits in a category are hardly ever unique, but the universal property which they enjoy makes then 'unique up to canonical isomorphism'. We want to fix our language for talking about this unambiguously. We recall that a *colimit* of a functor $F : I \longrightarrow \mathcal{C}$ is a pair $(\bar{F},\varphi)$ consisting of an object $\bar{F}$ of $\mathcal{C}$ and a natural transformation $\varphi : F \longrightarrow c\bar{F}$ from $F$ to the constant functor with value $\bar{F}$ which is initial among all natural transformations from $F$ to constant functors. We often follow the standard abuse of language and call the object $\bar{F}$ a colimit, or even *the* colimit, of the functor $F$ and denote it by $\operatorname{colim}_I F$. When we need to refer to the natural transformation $\varphi$ which is part of the data of a colimit, we refer to the component $\varphi_i : F(i) \longrightarrow \operatorname{colim}_I F$ at an object $i \in I$ as the *canonical morphism* from the object $F(i)$ to the colimit. Dually for limits.

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CHAPTER I

# Basic definitions and examples

## 1. Symmetric spectra

DEFINITION 1.1. A *symmetric spectrum* consists of the following data:

- a sequence of pointed spaces $X_n$ for $n \geq 0$
- a basepoint preserving continuous left action of the symmetric group $\Sigma_n$ on $X_n$ for each $n \geq 0$
- based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the composite

$$(1.2) \qquad X_n \wedge S^m \xrightarrow{\sigma_n \wedge \mathrm{Id}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge \mathrm{Id}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge \mathrm{Id}} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant. We often denote this composite map by $\sigma^m$. Here the symmetric group $\Sigma_m$ acts by permuting the coordinates of $S^m$, and $\Sigma_n \times \Sigma_m$ acts on the target by restriction of the $\Sigma_{n+m}$-action. We refer to the space $X_n$ as the *nth level* of the symmetric spectrum $X$.

A *morphism* $f : X \longrightarrow Y$ of symmetric spectra consists of $\Sigma_n$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge \mathrm{Id}_{S^1})$ for all $n \geq 0$. The category of symmetric spectra is denoted by $\mathcal{S}p^\Sigma$.

DEFINITION 1.3. A *symmetric ring spectrum* $R$ consists of the following data:

- a sequence of pointed spaces $R_n$ for $n \geq 0$
- a basepoint preserving continuous left action of the symmetric group $\Sigma_n$ on $R_n$ for each $n \geq 0$
- $\Sigma_n \times \Sigma_m$-equivariant *multiplication maps*
$$\mu_{n,m} \; : \; R_n \wedge R_m \longrightarrow R_{n+m}$$
for $n, m \geq 0$, and
- two *unit maps*
$$\iota_0 \; : \; S^0 \longrightarrow R_0 \quad \text{and} \quad \iota_1 \; : \; S^1 \longrightarrow R_1 \; .$$

This data is subject to the following conditions:

(Associativity) The square

$$\begin{tikzcd} R_n \wedge R_m \wedge R_p \arrow[r, "\mathrm{Id} \wedge \mu_{m,p}"] \arrow[d, "\mu_{n,m} \wedge \mathrm{Id}"'] & R_n \wedge R_{m+p} \arrow[d, "\mu_{n,m+p}"] \\ R_{n+m} \wedge R_p \arrow[r, "\mu_{n+m,p}"'] & R_{n+m+p} \end{tikzcd}$$

commutes for all $n, m, p \geq 0$.

(Unit) The two composites

$$R_n \cong R_n \wedge S^0 \xrightarrow{\mathrm{Id} \wedge \iota_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n$$

$$R_n \cong S^0 \wedge R_n \xrightarrow{\iota_0 \wedge \mathrm{Id}} R_0 \wedge R_n \xrightarrow{\mu_{0,n}} R_n$$

are the identity for all $n \geq 0$.

(Centrality) The diagram

$$\begin{tikzcd} R_n \wedge S^1 \arrow[r, "\mathrm{Id} \wedge \iota_1"] \arrow[d, "\tau"'] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\chi_{n,1}"] \\ S^1 \wedge R_n \arrow[r, "\iota_1 \wedge \mathrm{Id}"'] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"'] & R_{1+n} \end{tikzcd}$$

commutes for all $n \geq 0$. Here $\chi_{n,m} \in \Sigma_{n+m}$ denotes the shuffle permutation which moves the first $n$ elements past the last $m$ elements, keeping each of the two blocks in order; in formulas,

$$\chi_{n,m}(i) \;=\; \begin{cases} i+m & \text{for } 1 \leq i \leq n, \\ i-n & \text{for } n+1 \leq i \leq n+m. \end{cases}$$

A symmetric ring spectrum $R$ is *commutative* if the square

$$\begin{tikzcd} R_n \wedge R_m \arrow[r, "\text{twist}"] \arrow[d, "\mu_{n,m}"'] & R_m \wedge R_n \arrow[d, "\mu_{m,n}"] \\ R_{n+m} \arrow[r, "\chi_{n,m}"'] & R_{m+n} \end{tikzcd}$$

commutes for all $n, m \geq 0$.

A *morphism* $f : R \longrightarrow S$ of symmetric ring spectra consists of $\Sigma_n$-equivariant based maps $f_n : R_n \longrightarrow S_n$ for $n \geq 0$, which are compatible with the multiplication and unit maps in the sense that $f_{n+m} \circ \mu_{n,m} = \mu_{n,m} \circ (f_n \wedge f_m)$ for all $n, m \geq 0$, and $f_0 \circ \iota_0 = \iota_0$ and $f_1 \circ \iota_1 = \iota_1$.

Definition 1.4. A *right module* $M$ over a symmetric ring spectrum $R$ consists of the following data:

- a sequence of pointed spaces $M_n$ for $n \geq 0$
- a basepoint preserving continuous left action of the symmetric group $\Sigma_n$ on $M_n$ for each $n \geq 0$, and
- $\Sigma_n \times \Sigma_m$-equivariant *action maps* $\alpha_{n,m} : M_n \wedge R_m \longrightarrow M_{n+m}$ for $n, m \geq 0$.

The action maps have to be associative and unital in the sense that the following diagrams commute

$$\begin{tikzcd} M_n \wedge R_m \wedge R_p \arrow[r, "\mathrm{Id} \wedge \mu_{m,p}"] \arrow[d, "\alpha_{n,m} \wedge \mathrm{Id}"'] & M_n \wedge R_{m+p} \arrow[d, "\alpha_{n,m+p}"] \\ M_{n+m} \wedge R_p \arrow[r, "\alpha_{n+m,p}"'] & M_{n+m+p} \end{tikzcd} \qquad \begin{tikzcd} M_n \cong M_n \wedge S^0 \arrow[r, "\mathrm{Id} \wedge \iota_0"] \arrow[dr, equal] & M_n \wedge R_0 \arrow[d, "\alpha_{n,0}"] \\ & M_n \end{tikzcd}$$

for all $n, m, p \geq 0$. A *morphism* $f : M \longrightarrow N$ of right $R$-modules consists of $\Sigma_n$-equivariant based maps $f_n : M_n \longrightarrow N_n$ for $n \geq 0$, which are compatible with the action maps in the sense that $f_{n+m} \circ \alpha_{n,m} = \alpha_{n,m} \circ (f_n \wedge \mathrm{Id})$ for all $n, m \geq 0$. We denote the category of right $R$-modules by mod- $R$.

The *k-th homotopy group* of a symmetric spectrum $X$ is defined as the colimit

$$\pi_k X \;=\; \mathrm{colim}_n \, \pi_{k+n} X_n$$

taken over the maps

$$\pi_{k+n} X_n \xrightarrow{-\wedge S^1} \pi_{k+n+1} \left(X_n \wedge S^1\right) \xrightarrow{(\sigma_n)_*} \pi_{k+n+1} X_{n+1} \;. \tag{1.5}$$

Homotopy groups of symmetric spectra are abelian groups, and for symmetric ring spectra they often form graded rings. More precisely, the underlying symmetric spectrum of $R$ has to be 'semistable' (see Theorem 4.44) for $\pi_* R$ to form a graded ring, see Theorem 4.54. In general the homotopy groups support a more sophisticated algebraic structure, namely an algebra over the 'injection operad'.

REMARK 1.6. We have stated the axioms for symmetric ring spectra in terms of a minimal amount of data and conditions. Now we put these conditions into perspective. We consider a symmetric ring spectrum $R$.

(i) It will be useful to have the following notation for *iterated multiplication maps*. For natural numbers $n_1, \dots, n_i \geq 0$ we denote by

$$\mu_{n_1,\dots,n_i} \ : \ R_{n_1} \wedge \dots \wedge R_{n_i} \ \longrightarrow \ R_{n_1+\dots+n_i}$$

the map obtained by composing multiplication maps smashed with suitable identity maps; by associativity, the parentheses in the multiplications don't matter. More formally we can define the iterated multiplication maps inductively, setting

$$\mu_{n_1,\dots,n_i} \ = \ \mu_{n_1,n_2+\dots+n_i} \circ (\mathrm{Id}_{R_{n_1}} \wedge \mu_{n_2,\dots,n_i}) \ .$$

(ii) We can define higher-dimensional unit maps $\iota_m : S^m \longrightarrow R_m$ for $m \geq 2$ as the composite

$$S^m = S^1 \wedge \dots \wedge S^1 \xrightarrow{\iota_1 \wedge \dots \wedge \iota_1} R_1 \wedge \dots \wedge R_1 \xrightarrow{\mu_{1,\dots,1}} R_m \ .$$

Centrality then implies that $\iota_m$ is $\Sigma_m$-equivariant, and it implies that the diagram

$$\begin{tikzcd} R_n \wedge S^m \arrow[r, "\mathrm{Id} \wedge \iota_m"] \arrow[d, "\mathrm{twist}"'] & R_n \wedge R_m \arrow[r, "\mu_{n,m}"] & R_{n+m} \arrow[d, "\chi_{n,m}"] \\ S^m \wedge R_n \arrow[r, "\iota_m \wedge \mathrm{Id}"'] & R_m \wedge R_n \arrow[r, "\mu_{m,n}"'] & R_{m+n} \end{tikzcd}$$

commutes for all $n, m \geq 0$, generalizing the original centrality condition.

(iii) As the terminology suggests, the symmetric ring spectrum $R$ has an underlying symmetric spectrum. In fact, the multiplication maps $\mu_{n,m}$ make $R$ into a right module over itself, and more generally, every right $R$-module $M$ has an underlying symmetric spectrum as follows. We keep the spaces $M_n$ and symmetric group actions and define the missing structure maps $\sigma_n : M_n \wedge S^1 \longrightarrow M_{n+1}$ as the composite $\alpha_{n,1} \circ (\mathrm{Id}_{M_n} \wedge \iota_1)$. Associativity implies that the iterated structure map $\sigma^m : M_n \wedge S^m \longrightarrow M_{n+m}$ equals the composite

$$M_n \wedge S^m \ \xrightarrow{\mathrm{Id} \wedge \iota_m} \ M_n \wedge R_m \ \xrightarrow{\alpha_{n,m}} \ M_{n+m} \ .$$

So the iterated structure map is $\Sigma_n \times \Sigma_m$-equivariant by part (ii) and the equivariance hypothesis on $\alpha_{n,m}$, and we have in fact obtained a symmetric spectrum.

The forgetful functors which associates to a symmetric ring spectrum or module spectrum its underlying symmetric spectrum have left adjoints. We will construct the left adjoints in Example 3.10 below after introducing the smash product of symmetric spectra. The left adjoints associate to a symmetric spectrum $X$ the 'free $R$-module' $X \wedge R$ respectively the 'free symmetric ring spectrum' $TX$ generated by $X$, which we will refer to it as the *tensor algebra*.

(iv) If the symmetric ring spectrum $R$ is commutative, then centrality is automatic.

(v) Using the internal smash product of symmetric spectra introduced in Section 3, we can identify the 'explicit' definition of a symmetric ring spectrum which we just gave with a more 'implicit' definition of a symmetric spectrum $R$ together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$ (where $\mathbb{S}$ is the sphere spectrum, see Example 2.1) which are suitably associative and unital. The 'explicit' and 'implicit' definitions of symmetric ring spectra coincide in the sense that they define isomorphic categories, see Theorem 3.8.

We will often use a variation on the notions of symmetric spectrum and symmetric ring spectrum where topological spaces are replaced by simplicial sets. We can go back and forth between the two concepts using the adjoint functors of geometric realization and singular complex, as we explain below.

DEFINITION 1.7 (Symmetric spectra of simplicial sets). A *symmetric spectrum of simplicial sets* consists of the following data:

- a sequence of pointed simplicial sets $X_n$ for $n \geq 0$
- a basepoint preserving simplicial left action of the symmetric group $\Sigma_n$ on $X_n$ for each $n \geq 0$
- pointed morphisms $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$,

such that for all $n, m \geq 0$, the composite

$$X_n \wedge S^m \xrightarrow{\sigma_n \wedge \mathrm{Id}} X_{n+1} \wedge S^{m-1} \xrightarrow{\sigma_{n+1} \wedge \mathrm{Id}} \cdots \xrightarrow{\sigma_{n+m-2} \wedge \mathrm{Id}} X_{n+m-1} \wedge S^1 \xrightarrow{\sigma_{n+m-1}} X_{n+m}$$

is $\Sigma_n \times \Sigma_m$-equivariant. Here $S^1$ denotes the 'small simplicial circle' $S^1 = \Delta[1]/\partial\Delta[1]$ and $S^m = S^1 \wedge \ldots \wedge S^1$ is the $m$th smash power, with $\Sigma_m$ permuting the factors.

We similarly define a *symmetric ring spectrum of simplicial sets* by replacing 'space' by 'simplicial set' in Definition 1.3, while also replacing the topological circle $S^1$ by the simplicial circle $S^1 = \Delta[1]/\partial\Delta[1]$ and replacing $S^m$ by the $m$-fold smash power $S^m = S^1 \wedge \ldots \wedge S^1$.

As we already mentioned we can apply the adjoint functors 'geometric realization', denoted $|-|$, and 'singular complex', denoted $\mathcal{S}$, levelwise to go back and forth between topological and simplicial symmetric spectra. We use that geometric realization is a 'strong symmetric monoidal' functor, i.e., there is natural, unital, associative and commutative homeomorphism

$$r_{A,B} \ : \ |A| \wedge |B| \ \cong \ |A \wedge B| \tag{1.8}$$

for pointed simplicial sets $A$ and $B$. Indeed, the canonical continuous map $|A \times B| \longrightarrow |A| \times |B|$ is a homeomorphism (since we work in the category of compactly generated topological spaces) and the homeomorphism $r_{A,B}$ is gotten from there by passing to quotients.

We already allowed ourselves the freedom to use the same symbols for the topological and simplicial spheres. The justification is that the geometric realization of the simplicial $S^m$ is homeomorphic to the topological $S^m$. To be completely explicit, we choose a homeomorphism $h : S^1 \longrightarrow |S^1|$ [choose one...] and then obtain a $\Sigma_m$-equivariant homeomorphism as the composite

$$S^m \cong S^1 \wedge \cdots \wedge S^1 \xrightarrow{h^{(m)}} |S^1| \wedge \cdots \wedge |S^1| \xrightarrow{r_{S^1,\ldots,S^1}} |S^1 \wedge \cdots \wedge S^1| \ = |S^m| \ .$$

Now we can finally define the adjoint functors 'geometric realization' and 'singular complex' for symmetric spectra.

If $Y$ is a symmetric spectrum of simplicial sets we define a symmetric spectrum $|Y|$ of topological spaces by $|Y|_n = |Y_n|$ with structure maps

$$|Y_n| \wedge S^1 \ \xrightarrow{\mathrm{Id} \wedge h} \ |Y_n| \wedge |S^1| \ \xrightarrow{r_{Y_n,S^1}} \ |Y_n \wedge S^1| \ \xrightarrow{|\sigma_n|} \ |Y_{n+1}| \ .$$

Commutativity of the isomorphism (1.8) guarantees that the equivariance condition for the iterated structure map $\sigma^m$ is inherited by the realization $|Y|$.

Adjoint to the homeomorphism (1.8) is a 'lax symmetric monoidal' transformation of pointed simplicial sets, i.e., a natural, unital, associative and commutative morphism $\mathcal{S}(X) \wedge \mathcal{S}(Y) \longrightarrow \mathcal{S}(X \wedge Y)$ for pointed spaces $X$ and $Y$. So if $X$ is a symmetric spectrum of topological spaces, then we get a symmetric spectrum $\mathcal{S}(X)$ of simplicial sets by $\mathcal{S}(X)_n = \mathcal{S}(X_n)$ with structure map

$$\mathcal{S}(X_n) \wedge S^1 \ \xrightarrow{\mathrm{Id} \wedge \hat{h}} \ \mathcal{S}(X_n) \wedge \mathcal{S}(S^1) \ \longrightarrow \ \mathcal{S}(X_n \wedge S^1) \ \xrightarrow{\mathcal{S}(\sigma_n)} \ \mathcal{S}(X_{n+1}) \ .$$

Here $\hat{h} : S^1 \longrightarrow \mathcal{S}(S^1)$ is the morphism of pointed simplicial sets which is adjoint to the inverse homeomorphism $h^{-1} : |S^1| \longrightarrow S^1$. We use the adjunction unit and counit between $|-|$ and $\mathcal{S}$ levelwise to make geometric realization and singular complex into adjoint functors between topological and simplicial symmetric spectra.

Geometric realization and singular complex are lax symmetric monoid functors with respect to the smash products of pointed spaces and pointed simplicial sets (geometric realization is even strong symmetric monoidal, i.e., commutes with the smash product up to homeomorphism). So both constructions preserve multiplications, so they take ring spectra to ring spectra and preserve commutativity.

The homotopy groups of a symmetric spectrum based on simplicial sets $Y$ are defined as the homotopy groups of the geometric realization $|Y|$.

Now we introduce an important class of symmetric spectra.

DEFINITION 1.9. A symmetric spectrum of topological spaces $X$ is an *$\Omega$-spectrum* if for all $n \geq 0$ the map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ which is adjoint to the structure map $\sigma_n \colon X_n \wedge S^1 \to X_{n+1}$ is a weak homotopy equivalence. The symmetric spectrum $X$ is a *positive $\Omega$-spectrum* if the map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence for all positive values of $n$ (but not necessarily for $n = 0$).

A symmetric spectrum of simplicial sets $Y$ is an *$\Omega$-spectrum* respectively *positive $\Omega$-spectrum* if the geometric realization $|Y|$ is an $\Omega$-spectrum, respectively positive $\Omega$-spectrum, of topological spaces.

A symmetric spectrum of simplicial sets $Y$ is thus an $\Omega$-spectrum if and only if for all $n \geq 0$ the map $|Y_n| \longrightarrow \Omega|Y_{n+1}|$ which is adjoint to the composite

$$|Y_n| \wedge S^1 \xrightarrow{\cong} |Y_n \wedge S^1| \xrightarrow{|\sigma_n|} |Y_{n+1}|$$

is a weak homotopy equivalence. Our definition of '$\Omega$-spectrum' differs slightly from other sources in that we do *not* require that each simplicial set $Y_n$ has to be a Kan complex. If $Y$ is a symmetric spectrum of simplicial sets in which all the $Y_n$'s are Kan, then the natural maps $|\Omega Y_n| \longrightarrow \Omega|Y_n|$ adjoint to

$$|\Omega Y_n| \wedge S^1 \ \longrightarrow \ |(\Omega Y_n) \wedge S^1| \ \xrightarrow{|\text{evaluate}|} \ |Y_n|$$

are weak equivalences, and so $Y$ is an $\Omega$-spectrum in our sense if and only if the morphisms of simplicial sets $\tilde{\sigma}_n : Y_n \longrightarrow \Omega Y_{n+1}$ adjoint to the structure maps are weak equivalences.

For every $\Omega$-spectrum $X$ and all $k, n \geq 0$, the canonical map $\pi_k X_n \longrightarrow \pi_{k-n} X$ is a bijection. Indeed, the homotopy groups of $\Omega X_{n+1}$ are isomorphic to the homotopy groups of $X_{n+1}$, shifted by one dimension. So the colimit system which defines $\pi_{k-n} X$ is isomorphic to the colimit system

$$\pi_k\, X_n \ \longrightarrow \ \pi_k\,(\Omega X_{n+1}) \ \longrightarrow \ \pi_k\,(\Omega^2 X_{n+2}) \ \longrightarrow \ \cdots \ , \tag{1.10}$$

where the maps in the system are induced by the maps $\tilde{\sigma}_n$ adjoint to the structure maps. In an $\Omega$-spectrum, the maps $\tilde{\sigma}_n$ are weak equivalences, so all maps in the sequence (1.10) are bijective, hence so is the map from each term to the colimit $\pi_{k-n} X$.

Several examples of $\Omega$-spectra will come up in the next section, for example Eilenberg-Mac Lane spectra (Example 2.7) and spectra arising from very special $\Gamma$-spaces by evaluation on spheres (Example 2.39). Examples which arise naturally as positive $\Omega$-spectra are the spectra of topological $K$-theory (Example 2.10) and algebraic $K$-theory $K(\mathcal{C})$ (Example 2.11) and spectra arising from special (but not necessarily very special) $\Gamma$-spaces by evaluation on spheres. The $\Omega$-spectra with the additional property of being 'injective' form the objects of the stable homotopy category (see Chapter II).

REMARK 1.11 (Coordinate free symmetric spectra). There is an equivalent definition of symmetric spectra which is, in a certain sense, 'coordinate free'; the reason for calling this 'coordinate free' will hopefully become clear after our discussion of orthogonal spectra in Example 2.40.

If $A$ is a finite set we denote by $\mathbb{R}^A$ the set of functions from $A$ to $\mathbb{R}$ with pointwise structure as a $\mathbb{R}$-vector space. We let $S^A$ denote the one-point compactification of $\mathbb{R}^A$, a sphere of dimension equal to the cardinality of $A$. A *coordinate free symmetric spectrum* consists of the following data:

- a pointed space $X_A$ for every finite set $A$
- a based continuous map $\alpha_* : X_A \wedge S^{B-\alpha(A)} \longrightarrow X_B$ for every injective map $\alpha : A \longrightarrow B$ of finite sets, where $B - \alpha(A)$ is the complement of the image of $\alpha$.

This data is subject to the following conditions:

- (Unitality) For every finite set $A$, the composite

$$X_A \cong X_A \wedge S^{\emptyset} \xrightarrow{(\mathrm{Id}_A)_*} X_A$$

is the identity.

- (Associativity) For every pair of composable injections $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ the diagram

$$\begin{tikzcd}
X_A \wedge S^{B-\alpha(A)} \wedge S^{C-\beta(B)} \arrow[r, "\mathrm{Id}\wedge\beta^!"] \arrow[d, "\alpha_*\wedge\mathrm{Id}"'] & X_A \wedge S^{C-\beta(\alpha(A))} \arrow[d, "(\beta\alpha)_*"] \\
X_B \wedge S^{C-\beta(B)} \arrow[r, "\beta_*"'] & X_C
\end{tikzcd}$$

commutes. In the top vertical map we use the homeomorphism $\beta^! : S^{B-\alpha(A)} \wedge S^{C-\beta(B)} \cong S^{C-\beta(\alpha(A))}$ which is one-point compactified from the linear isomorphism $\mathbb{R}^{B-\alpha(A)} \times \mathbb{R}^{C-\beta(B)} \cong \mathbb{R}^{C-\beta(\alpha(A))}$ which uses $\beta$ on the basis elements indexed by $B - \alpha(A)$ and the identity on the basis elements indexed by $C - \beta(B)$.

A coordinate free symmetric spectrum $X$ gives rise to a symmetric spectrum in the sense of Definition 1.1 as follows. For this we let $\mathbf{n} = \{1, \dots, n\}$ denote the 'standard' set with $n$ elements and identify $\mathbb{R}^{\mathbf{n}}$ with $\mathbb{R}^n$ and $S^{\mathbf{n}}$ with $S^n$.

We set $X_n = X_{\mathbf{n}}$. A permutation $\gamma \in \Sigma_n$ acts on $X_n$ as the composite

$$X_{\mathbf{n}} \cong X_{\mathbf{n}} \wedge S^{\emptyset} \xrightarrow{\gamma_*} X_{\mathbf{n}} \ .$$

The associativity condition in particular shows that this is in fact an associative action. We define the structure map $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ as the map

$$\iota_* \ : \ X_{\mathbf{n}} \wedge S^1 \cong X_{\mathbf{n}} \wedge S^{\{n+1\}} \longrightarrow X_{\mathbf{n+1}}$$

where $\iota : \mathbf{n} \longrightarrow \mathbf{n+1}$ is the inclusion, and the homeomorphism $S^{\{n+1\}} \cong S^1$ arises from the linear isomorphism $\mathbb{R}^{\{n+1\}} \cong \mathbb{R}^1$ respecting the preferred bases.

The associativity condition implies that the iterated structure map $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$ equals the composite

$$X_{\mathbf{n}} \wedge S^m \cong X_{\mathbf{n}} \wedge S^{\{n+1,\dots,n+m\}} \xrightarrow{\iota^m_*} X_{\mathbf{n+m}}$$

where $\iota^m : \mathbf{n} \longrightarrow \mathbf{n+m}$ is the inclusion. The equivariance property is seen as follows: for $\gamma \in \Sigma_n$ we have $\iota^m \circ \gamma = (\gamma \times 1) \circ \iota^m$, and associativity for this injection $\mathbf{n} \longrightarrow \mathbf{n+m}$ amounts to $\Sigma_n \times 1$-equivariance. For $\tau \in \Sigma_m$ we have $\iota^m = (1 \times \tau) \circ \iota^m$, and associativity for this injection $\mathbf{n} \longrightarrow \mathbf{n+m}$ amounts to $1 \times \Sigma_m$-equivariance. Exercise 5.1 shows that this 'forgetful' functor from coordinate free symmetric spectra to symmetric spectra is an equivalence of categories. [Say how ring spectra are formulated in this language]

Remark 1.12 (Manipulations rules for coordinates). Natural numbers occurring as levels of a symmetric spectrum or as dimensions of homotopy groups are really placeholders for sphere coordinates. The role of the symmetric group actions on the spaces of a symmetric spectrum is to keep track of how such coordinates are shuffled. Permutations will come up over and over again in constructions and results about symmetric spectra, and there is a very useful small set of rules which predict when to expect permutations. I recommend being very picky about the order in which dimensions or levels occur when performing constructions with symmetric spectra, as this predicts necessary permutations and helps to prevent mistakes. Sometimes missing a permutation just means missing a sign; in particular missing an even permutation may not have any visible effect. But in general the issue is more serious; for symmetric spectra which are not semistable, missing a permutation typically misses a nontrivial operation.

A first example of this are the centrality and commutativity conditions for symmetric ring spectra, which use shuffle permutations $\chi_{n,1}$ and $\chi_{n,m}$. A good way of remembering when to expect a shuffle is to carefully distinguish between indices such as $n + m$ and $m + n$. Of course these two numbers are equal, but the fact that one arises naturally instead of the other reminds us that a shuffle permutation should be inserted. A shuffle required whenever identifying $n + m$ with $m + n$ is just one rule, and here are some more.

**Main rule:** When manipulating expressions which occur as levels of symmetric spectra or dimensions of spheres, be very attentive for how these expressions arise naturally and when you use the basic rules of arithmetic of natural numbers. When using the basic laws of addition and multiplication of natural numbers in such a context, add permutations according to the following rules (i)-(v).

(i) Do not worry about associativity of addition or multiplication, or the fact that 0 respectively 1 are units for those operations. No permutations are required.

(ii) Whenever using commutativity of addition as in $n + m = m + n$, add a shuffle permutation $\chi_{n,m} \in \Sigma_{n+m}$.

(iii) Whenever using commutativity of multiplication as in $nm = mn$, add a *multiplicative shuffle* $\chi^{\times}_{n,m} \in \Sigma_{nm}$ defined by

$$\chi^{\times}_{n,m}(j + (i-1)n) \ = \ i + (j-1)m$$

for $1 \leq j \leq n$ and $1 \leq i \leq m$.

(iv) Do not worry about left distributivity as in $p(n+m) = pn + pm$. No permutation is required.

(v) Whenever using right distributivity as in $(n+m)q = nq + mq$, add the permutation

$$(\chi^{\times}_{q,n} \times \chi^{\times}_{q,m}) \circ \chi^{\times}_{n+m,q} \ \in \ \Sigma_{(n+m)q} \ .$$

Rule (v) also requires us to throw in permutations whenever we identify a product $nq$ with an iterated sum $q + \cdots + q$ ($n$ copies) since we use right distributivity in the process. However, no permutations are needed when instead identifying $nq$ with a sum of $q$ copies of $n$, since that only uses left distributivity.

The heuristic rules (i) through (v) above are a great help in guessing when to expect coordinate or level permutations when working with symmetric spectra. But the rules are more than heuristics, and are based on the following rigorous mathematics. Typically, there are 'coordinate free' constructions in the background (compare Remark 1.11) which are indexed by finite sets $A$ which are not identified with any of the standard finite sets $\mathbf{n} = \{1, \ldots, n\}$. The outcome of such constructions may naturally be indexed by sets which are built by forming disjoint unions or products. The permutations arise because in contrast to the arithmetic rules for $+$ and $\cdot$, their analogues for disjoint union and cartesian product of sets only holds up to isomorphism, and one can arrange to make some, but not all, of the required isomorphisms be identity maps.

In more detail, when we want to restrict a 'coordinate free' construction to symmetric spectra, we specialize to standard finite sets $\mathbf{n}$; however, if the coordinate free construction involves disjoint union or cartesian product, we need to identify the unions or products of standard finite sets in a consistent way with the standard finite set of the same cardinality. A consistent way to do that amounts to what is called a structure of *bipermutative category* on the category of standard finite sets. So we define binary functors $+$ and $\cdot$ on standard finite sets resembling addition and multiplication of natural numbers as closely as possible.

We let $\mathcal{F}in$ denote the category of standard finite sets whose objects are the sets $\mathbf{n}$ for $n \geq 0$ and whose morphisms are all set maps. We define the sum functor $+ : \mathcal{F}in \times \mathcal{F}in \longrightarrow \mathcal{F}in$ by addition on objects and by 'disjoint union' on morphisms. More precisely, for morphisms $f : \mathbf{n} \longrightarrow \mathbf{n}'$ and $g : \mathbf{m} \longrightarrow \mathbf{m}'$ we define $f + g : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{n}' + \mathbf{m}'$ by

$$(f+g)(i) \ = \ \begin{cases} f(i) & \text{if } 1 \leq i \leq n, \text{ and} \\ g(i-n) + n' & \text{if } n+1 \leq i \leq n+m. \end{cases}$$

This operation is strictly associative and the empty set $\mathbf{0}$ is a strict unit. The symmetry isomorphism is the shuffle map $\chi_{n,m} : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$.

We define the product functor $\cdot : \mathcal{F}in \times \mathcal{F}in \longrightarrow \mathcal{F}in$ by multiplication on objects and by 'cartesian product' on morphisms. To make sense of this we have to linearly order the product of the sets $\mathbf{n}$ and $\mathbf{m}$. There are two choices which are more obvious than others, namely lexicographically with either the first or the second coordinate defined as the more important one. Both choices work fine, and we will prefer the first coordinate. More precisely, for morphisms $f : \mathbf{n} \longrightarrow \mathbf{n}'$ and $g : \mathbf{m} \longrightarrow \mathbf{m}'$ we define $f \cdot g : \mathbf{n} \cdot \mathbf{m} \longrightarrow \mathbf{n}' \cdot \mathbf{m}'$ by

$$(f \cdot g)(j + (i-1)n) \ = \ f(j) + (g(i) - 1)n'$$

for $1 \le j \le n$ and $1 \le i \le m$. The product $\cdot$ is also strictly associative and the set $\mathbf{1}$ is a strict unit. The commutativity isomorphism is the multiplicative shuffle $\chi^{\times}_{nm} : \mathbf{n} \cdot \mathbf{m} \longrightarrow \mathbf{m} \cdot \mathbf{n}$.

This choice of ordering the product of $\mathbf{n}$ and $\mathbf{m}$ has the effect of making $\mathbf{n} \cdot \mathbf{m}$ 'naturally' the same as $\mathbf{n} + \cdots + \mathbf{n}$ ($m$ copies), because we have

$$f \cdot \mathrm{Id}_{\mathbf{m}} \; = \; f + \cdots + f \quad (m \text{ copies}).$$

Since $\mathbf{p} \cdot \mathbf{k}$ 'is' $\mathbf{p} + \cdots + \mathbf{p}$ ($k$ times), we can take the left distributivity isomorphism $\mathbf{p} \cdot (\mathbf{n} + \mathbf{m}) = (\mathbf{p} \cdot \mathbf{n}) + (\mathbf{p} \cdot \mathbf{m})$ as the identity (compare rule (iv)).

In contrast, $\mathrm{Id}_{\mathbf{n}} \cdot g$ is in general *not* equal to $g + \cdots + g$ ($n$ copies), but rather we have

$$\mathrm{Id}_{\mathbf{n}} \cdot g \; = \; \chi^{\times}_{m',n}(g + \cdots + g)\chi^{\times}_{n,m}$$

for a morphism $g : \mathbf{m} \longrightarrow \mathbf{m}'$. However, then right distributivity isomorphism cannot be taken as the identity; since the coherence diagram

$$\begin{tikzcd}
\mathbf{q} \cdot (\mathbf{n} + \mathbf{m}) \arrow[r, "\chi^{\times}_{q,n+m}"] \arrow[d, equal, "\text{left dist.}"'] & (\mathbf{n} + \mathbf{m}) \cdot \mathbf{q} \arrow[d, "\text{right dist.}"] \\
\mathbf{q} \cdot \mathbf{n} + \mathbf{q} \cdot \mathbf{m} \arrow[r, "\chi^{\times}_{q,n} + \chi^{\times}_{q,m}"'] & \mathbf{n} \cdot \mathbf{q} + \mathbf{m} \cdot \mathbf{q}
\end{tikzcd}$$

is supposed to commute, we are forced to define the right distributivity isomorphism $(\mathbf{n} + \mathbf{m}) \cdot \mathbf{q} \cong (\mathbf{n} \cdot \mathbf{q}) + (\mathbf{m} \cdot \mathbf{q})$ as $(\chi^{\times}_{q,n} \times \chi^{\times}_{q,m}) \circ \chi^{\times}_{n+m,q}$, which explains rule (v) above.

## 2. Examples

In this section we give examples of symmetric spectra and symmetric ring spectra, which we have grouped into three subsections. Section 2.1 contains basic examples of important stable homotopy types which one can write down in closed form as a symmetric spectrum. We discuss in particular the sphere spectrum (2.1), suspension spectra (2.6), Eilenberg-Mac Lane spectra (2.7), Thom spectra (2.8, 2.9 and 2.42), topological $K$-theory (2.10) and algebraic $K$-theory spectra (2.11).

In Section 2.2 we discuss constructions which produce new symmetric spectra and ring spectra from old ones, or from a symmetric spectrum and a space. We define free (2.12) and semifree symmetric spectra (2.13), limits and colimits (2.14), smash product with a space (2.15), suspension (2.16), shift (2.18), twisted smash product with a $\Sigma_m$-space (2.20), function spectra (2.22), loop spectra (2.23), mapping spaces (2.24), internal Hom spectra (2.25), endomorphism ring spectra (2.26), mapping telescope and diagonal of a sequence (2.27), smash product with an $I$-space (2.31), monoid ring spectra (2.32), ring spectra from multiplicative $I$-spaces (2.33), matrix ring spectra (2.34), inverting an integer (2.35) or an element in $\pi_0$ (2.36) of a symmetric ring spectrum and adjoining roots of unity to a symmetric ring spectrum (2.37).

In Section 2.3 we review other kinds of spectra and various forgetful functors between them. We discuss continuous functors (2.38), $\Gamma$-spaces (2.39), orthogonal spectra (2.40), unitary spectra (2.41) and $S$-modules (2.43). The main point here is that an object in any of these other categories also gives rise to a symmetric spectrum.

### 2.1. Basic examples.

Example 2.1 (Sphere spectrum). The symmetric *sphere spectrum* $\mathbb{S}$ is given by $\mathbb{S}_n = S^n$, where the symmetric group permutes the coordinates and $\sigma_n : S^n \wedge S^1 \longrightarrow S^{n+1}$ is the canonical isomorphism. This is a commutative symmetric ring spectrum with identity as unit map and the canonical isomorphism $S^n \wedge S^m \longrightarrow S^{n+m}$ as multiplication map. The sphere spectrum is the *initial* symmetric ring spectrum: if $R$ is any symmetric ring spectrum, then a unique morphism of symmetric ring spectra $\mathbb{S} \longrightarrow R$ is given by the collection of unit maps $\iota_n : S^n \longrightarrow R_n$ (compare 1.6 (ii)). Being initial, the sphere spectrum plays the same formal role for symmetric ring spectra as the integers $\mathbb{Z}$ play for rings. This motivates the notation '$\mathbb{S}$' using the `\mathbb` font. The category of right $\mathbb{S}$-modules is isomorphic to the category of symmetric spectra, via the forgetful functor mod-$\mathbb{S} \longrightarrow \mathcal{S}p^\Sigma$ (see Remark 1.6 (ii)). Indeed, if $X$ is a symmetric spectrum then the associativity condition shows that there is at most one way to define action maps

$$\alpha_{n,m} \ : \ X_n \wedge S^m \ \longrightarrow \ X_{n+m} \ ,$$

namely as the iterated structure map $\sigma^m$, and these do define the structure of right $\mathbb{S}$-module on $X$.

The homotopy group $\pi_k\mathbb{S} = \operatorname{colim}_n \pi_{k+n}S^n$ is called the *k-th stable homotopy group of spheres*, or the *k-th stable stem*. Since $S^n$ is $(n-1)$-connected, the group $\pi_k\mathbb{S}$ is trivial for negative values of $k$. The degree of a self-map of a sphere provides an isomorphism $\pi_0\mathbb{S} \cong \mathbb{Z}$.

For $k \geq 1$, the homotopy group $\pi_k\mathbb{S}$ is finite. This is a direct consequence of Serre's calculation of the homotopy groups of spheres modulo torsion, which we recall without giving a proof, and Freudenthal's suspension theorem.

Theorem 2.2 (Serre). *Let $m > n \geq 1$. Then*

$$\pi_m S^n = \begin{cases} \text{(finite group)} \oplus \mathbb{Z} & \text{if } n \text{ is even and } m = 2n-1 \\ \text{(finite group)} & \text{else.} \end{cases}$$

*Thus for $k \geq 1$, the stable stem $\pi_k^s = \pi_k\mathbb{S}$ is finite.*

As a concrete example, we inspect the colimit system defining $\pi_1\mathbb{S}$, the first stable homotopy group of spheres. Since the universal cover of $S^1$ is the real line, which is contractible, the theory of covering spaces shows that the groups $\pi_n S^1$ are trivial for $n \geq 2$. The Hopf map

$$\eta \ : \ S^3 \subseteq \mathbb{C}^2 - \{0\} \xrightarrow{\text{proj.}} \mathbb{C}P^1 \cong S^2$$

is a locally trivial fibre bundle with fibre $S^1$, so it gives rise to long exact sequence of homotopy groups. Since the fibre $S^1$ has no homotopy above dimension one, the group $\pi_3 S^2$ is free abelian of rank one, generated by the class of $\eta$. By Freudenthal's suspension theorem the suspension homomorphism $-\wedge S^1 : \pi_3 S^2 \longrightarrow \pi_4 S^3$ is surjective and from $\pi_4 S^3$ on the suspension homomorphism is an isomorphism. So the first stable stem $\pi_1^s$ is cyclic, generated by the image of $\eta$, and its order equals the order of the suspension of $\eta$. On the one hand, $\eta$ itself is stably essential, since the Steenrod operation $Sq^2$ acts non-trivially on the mod-2 cohomology of the mapping cone of $\eta$, which is homeomorphic to $\mathbb{C}P^2$.

On the other hand, twice the suspension of $\eta$ is null-homotopic. To see this we consider the commutative square

$$\begin{tikzcd} (x,y) \arrow[d, mapsto] & S^3 \arrow[r, "\eta"] \arrow[d] & \mathbb{C}P^1 \arrow[d] & {[x:y]} \arrow[d, mapsto] \\ (\bar{x},\bar{y}) & S^3 \arrow[r, "\eta"'] & \mathbb{C}P^1 & {[\bar{x}:\bar{y}]} \end{tikzcd}$$

in which the vertical maps are induced by complex conjugation in both coordinates of $\mathbb{C}^2$. The left vertical map has degree 1, so it is homotopic to the identity, whereas complex conjugation on $\mathbb{C}P^1 \cong S^2$ has degree $-1$. So $(-1) \circ \eta$ is homotopic to $\eta$. Thus the suspension of $\eta$ is homotopic to the suspension of $(-1) \circ \eta$, which by the following lemma is homotopic to the negative of $\eta \wedge S^1$.

Lemma 2.3. *Let $Y$ be a pointed space, $m \geq 0$ and $f : S^m \longrightarrow S^m$ a based map of degree $k$. Then for every homotopy class $x \in \pi_n(Y \wedge S^m)$ the classes $(\mathrm{Id}_Y \wedge f)_*(x)$ and $k \cdot x$ become equal in $\pi_{n+1}(Y \wedge S^{m+1})$ after one suspension.*

Proof. Let $d_k : S^1 \longrightarrow S^1$ be any pointed map of degree $k$. Then the maps $f \wedge S^1, S^m \wedge d_k : S^{m+1} \longrightarrow S^{m+1}$ have the same degree $k$, hence they are based homotopic. Suppose $x$ is represented by $\varphi : S^n \longrightarrow Y \wedge S^m$. Then the suspensions of $(Y \wedge f)_*(x)$ is represented by $(Y \wedge f \wedge S^1) \circ (\varphi \wedge S^1)$ which is homotopic to $(Y \wedge S^m \wedge d_k) \circ (\varphi \wedge S^1) = (\varphi \wedge S^1) \circ (S^n \wedge d_k)$. Precomposition with the degree $k$ map $S^n \wedge d_k$ of $S^{n+1}$ induces multiplication by $k$, so the last map represents the suspension of $k \cdot x$. □

The conclusion of Lemma 2.3 does not in general hold without the extra suspension, i.e., $(Y \wedge f)_*(x)$ need not equal $(-1)^k x$ in $\pi_n(Y \wedge S^m)$: as we showed above, $(-1) \circ \eta$ is homotopic to $\eta$, which is *not* homotopic to $-\eta$ since $\eta$ generates the infinite cyclic group $\pi_3 S^2$.

As far as we know, the stable homotopy groups of spheres don't follow any simple pattern. Much machinery of algebraic topology has been developed to calculate homotopy groups of spheres, both unstable and stable, but no one expects to ever get explicit formulae for all stable homotopy groups of spheres. The Adams spectral sequence based on mod-$p$ cohomology and the Adams-Novikov spectral sequence based on $MU$ (complex cobordism) or $BP$ (the Brown-Peterson spectrum at a fixed prime $p$) are the most effective tools for explicit calculations as well as for discovering systematic phenomena.

EXAMPLE 2.4 (Multiplication in stable stems). The stable stems $\pi_*^s = \pi_* \mathbb{S}$ form a graded commutative ring which acts on the homotopy groups of every other symmetric spectrum $X$. We denote the action simply by a 'dot'

$$\cdot \; : \; \pi_k X \times \pi_l \mathbb{S} \; \longrightarrow \; \pi_{k+l} X;$$

the definition is essentially straightforward, but there is one subtlety in showing that the product is well-defined. I repeat an earlier warning. It is tempting to try to define a product on the homotopy groups of a symmetric ring spectrum $R$ in a similar fashion, by smashing representatives and shuffling sphere coordinates into their natural order. This will indeed give an associative product in many natural cases, namely whenever the underlying symmetric spectrum of $R$ is 'semistable', see Theorem 4.54. However, if $R$ is not semistable, then smashing of representatives does not descend to a well-defined product on stable homotopy groups! In that case the algebraic structure that the homotopy groups of $R$ enjoy is more subtle.

Suppose $f : S^{k+n} \longrightarrow X_n$ and $g : S^{l+m} \longrightarrow S^m$ represent classes in $\pi_k X$ respectively $\pi_l \mathbb{S}$. Then we agree that the composite

$$S^{k+l+n+m} \xrightarrow{\mathrm{Id} \wedge \chi_{l,n} \wedge \mathrm{Id}} S^{k+n+l+m} \xrightarrow{f \wedge g} X_n \wedge S^m \xrightarrow{\sigma^m} X_{n+m} \tag{2.5}$$

represents the product of $[f]$ and $[g]$. The shuffle permutation $\chi_{l,n}$ is predicted by the principle that all natural number must occur in the 'natural order' compare Remark 1.12. If we simply smash $f$ and $g$ the dimension of the sphere of origin is naturally $(k+n)+(l+m)$, but in order to represent an element of $\pi_{k+l}X$ they should occur in the order $(k+l)+(n+m)$, whence the shuffle permutation (which here simply introduces the sign $(-1)^{ln}$).

We check that the multiplication is well-defined. If we replace $g : S^{l+m} \longrightarrow S^m$ by its suspension $g \wedge S^1$, then the composite (2.5) changes to its suspension, composed with the structure map $\sigma_{n+m} : X_{n+m} \wedge S^1 \longrightarrow X_{n+m+1}$. So the resulting stable class is independent of the representative $g$ of the stable class in $\pi_l \mathbb{S}$. Independence of the representative for $\pi_k X$ is slightly more subtle. If we replace $f : S^{k+n} \longrightarrow X_n$ by the representative $\sigma_n \circ (f \wedge S^1) : S^{k+n+1} \longrightarrow X_{n+1}$, then the composite (2.5) changes to $\sigma^{1+m}(f \wedge \mathrm{Id} \wedge g)(\mathrm{Id} \wedge \chi_{l,n+1} \wedge \mathrm{Id})$, which is the lower left composite in the commutative diagram

$$\begin{tikzcd}
S^{k+l+n+1+m} \arrow[r, "\mathrm{Id} \wedge \chi_{l,n} \wedge \chi_{1,m}"] \arrow[dr, "\mathrm{Id} \wedge \chi_{l,n+1} \wedge \mathrm{Id}"'] & S^{k+n+l+m+1} \arrow[r, "f \wedge g \wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id} \wedge \chi_{l+m,1}"] & X_n \wedge S^{m+1} \arrow[d, "\mathrm{Id} \wedge \chi_{m,1}"] & \\
& S^{k+n+1+l+m} \arrow[r, "f \wedge \mathrm{Id} \wedge g"'] & X_n \wedge S^{1+m} \arrow[r, "\sigma^{1+m}"'] & X_{n+1+m}
\end{tikzcd}$$

By Lemma 2.3 the map $\mathrm{Id} \wedge \chi_{m,1}$ induces multiplication by $(-1)^m$ on homotopy groups *after one suspension.* This cancels the sign coming from the shuffle factor $\chi_{1,m}$ in the initial horizontal map. So the composite is homotopic, after one suspension, to the composite $\sigma^{1+m}(f \wedge g \wedge \mathrm{Id})(\mathrm{Id} \wedge \chi_{l,n} \wedge \mathrm{Id})$, which represents the same stable class as (2.5).

Now we verify that the dot product is biadditive. We only show the relation $(x + x') \cdot y = x \cdot y + x' \cdot y$, and additivity in $y$ is similar. Suppose as before that $f, f' : S^{k+n} \longrightarrow X_n$ and $g : S^{l+m} \longrightarrow S^m$ represent classes in $\pi_k X$ respectively $\pi_l \mathbb{S}$. Then the sum of $f$ and $f'$ in $\pi_{k+n} X_n$ is represented by the composite

$$S^{k+n} \xrightarrow{\text{pinch}} S^{k+n} \vee S^{k+n} \xrightarrow{f \vee f'} X_n \; .$$

In the square

$$\begin{tikzcd}
S^{k+l+n+m} \arrow[r, "1\wedge\chi_{l,n}\wedge 1"] \arrow[dd, "\text{pinch}"'] & S^{k+n+l+m} \arrow[d, "\text{pinch}\wedge\text{Id}"'] \arrow[dr, "(f+f')\wedge g"] & \\
 & (S^{k+n}\vee S^{k+n}) \wedge S^{l+m} \arrow[r, "(f\vee f')\wedge g"'] & X_n \wedge S^m \\
S^{k+l+n+m}\vee S^{k+l+n+m} \arrow[r, "(1\wedge\chi_{l,n}\wedge 1)\vee(1\wedge\chi_{l,n}\wedge 1)"'] & S^{k+n+l+m}\vee S^{k+n+l+m} \arrow[u, "\cong"'] \arrow[ur, "(f\wedge g)\vee(f'\wedge g)"'] &
\end{tikzcd}$$

the right part commutes on the nose and the left square commutes up to homotopy. After composing with the iterated structure map $\sigma^m : X_n \wedge S^m \longrightarrow X_{n+m}$, the composite around the top of the diagram becomes $(f + f') \cdot g$, whereas the composite around the bottom represents $fg + f'g$. This proves additivity of the dot product in the left variable.

If we specialize to $X = \mathbb{S}$ then the product provides a biadditive graded pairing $\cdot : \pi_k\mathbb{S} \times \pi_l\mathbb{S} \longrightarrow \pi_{k+l}\mathbb{S}$ of the stable homotopy groups of spheres. We claim that for every symmetric spectrum $X$ the diagram

$$\begin{tikzcd}
\pi_k X \times \pi_l\mathbb{S} \times \pi_j\mathbb{S} \arrow[r, "\cdot\times\text{Id}"] \arrow[d, "\text{Id}\times\cdot"'] & \pi_{k+l}X \times \pi_j\mathbb{S} \arrow[d, "\cdot"] \\
\pi_k X \times \pi_{l+j}\mathbb{S} \arrow[r, "\cdot"'] & \pi_{k+l+j}X
\end{tikzcd}$$

commutes, so the product on the stable stems and the action on the homotopy groups of a symmetric spectrum are associative. After choosing representing maps $S^{k+n} \longrightarrow X_n$, $S^{l+m} \longrightarrow S^m$ and $S^{j+q} \longrightarrow S^q$ and unraveling all the definitions, this associativity ultimately boils down to the equality

$$(k \times \chi_{l,n} \times m \times j \times q) \circ (k \times l \times \chi_{j,n+m} \times q) \;=\; (k \times n \times l \times \chi_{j,m} \times q) \circ (k \times \chi_{l+j,n} \times m \times q)$$

in the symmetric group $\Sigma_{k+l+j+q+n+m}$ and commutativity of the square

$$\begin{tikzcd}
X_n \wedge S^m \wedge S^q \arrow[r, "\sigma^m\wedge\text{Id}"] \arrow[d, "\text{Id}\wedge\cong"'] & X_{n+m} \wedge S^q \arrow[d, "\sigma^q"] \\
X_n \wedge S^{m+q} \arrow[r, "\sigma^{m+q}"'] & X_{n+m+q}
\end{tikzcd}$$

Finally, the multiplication in the homotopy groups of spheres is commutative in the graded sense, i.e., we have $xy = (-1)^{kl}yx$ for $x \in \pi_k\mathbb{S}$ and $y \in \pi_l\mathbb{S}$. Indeed, for representing maps $f : S^{k+n} \longrightarrow S^n$ and $g : S^{l+m} \longrightarrow S^m$ the square

$$\begin{tikzcd}
S^{k+l+n+m} \arrow[r, "\text{Id}\wedge\chi_{l,n}\wedge\text{Id}"] \arrow[d, "\chi_{k,l}\times\chi_{n,m}"'] & S^{k+n+l+m} \arrow[r, "f\wedge g"] & S^{n+m} \arrow[d, "\chi_{n,m}"] \\
S^{l+k+m+n} \arrow[r, "\text{Id}\wedge\chi_{k,m}\wedge\text{Id}"'] & S^{l+m+k+n} \arrow[r, "g\wedge f"'] & S^{m+n}
\end{tikzcd}$$

commutes. The two vertical coordinate permutations induce the signs $(-1)^{kl+nm}$ respectively (after one suspension) $(-1)^{nm}$ on homotopy groups. Since the upper horizontal composite represents $xy$ and the lower composite represents $yx$, this proves the relation $xy = (-1)^{kl}yx$.

The following table gives the stable homotopy groups of spheres through dimension 8:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\pi_n\mathbb{S}$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | $\mathbb{Z}/24$ | 0 | 0 | $\mathbb{Z}/2$ | $\mathbb{Z}/240$ | $(\mathbb{Z}/2)^2$ |
| generator | $\iota$ | $\eta$ | $\eta^2$ | $\nu$ | | | $\nu^2$ | $\sigma$ | $\eta\sigma,\ \varepsilon$ |

Here $\nu$ and $\sigma$ are the Hopf maps which arises unstably as fibre bundles $S^7 \longrightarrow S^4$ respectively $S^{15} \longrightarrow S^8$. The element $\varepsilon$ in the 8-stem can be defined using Toda brackets (see Construction 4.71) as $\varepsilon = \eta\sigma + \langle \nu, \eta, \nu \rangle$. The table records all multiplicative relations in this range except for $\eta^3 = 12\nu$. A theorem of Nishida's says that every homotopy element of positive dimension is nilpotent.

Example 2.6 (Suspension spectra). Every pointed space $K$ gives rise to a *suspension spectrum* $\Sigma^\infty K$ via

$$(\Sigma^\infty K)_n \;=\; K \wedge S^n$$

with structure maps given by the canonical isomorphism $(K \wedge S^n) \wedge S^1 \xrightarrow{\cong} K \wedge S^{n+1}$; we then have $\mathbb{S} \cong \Sigma^\infty S^0$. The homotopy group

$$\pi_k^s K \;=\; \pi_k\left(\Sigma^\infty K\right) \;=\; \operatorname{colim}_n \, \pi_{k+n}(K \wedge S^n)$$

is called the $k$th *stable homotopy group* of $K$.

Since $K \wedge S^n$ is $(n-1)$-connected, the suspension spectrum $\Sigma^\infty K$ is connective. The Freudenthal suspension theorem implies that for every suspension spectrum, the colimit system for a specific homotopy group always stabilizes. A symmetric spectrum $X$ is isomorphic to a suspension spectrum (necessarily that of its zeroth space $X_0$) if and only if every structure map $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ is a homeomorphism.

Example 2.7 (Eilenberg-Mac Lane spectra). For an abelian group $A$, the *Eilenberg-Mac Lane spectrum* $HA$, based on simplicial sets, is defined by

$$(HA)_n \;=\; A \otimes \tilde{\mathbb{Z}}[S^n] \ ,$$

i.e., the underlying simplicial set of the dimensionwise tensor product of $A$ with the reduced free simplicial abelian group generated by the simplicial $n$-sphere. The symmetric group acts by permuting the smash factors of $S^n$. The geometric realization of $(HA)_n$ is an Eilenberg-Mac Lane space of type $(A, n)$, i.e., it has only one non-trivial homotopy group in dimensions $n$, which is isomorphic to $A$. The loop space of the next space $(HA)_{n+1}$ is also an Eilenberg-Mac Lane space of type $(A, n)$, and in fact the map $\tilde{\sigma}_n : (HA)_n \longrightarrow \Omega(HA)_{n+1}$ adjoint to the structure map is a weak equivalence for all $n \geq 0$. In other words, $HA$ is an $\Omega$-spectrum.

It follows that the homotopy groups of the symmetric spectrum $HA$ are concentrated in dimension zero, where we have a natural isomorphism $A = \pi_0(HA)_0 \cong \pi_0 HA$. If $A$ is not just an abelian group but also has a ring structure, then $HA$ becomes a symmetric ring spectrum via the multiplication map

$$\begin{aligned}(HA)_n \wedge (HA)_m \;=\; &(A \otimes \mathbb{Z}[S^n]) \wedge (A \otimes \mathbb{Z}[S^m]) \\ &\longrightarrow\; A \otimes \mathbb{Z}[S^{n+m}] \;=\; (HA)_{n+m}\end{aligned}$$

given by

$$\left(\sum_i a_i \cdot x_i\right) \wedge \left(\sum_j b_j \cdot y_j\right) \;\longmapsto\; \sum_{i,j} (a_i \cdot b_j) \cdot (x_i \wedge y_j) \ .$$

The unit maps $S^m \longrightarrow (HA)_m$ are given by the inclusion of generators.

We shall see in Example 3.11 below that the Eilenberg-Mac Lane functor $H$ can be made into a lax symmetric monoidal functor with respect to the tensor product of abelian groups and the smash product of symmetric spectra; this also explains why $H$ takes rings (monoids in the category of abelian with respect to tensor product) to ring spectra (monoids in the category of symmetric spectra with respect to smash product).

Example 2.8 (Real cobordism spectra). We define a commutative symmetric ring spectrum $MO$ whose stable homotopy groups are isomorphic to the ring of cobordism classes of closed smooth manifolds. We set

$$MO_n \;=\; EO(n)^+ \wedge_{O(n)} S^n \ ,$$

the Thom space of the tautological vector bundle $EO(n) \times_{O(n)} \mathbb{R}^n$ over $BO(n) = EO(n)/O(n)$. Here $O(n)$ is the $n$-th orthogonal group consisting of Euclidean automorphisms of $\mathbb{R}^n$. The space $EO(n)$ is the geometric realization of the simplicial space which in dimension $k$ is the $(k+1)$-fold product of copies of

$O(n)$, and where face maps are projections. Thus $EO(n)$ is contractible and has a right action by $O(n)$. The right $O(n)$-action is used to form the orbit space $MO_n$, where we remember that $S^n$ is the one-point compactification of $\mathbb{R}^n$, so it comes with a left $O(n)$-action.

The symmetric group $\Sigma_n$ acts on $O(n)$ by conjugation with the permutation matrices. Since the '$E$'-construction is natural in topological groups, this induces an action of $\Sigma_n$ on $EO(n)$. If we let $\Sigma_n$ act on the sphere $S^n$ by coordinate permutations and diagonally on $EO(n)^+ \wedge S^n$, then the action descends to the quotient space $MO_n$.

The unit of the ring spectrum $MO$ is given by the maps

$$S^n \cong O(n)^+ \wedge_{O(n)} S^n \longrightarrow EO(n)^+ \wedge_{O(n)} S^n = MO_n$$

using the 'vertex map' $O(n) \longrightarrow EO(n)$. There are multiplication maps

$$MO_n \wedge MO_m \longrightarrow MO_{n+m}$$

which are induced from the identification $S^n \wedge S^m \cong S^{n+m}$ which is equivariant with respect to the group $O(n) \times O(m)$, viewed as a subgroup of $O(n+m)$ by block sum of matrices. The fact that these multiplication maps are associative and commutative uses that

- for topological groups $G$ and $H$, the simplicial model of $EG$ comes with a natural, associative and commutative isomorphism $E(G \times H) \cong EG \times EH$;
- the group monomorphisms $O(n) \times O(m) \longrightarrow O(n+m)$ by orthogonal direct sum are strictly associative, and the following diagram commutes

$$\begin{tikzcd} O(n) \times O(m) \arrow[r] \arrow[d, "\text{twist}"'] & O(n+m) \arrow[d, "\text{conj. by } \chi_{n,m}"] \\ O(m) \times O(n) \arrow[r] & O(m+n) \end{tikzcd}$$

  where the right vertical map is conjugation by the permutation matrix of the shuffle permutation $\chi_{n,m}$.

Essentially the same construction gives commutative symmetric ring spectra $MSO$ representing oriented bordism and $MSpin$ representing spin bordism. For $MSO$ this uses that conjugation of $O(n)$ by a permutation matrix restricts to an automorphism of $SO(n)$ and the block sum of two special orthogonal transformations is again special. For $MSpin$ it uses that the block sum pairing and the $\Sigma_n$-action uniquely lift from the groups $SO(n)$ to their universal covers $Spin(n)$.

We intend to discuss these and other examples of Thom spectra in more detail in a later chapter.

Example 2.9 (Complex cobordism spectra). The cobordism ring spectra $MU$ and $MSU$ representing complex bordism, or symplectic bordism $MSp$ have to be handled slightly differently from real Thom spectra such as $MO$ in the previous example. The point is that $MU$ and $MSU$ are most naturally indexed on 'even spheres', i.e., one-point compactifications of complex vector spaces, and $MSp$ is most naturally indexed on spheres of dimensions divisible by 4. However, a small variation gives $MU$, $MSU$ and $MSp$ as commutative symmetric ring spectra, as we shall now explain. The complex cobordism spectrum $MU$ plays an important role in stable homotopy theory because of its relationship to the theory of formal groups laws. Thus module and algebra spectra over $MU$ are important, and we plan to study these in some detail later.

We first consider the collection of pointed spaces $\overline{MU}$ with

$$(\overline{MU})_n = EU(n)^+ \wedge_{U(n)} S^{2n} ,$$

the Thom space of the tautological complex vector bundle $EU(n) \times_{U(n)} \mathbb{C}^n$ over $BU(n) = EU(n)/U(n)$. Here $U(n)$ is the $n$-th unitary group consisting of Euclidean automorphisms of $\mathbb{C}^n$. The $\Sigma_n$-action arises from conjugation by permutation matrices and the permutation of complex coordinates, similarly as in the case of $MO$ above.

There are multiplication maps

$$(\overline{MU})_p \wedge (\overline{MU})_q \longrightarrow (\overline{MU})_{p+q}$$

which are induced from the identification $\mathbb{C}^p \oplus \mathbb{C}^q \cong \mathbb{C}^{p+q}$ which is equivariant with respect to the group $U(p)\times U(q)$, viewed as a subgroup of $U(p+q)$ by direct sum of linear maps. There is a unit map $\iota_0 : S^0 \longrightarrow (\overline{MU})_0$, but instead of a unit map from the circle $S^1$, we only have a unit map $S^2 \longrightarrow (\overline{MU})_1$. Thus we do *not* end up with a symmetric spectrum since we only get structure maps $(\overline{MU})_n \wedge S^2 \longrightarrow (\overline{MU})_{n+1}$ involving the 2-sphere. In other words, $\overline{MU}$ has the structure of what could be called an 'even symmetric ring spectrum' ($\overline{MU}$ is really a unitary ring spectrum, as we shall define in Example 2.41 below).

In order to get an honest symmetric ring spectrum we now use a general construction which turns a commutative monoid $\Phi R$ in the category of symmetric sequences into a new such monoid by appropriately looping all the spaces involved. We set

$$\Phi(R)_n \;=\; \mathrm{map}(S^n, R_n)$$

an let the symmetric group act by conjugation. Then the product of $R$ combined with smashing maps gives $\Sigma_n \times \Sigma_m$-equivariant maps

$$\begin{array}{ccc} \Phi(R)_n \wedge \Phi(R)_m = \mathrm{map}(S^n, R_n) \wedge \mathrm{map}(S^m, R_m) & \longrightarrow & \mathrm{map}(S^{n+m}, R_{n+m}) = \Phi(R)_{n+m} \\ f \wedge g & \longmapsto & f \cdot g = \mu_{n,m} \circ (f \wedge g) \ . \end{array}$$

Now we apply this construction to $\overline{MU}$ and obtain a commutative monoid $MU = \Phi(\overline{MU})$ in the category of symmetric sequences. We make $MU$ into a symmetric ring spectrum via the unit map $S^1 \longrightarrow (MU)_1 = \mathrm{map}(S^1, (\overline{MU})_1)$ which is adjoint to

$$\iota \;:\; S^2 \cong U(1)^+ \wedge_{U(1)} S^2 \;\longrightarrow\; EU(1)^+ \wedge_{U(1)} S^2 = (\overline{MU})_1$$

using the 'vertex map' $U(1) \longrightarrow EU(1)$. More precisely, we use the decomposition $\mathbb{C} = \mathbb{R}\cdot 1 \oplus \mathbb{R}\cdot i$ to view $S^2$ as the smash product of a 'real' and 'imaginary' circle, and then we view the source of the unit map $S^1 \longrightarrow (MU)_1 = \mathrm{map}(S^1, (\overline{MU})_1)$ as the real circle, and we think of the imaginary circle as parameterizing the loop coordinate in the target $(MU)_1$. Since the multiplication of $MU$ is commutative, the centrality condition is automatically satisfied. Then the iterated unit map

$$S^n \;\longrightarrow\; (MU)_n = \Omega^n(\overline{MU})_n$$

is given by

$$(x_1, \dots, x_n) \quad\longmapsto\quad ((y_1, \dots, y_n) \mapsto \mu(\iota(x_1, y_1), \dots, \iota(x_n, y_n)))$$

where $\mu : (\overline{MU})_1^{\wedge n} \longrightarrow (\overline{MU})_n$ is the iterated multiplication map.

The homotopy groups of $MU$ are given by

$$\pi_k MU = \mathrm{colim}_n\, \pi_{k+n}\, \mathrm{map}(S^n, (\overline{MU}_n)) \cong \mathrm{colim}_n\, \pi_{k+2n}(EU(n)^+ \wedge_{U(n)} S^{2n}) \ ;$$

so by Thom's theorem they are isomorphic to the ring of cobordism classes of stably almost complex $k$-manifolds. So even though the individual spaces $MU_n = \mathrm{map}(S^n, EU(n)^+ \wedge_{U(n)} S^{2n})$ are not Thom spaces, the symmetric spectrum which they form altogether has the 'correct' homotopy groups (and in fact, the correct stable homotopy type).

Essentially the same construction gives a commutative symmetric ring spectrum $MSU$. The symplectic bordism and $MSp$ can also be handled similarly: it first arises as a commutative monoid $\overline{MSp}$ in symmetric sequences with structure maps $(\overline{MSp})_n \wedge S^4 \longrightarrow (\overline{MSp})_{n+1}$ and a unit map $S^4 \longrightarrow (\overline{MSp})_1$. If we apply the construction $\Phi$ three times, we obtain a commutative symmetric ring spectrum $MSp = \Phi^3(\overline{MSp})$ representing symplectic bordism.

Example 2.10 (Topological $K$-theory). We define the commutative symmetric ring spectrum $KU$ of *complex topological $K$-theory.* We set

$$KU_n \;=\; \hom(q^n(C_0(\mathbb{R}^n)), \mathcal{K}^{(n)})$$

where:

- $C_0(\mathbb{R}^n)$ is the $C^*$-algebra of continuous complex valued functions on $\mathbb{R}^n$ which vanish at infinity;

- $q^n$ is the $n$-th iterate of a functor $q$ which associates to a $C^*$-algebra its *Cuntz algebra*; $qA$ is the kernel of the fold morphism $A * A \longrightarrow A$, where the star is the categorical coproduct of $C^*$-algebras (a certain completion of the algebraic coproduct);
- $\mathcal{K}$ is the $C^*$-algebra of compact operators on a fixed separable Hilbert space, and $\mathcal{K}^{(n)}$ is the spatial tensor product of $n$ copies of $\mathcal{K}$;
- hom is the space of $*$-homomorphisms between two $C^*$-algebras with the subspace topology of the compact open topology on the space of all continuous maps; the basepoint is the zero map.

Here $C^*$-algebras are not necessarily unital, and homomorphisms need not preserve units, if they exist. There is an adjunction as follows: if $K$ is a locally compact space and $A$ and $B$ are $C^*$-algebras, then

$$\mathcal{T}(\bar{K}, \hom(A,B)) \;\cong\; \hom(A, C_0K \otimes B)$$

where $\bar{K}$ is the one-point compactification.

There is an action of the symmetric group on $q^nA$, but it is not obvious from what we have said so far [define it]. The symmetric group also acts on $C_0(\mathbb{R}^n)$ by permutation of coordinates, on $\mathcal{K}^{(n)}$ by permutation of tensor factors, and on the mapping space $KU_n$ by conjugation.

In level 0 we have $q^0A = A$, so $q^0(C_0(\mathbb{R}^n)) = \mathbb{C}$, and $\mathcal{K}^{(0)} = \mathbb{C}$. Thus we have $KU_0 = \hom(\mathbb{C},\mathbb{C})$ which consists of two elements, the zero and the identity homomorphism. So we define $\iota_0 : S^0 :\longrightarrow KU_0$ as the homeomorphism which sends the basepoint to the zero homomorphism and the non-basepoint to the identity of $\mathbb{C}$. We have $KU_1 = \hom(q(C_0(\mathbb{R})), \mathcal{K})$ [identify with infinite unitary group] and via the adjunction, the unit map $\iota_1 : S^1 \longrightarrow KU_1$ corresponds to a $C^*$-homomorphism $\hat{\iota}_1 : q(C_0(\mathbb{R})) \longrightarrow C_0(\mathbb{R}) \otimes \mathcal{K}$ which is defined as the composite

$$q(C_0(\mathbb{R})) \;\longrightarrow\; C_0(\mathbb{R}) \;\longrightarrow\; C_0(\mathbb{R}) \otimes \mathcal{K}$$

where the first map is the restriction of the morphism $1 * 0 : C_0(\mathbb{R}) * C_0(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$ to the kernel of the fold map and the second map sends $f$ to $f \otimes e$ where $e \in \mathcal{K}$ is a fixed rank 1 projection. [this should be a positive $\Omega$-spectrum. Necessary modifications to yield $KO$ and $KT$. How about $KSp$ ? $C_2$-action on $KU$?]

The *Bott periodicity theorem* says that there is a homotopy equivalence $\Omega^2 BU \;\simeq\; \mathbb{Z} \times BU$. It implies that the homotopy ring of $KU$ contains a unit in dimension 2, the *Bott class* $u \in \pi_2 KU$. There is an isomorphism of graded rings $\pi_* KU \cong \mathbb{Z}[u, u^{-1}]$.

There is a variant, the symmetric spectrum of *real topological $K$-theory $KO$*. The real version has a Bott-periodicity of order 8, i.e., there is a homotopy equivalence

$$\Omega^8 BO \;\simeq\; \mathbb{Z} \times BO \ ,$$

which gives a unit $\beta \in \pi_8 KO$.

The following table (and Bott periodicity) gives the homotopy groups of the spectrum $KO$:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $\pi_n KO$ | $\mathbb{Z}$ | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $\mathbb{Z}$ |
| generator | $\iota$ | $\eta$ | $\eta^2$ | | $\xi$ | | | | $\beta$ |

Here $\eta$ is the Hurewicz image of the Hopf map, i.e., the image of the class $\eta \in \pi_1\mathbb{S}$ under the unique homomorphism of ring spectra $\mathbb{S} \longrightarrow KO$. There is a homomorphism of ring spectra $KO \longrightarrow KU$, the 'complexification map', which is injective on homotopy groups in dimensions divisible by 4, and bijective in dimensions divisible by 8. The elements $\xi$ and $\beta$ can be defined by the property that they hit $2u^2 \in \pi_4 KU$ respectively $u^4 \in \pi_8 KU$ under this complexification map. Thus there is the multiplicative relation $\xi^2 = 4\beta$.

Example 2.11 (Algebraic $K$-theory). There are various formalisms which associate to a category with suitable extra structure an algebraic $K$-theory space. These spaces are typically infinite loop spaces in a natural way, i.e., they arise from an $\Omega$-spectrum. One very general framework is Waldhausen's $S_\cdot$-*construction* which accepts *categories with cofibrations and weak equivalences* as input and which produces symmetric spectra which are positive $\Omega$-spectra.

We consider a category $\mathcal{C}$ with cofibrations and weak equivalences in the sense of Waldhausen [**66**]. For any finite set $Q$ we denote by $\mathcal{P}(Q)$ the power set of $Q$ viewed as a poset under inclusions, and thus as a category. A *$Q$-cube* in $\mathcal{C}$ is a functor $X : \mathcal{P}(Q) \longrightarrow \mathcal{C}$. Such a $Q$-cube $X$ is a *cofibration cube* if for all $S \subset T \subset Q$ the canonical map

$$\operatorname{colim}_{S \subset U \subsetneq T} X(U) \;\longrightarrow\; X(T)$$

is a cofibration in $\mathcal{C}$. (The colimit on the left can be formed by iterated pushouts along cofibrations, so it exists in $\mathcal{C}$.)

We view the ordered set $[n] = \{0 < 1 < \cdots < n\}$ as a category. If $n = \{n_s\}_{s \in Q}$ is a $Q$-tuple of non-negative integers, we denote by $[n]$ the product category of the categories $[n_s]$, $s \in Q$. For a morphism $i \to j$ in $[n]$ and a subset $U \subset Q$ we let $(i \to j)_U$ be the new morphisms in $[n]$ whose $s$th component is $i_s \to j_s$ if $s \in U$ and the identity $i_s \to i_s$ if $s \notin U$. Then for each morphism $i \to j$ in $[n]$, the assignment

$$U \;\mapsto\; (i \to j)_U$$

defines a $Q$-cube in the arrow category $\operatorname{Ar}[n]$.

For a finite set $Q$ and a $Q$-indexed tuple $n = \{n_s\}_{s \in Q}$ we define a category $S_n^Q \mathcal{C}$ as the full subcategory of the category of functors from $\operatorname{Ar}[n]$ to $\mathcal{C}$ consisting of the functors

$$A : \operatorname{Ar}[n] \longrightarrow \mathcal{C}\ , \quad (i \to j) \mapsto A_{i \to j}$$

with the following properties:

(i) if some component $i_s \to j_s$ of $i \to j$ is an identity (i.e., if $i_s = j_s$ for some $s \in Q$), then $A_{i \to j} = *$ is the distinguished zero object of $\mathcal{C}$;

(ii) for every pair of composable morphisms $i \to j \to k$ the cube

$$U \mapsto A_{(j \to k)_U \circ (i \to j)}$$

is a cofibration cube

(iii) for every pair of composable morphisms $i \to j \to k$ the square

$$\begin{tikzcd}
\operatorname{colim}_{U \subsetneq Q} A_{(j \to k)_U \circ (i \to j)} \arrow[r] \arrow[d] & A_{i \to k} \arrow[d] \\
* \arrow[r] & A_{j \to k}
\end{tikzcd}$$

is a pushout in $\mathcal{C}$.

The category $S_n^Q \mathcal{C}$ depends contravariantly on $[n]$, so that as $[n]$ varies, we get a $Q$-simplicial category $S_\cdot^Q \mathcal{C}$. We can make $S_\cdot^Q \mathcal{C}$ into a $Q$-simplicial object of categories with cofibrations and weak equivalences as follows. A morphism $f : A \longrightarrow A'$ is a *cofibration* in $S_n^Q \mathcal{C}$ if for every pair of composable morphisms $i \to j \to k$ the induced map of $Q$-cubes

$$\big(U \mapsto A_{(j \to k)_U \circ (i \to j)}\big) \;\longrightarrow\; \Big(U \mapsto A'_{(j \to k)_U \circ (i \to j)}\Big)$$

is a cofibration cube when viewed as a $(|Q|+1)$-cube in $\mathcal{C}$. A morphism $f : A \longrightarrow A'$ is a *weak equivalence* in $S_n^Q \mathcal{C}$ if for every morphism $i \longrightarrow j$ in $[n]$ the morphism $f_{i \longrightarrow j}$ is a weak equivalence in $\mathcal{C}$. If $Q$ has one element, the $S_\cdot^Q \mathcal{C}$ is isomorphic to $S_\cdot \mathcal{C}$ as defined by Waldhausen [**66**]. If $P \subset Q$ there is an isomorphism of $Q$-simplicial categories with cofibrations and weak equivalences

$$S_\cdot^Q \mathcal{C} \;\cong\; S_\cdot^{Q-P}(S_\cdot^P \mathcal{C})$$

[define]. So a choice of linear ordering of the set $Q$ specifies an isomorphism of categories

$$S_\cdot^Q \mathcal{C} \;\cong\; S_\cdot \cdots S_\cdot \mathcal{C}$$

to the $|Q|$-fold iterate of the $S_\cdot$-construction. Note that the permutation group of the set $Q$ acts on $S_\cdot^Q \mathcal{C}$ by permuting the indices.

Now we are ready to define the *algebraic $K$-theory spectrum* $K(\mathcal{C})$ of the category with cofibrations and weak equivalences $\mathcal{C}$. (This is really naturally a coordinate free symmetric spectrum in the sense of Remark 1.11.) It is the symmetric spectrum of simplicial sets with $n$th level given by

$$K(\mathcal{C})_n \;=\; N_\cdot\left(wS_\cdot^{\{1,\dots,n\}}\mathcal{C}\right) \;,$$

i.e., the nerve of the subcategory of weak equivalences in $S_\cdot^Q\mathcal{C}$ for the special case $Q = \{1,\dots,n\}$. The basepoint is the object of $S_\cdot^{\{1,\dots,n\}}\mathcal{C}$ given by the constant functor with values the distinguished zero object. The group $\Sigma_n$ of permutations of the set $\{1,\dots,n\}$, acts on $S_\cdot^{\{1,\dots,n\}}\mathcal{C}$ preserving weak equivalences, so it acts on the simplicial set $K(\mathcal{C})_n$. Note that $K(\mathcal{C})_0$ is the nerve of the category $w\mathcal{C}$ of weak equivalences in $\mathcal{C}$.

We still have to define the $\Sigma_n \times \Sigma_m$-equivariant structure maps

$$K(\mathcal{C})_n \wedge S^m \;\longrightarrow\; K(\mathcal{C})_{n+m} \;.$$

Consider a biexact functor $\wedge : \mathcal{C}\times\mathcal{D} \longrightarrow \mathcal{E}$ between categories with cofibrations and weak equivalences. For disjoint finite subsets $Q$ and $Q'$ we obtain a biexact functor of $(Q\cup Q')$-simplicial categories with cofibrations and weak equivalences

$$\wedge \;:\; S_\cdot^Q\mathcal{C} \times S_\cdot^{Q'}\mathcal{D} \;\longrightarrow\; S_\cdot^{Q\cup Q'}\mathcal{E}$$

by assigning

$$(A \wedge A')_{i\cup i' \to j\cup j'} \;=\; A_{i\to j} \wedge A'_{i'\to j} \;.$$

We specialize to $Q = \{1,\dots,n\}$ and $Q' = \{n+1,\dots,n+m\}$, restrict to weak equivalences and take nerves. This yields a $\Sigma_n \times \Sigma_m$-equivariant map $K(\mathcal{C})_n \times K(\mathcal{D})_m \longrightarrow K(\mathcal{E})_{n+m}$ which factors as

$$K(\mathcal{C})_n \wedge K(\mathcal{D})_m \;\longrightarrow\; K(\mathcal{E})_{n+m} \;.$$

These maps are associative for strictly associative pairings [explain].

The universal example of a category with cofibrations and weak equivalences which acts on any other such category is the category $\Gamma$ of finite pointed sets $n^+ = \{0,1,\dots,n\}$ with 0 as basepoint, and pointed set maps. Here the cofibrations are the injective maps and the weak equivalences are the bijections. The 'smash product' functor

$$\wedge \;:\; \Gamma\times\Gamma \;\longrightarrow\; \Gamma \;,\; (n^+, m^+) \mapsto (nm)^+$$

is biexact and strictly associative so it makes the symmetric sequence $\{K(\Gamma)_n\}_{n\geq 0}$ into a strict monoid of symmetric sequences. Here we identify $n^+ \wedge m^+$ with $(nm)^+$ using the lexicographic ordering. We define a map of simplicial sets

$$\iota_0 \;:\; S^0 \;\longrightarrow\; K(\Gamma)_0 = N_\cdot i\Gamma$$

by sending the basepoint to the identity map of the set $0^+$ and the non-basepoint to the identity map of the set $1^+$. The unit map

$$\iota_1 \;:\; S^1 \;\longrightarrow\; K(\Gamma)_1 = N_\cdot iS_\cdot^{\mathbf{1}}\Gamma$$

sends the non-degenerate 1-simplex of $S^1$ to the identity map of the set $1^+$, which is a 1-simplex of the nerve of the category $iS_\cdot^{\mathbf{1}}\Gamma$ [check; remark that $\mathbb{S} \longrightarrow K(\Gamma)$ is a $\pi_*$-isomorphism].

A biexact action

$$\wedge \;:\; \mathcal{C}\times\Gamma \;\longrightarrow\; \mathcal{C}$$

is given by sending $(A, n^+)$ to a choice of $n$-fold coproduct of the object $A$. This makes $K(\mathcal{C})$ into a module over the symmetric ring spectrum $K(\Gamma)$; in particular, the structure maps

$$K(\mathcal{C})_n \wedge S^1 \xrightarrow{\mathrm{Id}\wedge\iota_1} K(\mathcal{C})_n \wedge K(\Gamma)_1 \longrightarrow K(\mathcal{C})_{n+1}$$

make $K(\mathcal{C})$ into a symmetric spectrum.

### 2.2. Constructions.

Example 2.12 (Free symmetric spectra). Given a pointed space $K$ and $m \geq 0$, we define a symmetric spectrum $F_m K$ which is 'freely generated by $K$ in level $m$'. So technically we construct a left adjoint $F_m : \mathcal{T} \longrightarrow \mathcal{S}p^{\Sigma}$ to the forgetful functor which takes a symmetric spectrum $X$ to the pointed space $X_m$. The $n$th level of $F_m K$ is given, as a pointed $\Sigma_n$-space by

$$(F_m K)_n \;=\; \Sigma_n^+ \wedge_{1\times\Sigma_{n-m}} K \wedge S^{n-m} \ .$$

Here $1 \times \Sigma_{n-m}$ is the subgroup of $\Sigma_n$ of permutations which fix the first $m$ elements. The structure map $\sigma_n : (F_m K)_n \wedge S^1 \longrightarrow (F_m K)_{n+1}$ is given by

$$\begin{aligned} \left(\Sigma_n^+ \wedge_{1\times\Sigma_{n-m}} K \wedge S^{n-m}\right) \wedge S^1 &\longrightarrow \Sigma_{n+1}^+ \wedge_{1\times\Sigma_{n+1-m}} K \wedge S^{n-m+1} \\ [\tau, k \wedge x_1 \wedge \cdots \wedge x_{n-m}] \wedge x_{n-m+1} &\longmapsto [\tau, k \wedge x_1 \wedge \cdots \wedge x_{n-m} \wedge x_{n-m+1}] \end{aligned}$$

Free symmetric spectra generated in level zero are just suspension spectra, i.e., there is a natural isomorphism $F_0 K \cong \Sigma^\infty K$.

We calculate the 0th stable homotopy group of the symmetric spectrum $F_1 S^1$. Explicitly, $F_1 S^1$ is given by

$$(F_1 S^1)_n \;=\; \Sigma_n^+ \wedge_{1\times\Sigma_{n-1}} S^1 \wedge S^{n-1} \ .$$

So $(F_1 S^1)_n$ is a wedge of $n$ copies of $S^n$ and in the stable range, i.e., up to roughly dimensions $2n$, the homotopy groups of $(F_1 S^1)_n$ are a direct sum of $n$ copies of the homotopy groups of $S^n$. Moreover, in the stable range, the map in the colimit system (1.5) is a direct summand inclusion into $(n+1)$ copies of the homotopy groups of $S^n$. Thus in the colimit, the stable homotopy groups of the symmetric spectrum $F_1 S^1$ are a countably infinite direct sum of copies of the stable homotopy groups of spheres. In particular, $F_1 S^1$ is not $\pi_*$-isomorphic to the sphere spectrum $\mathbb{S}$, whose zeroth homotopy group is a single copy of the integers. Still, as we shall explain in Example 4.2 of Chapter II, $F_1 S^1$ represents the same stable homotopy type as the sphere spectrum $\mathbb{S}$. More generally, the free symmetric spectrum $F_n K$ is stably equivalent to an $n$-fold desuspension of the suspension spectrum of $K$.

Example 2.13 (Semifree symmetric spectra). There are somewhat 'less free' symmetric spectra starting from a pointed $\Sigma_m$-space $L$ which we want to install in level $m$, and then fill in the remaining data of a symmetric spectrum in the freest possible way. In other words, we claim that the forgetful functor

$$\mathcal{S}p \longrightarrow \Sigma_m\text{-}\mathcal{T} \ , \quad X \longmapsto X_m$$

has a left adjoint which we denote $G_m$; we refer to $G_m L$ as the *semifree symmetric spectrum* generated by $L$ in level $m$. In level $n$ we have

$$(G_m L)_n \;=\; \Sigma_n^+ \wedge_{\Sigma_m \times \Sigma_{n-m}} L \wedge S^{n-m} \ .$$

The structure map $\sigma_n : (G_m K)_n \wedge S^1 \longrightarrow (G_m K)_{n+1}$ is defined by the same tautological formula as in the previous example. In fact every free symmetric spectrum is semifree, i.e., there is a natural isomorphism $F_m K \cong G_m(\Sigma_m^+ \wedge K)$.

Example 2.14 (Limits and colimits). The category of symmetric spectra has all limits and colimits, and they are defined levelwise. Let us be a bit more precise and consider a functor $F : J \longrightarrow \mathcal{S}p^{\Sigma}$ from a small category $J$ to the category of symmetric spectra. Then we define a symmetric spectrum $\mathrm{colim}_J F$ in level $n$ by

$$(\mathrm{colim}_J F)_n \;=\; \mathrm{colim}_{j\in J} F(j)_n \ ,$$

the colimit being taken in the category of pointed $\Sigma_n$-spaces. The structure map is the composite

$$(\mathrm{colim}_{j\in J} F(j)_n) \wedge S^1 \cong \mathrm{colim}_{j\in J}(F(j)_n \wedge S^1) \xrightarrow{\mathrm{colim}_J \sigma_n} \mathrm{colim}_{j\in J} F(j)_{n+1} \ ;$$

here we exploit that smashing with $S^1$ is a left adjoint, and thus the natural map $\mathrm{colim}_{j\in J}(F(j)_n \wedge S^1) \longrightarrow (\mathrm{colim}_{j\in J} F(j)_n) \wedge S^1$ is a homeomorphism, whose inverse is the first map above.

The argument for inverse limits is similar, but we have to use that structure maps can also be defined in the adjoint form. We can take

$$(\lim_J F)_n \;=\; \lim_{j\in J} F(j)_n \ ,$$

and the structure map is adjoint to the composite

$$\lim_{j\in J} F(j)_n \xrightarrow{\lim_J \hat\sigma_n} \lim_{j\in J} \Omega(F(j)_{n+1}) \cong \Omega\left(\lim_{j\in J} F(j)_{n+1}\right) \ .$$

[Same for modules]

The inverse limit, calculated levelwise, of a diagram of symmetric *ring* spectra and homomorphisms is again a symmetric ring spectrum. In other words, symmetric ring spectra have limits and the forgetful functor to symmetric spectra preserves them. Symmetric ring spectra also have *co*-limits, but they are not preserved by the forgetful functor.

Example 2.15 (Smash products with spaces). If $K$ is pointed space and $X$ a symmetric spectrum, we define a new symmetric spectrum $K \wedge X$ by smashing $K$ levelwise with the terms of $X$, i.e., $(K \wedge X)_n = K \wedge X_n$. The structure map does not interact with the space $K$, i.e., it is given by the composite

$$(K \wedge X)_n \wedge S^1 \;=\; K \wedge X_n \wedge S^1 \xrightarrow{\mathrm{Id}\wedge\sigma_n} K \wedge X_{n+1} \;=\; (K \wedge X)_{n+1} \ .$$

For example, the spectrum $K \wedge \mathbb{S}$ is equal to the suspension spectrum $\Sigma^\infty K$. [Same for modules]

Example 2.16 (Suspension). A special case of the previous construction is the *suspension* $S^1 \wedge X$ of a symmetric spectrum $X$. So $S^1 \wedge X$ is defined by applying the functor $S^1 \wedge -$ levelwise, where the structure maps do not interact with the new suspension coordinate.

We claim that suspension simply shifts the homotopy groups of a symmetric spectrum $X$. The maps $S^1 \wedge - : \pi_{k+n} X_n \longrightarrow \pi_{1+k+n}(S^1 \wedge X_n)$ for varying $n$ form part of a diagram

(2.17)
$$\begin{tikzcd}
\pi_{k+n} X_n \arrow[r, "S^1\wedge -"] \arrow[d, "-\wedge S^1"'] & \pi_{1+k+n}(S^1 \wedge X_n) \arrow[d, "-\wedge S^1"] \arrow[dl, dotted, "(-1)^{k+n}\,\tau_*"] \\
\pi_{k+n+1}(X_n \wedge S^1) \arrow[d, "(\sigma_n)_*"'] & \pi_{1+k+n+1}(S^1 \wedge X_n \wedge S^1) \arrow[d, "(S^1\wedge\sigma_n)_*"] \\
\pi_{k+n+1} X_{n+1} \arrow[r, "S^1\wedge -"'] & \pi_{1+k+n+1}(S^1 \wedge X_{n+1})
\end{tikzcd}$$

whose outer rectangle commutes. This shows first of all that the maps $S^1 \wedge -$ are compatible with the stabilization process for the homotopy groups of $X$ respectively $S^1 \wedge X$, and thus induce a natural map

$$S^1 \wedge - \;:\; \pi_k X \;\longrightarrow\; \pi_{1+k}(S^1 \wedge X) \ .$$

We claim that this map is an isomorphism.

For this purpose we consider the diagonal dotted morphism which involves the twist isomorphism $\tau : X_n \wedge S^1 \longrightarrow S^1 \wedge X_n$ which interchanges the two factors in a smash product. This dotted morphism makes the upper left triangle in diagram (2.17) commute, which implies that the map $S^1 \wedge - : \pi_k X \longrightarrow \pi_{1+k}(S^1 \wedge X)$ is injective.

However, the dotted morphism does *not* in general make the lower right portion of diagram (2.17) commute as it stands, but it does after another suspension. Indeed, if we ignore the signs for a moment, then the two ways from $\pi_{1+k+n}(S^1 \wedge X_n)$ to $\pi_{1+k+n+1}(S^1 \wedge X_{n+1})$ send the class of a map $\alpha : S^{1+k+n} \longrightarrow S^1 \wedge X_n$

to the respective composites in the square

$$\begin{tikzcd}
S^{1+k+n+1} \arrow[r, "\alpha\wedge S^1"] \arrow[d, "S^1\wedge\chi_{k+n,1}"'] & S^1\wedge X_n\wedge S^1 \arrow[dddl, equal] \arrow[ddd, "S^1\wedge\sigma_n"] \\
S^{1+1+k+n} \arrow[d, "S^1\wedge\alpha"'] \arrow[r, phantom, "\not\circlearrowleft"] & \\
S^1\wedge S^1\wedge X_n \arrow[d, "S^1\wedge\tau"'] & \\
S^1\wedge X_n\wedge S^1 \arrow[r, "S^1\wedge\sigma_n"'] & S^1\wedge X_{n+1}.
\end{tikzcd}$$

The upper left part of this diagram does *not* commute ! The two composites from $S^{1+k+n+1}$ to $S^1\wedge X_n\wedge S^1$ differ by the automorphisms of $S^{1+k+n+1}$ and $S^1\wedge X_n\wedge S^1$ which interchanges the outer two sphere coordinates in each case. This coordinate change in the source induces multiplication by $-1$; the coordinate change in the target is a map of degree $-1$, so after a single suspension it also induces multiplication by $-1$ on homotopy groups (see Lemma 2.3).

EXAMPLE 2.18 (Shift). There is another construction for symmetric spectra which, like the suspension, reindexes the homotopy groups. The *shift* of a symmetric spectrum $X$ is given by

$$(\mathrm{sh}\, X)_n \;=\; X_{1+n}$$

with action of $\Sigma_n$ via the monomorphism $(1\times -):\Sigma_n\longrightarrow\Sigma_{1+n}$ which is explicitly given by $(1\times\gamma)(1)=1$ and $(1\times\gamma)(i)=\gamma(i-1)+1$ for $2\le i\le 1+n$. The structure maps of $\mathrm{sh}\, X$ are the reindexed structure maps for $X$.

For any symmetric spectrum $X$, integer $k$ and large enough $n$ we have

$$\pi_{(k+1)+n}(\mathrm{sh}\, X)_n \;=\; \pi_{k+(1+n)}X_{1+n}\ ,$$

and the maps in the colimit system for $\pi_{k+1}(\mathrm{sh}\, X)$ are the same as the maps in the colimit system for $\pi_k X$. Thus we get $\pi_{k+1}(\mathrm{sh}\, X)=\pi_k X$.

Warning: the suspension and shift construction both shift the homotopy groups, but, however, there is in general no morphism between $S^1\wedge X$ and $\mathrm{sh}\, X$ which induces an isomorphism of homotopy groups. This is closely related to the phenomenon of 'semistability', which we discuss in more detail in Section 4.5. There is an important natural morphism $\lambda: S^1\wedge X\longrightarrow \mathrm{sh}\, X$ for every symmetric spectrum $X$ whose level $n$ component $\lambda_n: S^1\wedge X_n\longrightarrow X_{1+n}$ is the composite

$$S^1\wedge X_n \xrightarrow[\text{twist}]{\cong} X_n\wedge S^1 \xrightarrow{\sigma_n} X_{n+1} \xrightarrow{\chi_{n,1}} X_{1+n}\ . \tag{2.19}$$

One should note that using only the structure map $\sigma_n$ without the twist isomorphism and permutation $\chi_{n,1}=(1,\dots,n,n+1)$ does not yield a morphism of symmetric spectra ! The morphism $\lambda$ is *not* in general a $\pi_*$-isomorphism, but when it is, the symmetric spectrum $X$ is called *semistable* (compare Theorem 4.44).

We can iterate the shift construction and get $(\mathrm{sh}^m X)_n = X_{m+n}$. In every level of the symmetric spectrum $\mathrm{sh}^m X$ the symmetric group $\Sigma_m$ acts via the 'inclusion' $(-\times 1):\Sigma_m\longrightarrow\Sigma_{m+n}$, and these actions are compatible with the structure maps. So in this way $\mathrm{sh}^m X$ becomes a $\Sigma_m$-symmetric spectrum.

EXAMPLE 2.20 (Twisted smash product). The twisted smash product starts from a number $m\ge 0$, a pointed $\Sigma_m$-space (or $\Sigma_m$-simplicial set) $L$ and a symmetric spectrum $X$ and produces a new symmetric spectrum which we denote $L\triangleright_m X$. This construction is a simultaneous generalization of semifree symmetric spectra (Example 2.13) and the smash product of a space and a spectrum (Example 2.15) and also provides a left adjoint to the shift construction (Example 2.18). Once the internal smash product of symmetric spectra is available, we will identify the twisted smash product $L \rhd_m X$ with the smash product of $G_m L$ and $X$, see Proposition 3.5 below.

We define the twisted smash product $L \rhd_m X$ as a point in levels smaller than $m$ and in general by

$$(L \rhd_m X)_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n \ .$$

The structure map $\sigma_{m+n} : (L \rhd_m X)_{m+n} \wedge S^1 \longrightarrow (L \rhd_m X)_{m+n+1}$ is obtained from $\mathrm{Id} \wedge \sigma_n : L \wedge X_n \wedge S^1 \longrightarrow L \wedge X_{n+1}$ by inducing up.

Here are some special cases. Taking $X = \mathbb{S}$ gives semifree and free symmetric spectra as

$$L \rhd_m \mathbb{S} \ = \ G_m L \quad \text{respectively} \quad (\Sigma^+_m \wedge K) \rhd_m \mathbb{S} \ = \ F_m K \ .$$

For $m = 0$ we get

$$K \rhd_0 X \ = K \wedge X \ ,$$

the levelwise smash product of $K$ and $X$. The twisted smash product has an associativity property in the form of a natural isomorphism

$$L \rhd_m (L' \rhd_n X) \ \cong \ (\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L') \rhd_{m+n} X \ .$$

The twisted smash product is related by various adjunctions to other constructions. As we noted at the end of Example 2.18, the $m$-fold shift of a symmetric spectrum $Z$ has an action of $\Sigma_m$ through spectrum automorphisms, i.e., $\mathrm{sh}^m Z$ is a $\Sigma_m$-symmetric spectrum. The levelwise smash product $L \wedge X$ (in the sense of Example 2.15) of the underlying space of $L$ and $X$ also is a $\Sigma_m$-symmetric spectrum through the action on $L$. Given a morphism $f : L \rhd_m X \longrightarrow Z$ of symmetric spectra, we can restrict the component in level $m+n$ to the summand $1 \wedge L \wedge X_n$ in $(L \rhd_m X)_{m+n}$ and obtain a $\Sigma_m \times \Sigma_n$-equivariant based map $\bar{f}_n = f_{m+n}(1 \wedge -) : L \wedge X_n \longrightarrow Z_{m+n} = (\mathrm{sh}^m Z)_n$. The compatibility of the $f_{m+n}$'s with the structure maps translates into the property that the maps $\bar{f} = \{\bar{f}_n\}_{n \geq 0}$ form a morphism of $\Sigma_m$-symmetric spectra from $L \wedge X$ to $\mathrm{sh}^m Z$. Conversely, every $\Sigma_m$-equivariant morphism $L \wedge X \longrightarrow \mathrm{sh}^m Z$ arises in this way from a morphism $f : L \rhd_m X \longrightarrow Z$. In other words, the assignment $f \mapsto \bar{f}$ is a natural bijection of functors

$$\mathcal{S}p^\Sigma(L \rhd_m X, Z) \ \cong \ \Sigma_m\text{-}\mathcal{S}p^\Sigma(L \wedge X, \mathrm{sh}^m Z) \ . \tag{2.21}$$

The case $m = 1$ and $L = S^0$ gives a bijection,

$$\mathcal{S}p^\Sigma(S^0 \rhd_1 X, Z) \ \cong \ \mathcal{S}p^\Sigma(X, \mathrm{sh}\, Z) \ ,$$

natural in the symmetric spectra $X$ and $Z$, which shows that $X \mapsto S^0 \rhd_1 X$ is left adjoint to shifting. [Same for modules]

Example 2.22 (Function spectra). If $X$ is a symmetric spectrum and $K$ a pointed space, we define a symmetric function spectrum $X^K$ by

$$(X^K)_n \ = \ X^K_n \ = \ \mathrm{map}(K, X_n)$$

with $\Sigma_n$-action induced by the action on $X_n$. The structure map is the composite

$$X^K_n \wedge S^1 \ \longrightarrow \ (X_n \wedge S^1)^K \ \longrightarrow \ X^K_{n+1}$$

where the first map is adjoint to the evaluation map $X^K_n \wedge S^1 \wedge K \longrightarrow X_n \wedge S^1$ and the second is application of $\mathrm{map}(K, -)$ to the structure map of $X$. [Same for modules]

We note that if $X$ is an $\Omega$-spectrum, then so is $X^K$, provided we also assume that

- $K$ is a CW-complex (when in the context of topological spaces), or
- $X$ is levelwise a Kan complex (when in the context of simplicial sets).

Indeed, under either hypothesis, the map mapping space functor $\mathrm{map}(K, -)$ takes the weak equivalence $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ to a weak equivalence

$$X^K_n = \mathrm{map}(K, X_n) \ \xrightarrow{\mathrm{map}(K, \tilde{\sigma}_n)} \ \mathrm{map}(K, \Omega X_{n+1}) \cong \Omega(X^K_{n+1}) \ .$$

If $R$ is a symmetric ring spectrum and $L$ an *unpointed* space, then $R^{L^+}$ is again a symmetric ring spectrum. The multiplication maps $R_n^{L^+} \wedge R_m^{L^+} \longrightarrow R_{n+m}^{L^+}$ are the composites

$$\mathrm{map}(L^+, R_n) \wedge \mathrm{map}(L^+, R_m) \xrightarrow{\wedge} \mathrm{map}(L^+ \wedge L^+, R_n \wedge R_m) \xrightarrow{\mathrm{map}(\mathrm{diagonal}, \mu_{n,m})} \mathrm{map}(L^+, R_{n+m})$$

using the diagonal map $L^+ \longrightarrow (L \times L)^+ \cong L^+ \wedge L^+$. Associativity of the multiplication on $R^{L^+}$ comes from associativity of $R$ and coassociativity of the diagonal map. The unit map $\iota_n : S^n \longrightarrow R_n^{L^+}$ is the composite of the unit map of $R$ with the map $R_n \cong R_n^{S^0} \longrightarrow R_n^{L^+}$ induced by the based map $L^+ \longrightarrow S^0$ which takes all of $L$ to the non-basepoint of $S^0$. If the multiplication of $R$ is commutative, then so is the multiplication of $R^{L^+}$, since the diagonal map is cocommutative.

In good cases (namely if $X$ is semistable), the homotopy groups of $X^K$ are the values of the reduced cohomology theory represented by $X$ on the space $K$, i.e,

$$\pi_k(X^K) \cong \tilde{X}^{-k}(K) .$$

A concrete example if $X = HA$, the Eilenberg-Mac Lane spectrum of an abelian group $A$. Then

$$\pi_k(HA^K) \cong \tilde{H}^{-k}(K; A) ,$$

the reduced singular cohomology of $K$ with coefficients in $A$. If $A$ is a ring, then $HA$ becomes a ring spectrum and this isomorphism takes the product of homotopy groups to the cup product in singular cohomology.

Example 2.23 (Loops). The loop $\Omega X$ of a symmetric spectrum $X$ is defined by applying the functor $\Omega = \mathrm{map}(S^1, -)$ levelwise, where the structure maps do not interact with the new loop coordinate. In other words, $\Omega X$ is the special case of a function spectrum as in the previous example. We claim that looping shifts the homotopy groups.

We use the isomorphism $\alpha : \pi_{k+n}\Omega(X_n) \cong \pi_{1+k+n}X_n$ defined by sending a representing continuous map $f : S^{k+n} \longrightarrow \Omega(X_n)$ to the class of the adjoint $\hat{f} : S^{1+k+n} \longrightarrow X_n$ given by $\hat{f}(s \wedge t) = f(t)(s)$, where $s \in S^1$, $t \in S^{k+n}$. As $n$ varies, these particular isomorphisms are compatible with stabilization maps, so they induce an isomorphism

$$\alpha \ : \ \pi_k(\Omega X) \xrightarrow{\cong} \pi_{1+k}X$$

on colimits. The composite

$$\pi_* X \xrightarrow{S^1 \wedge -} \pi_{1+*}(S^1 \wedge X) \xrightarrow{\alpha^{-1}} \pi_*(\Omega(S^1 \wedge X))$$

is the map induced by the adjunction unit $X \longrightarrow \Omega(S^1 \wedge X)$ on homotopy, where $S^1 \wedge -$ is the isomorphism of Example 2.16. Similarly, the composite

$$\pi_{1+*}(S^1 \wedge (\Omega X)) \xrightarrow{(S^1 \wedge -)^{-1}} \pi_*(\Omega X) \xrightarrow{\alpha} \pi_{1+*}X$$

is the map induced by the adjunction counit $S^1 \wedge \Omega X \longrightarrow X$ on homotopy.

What we said about the loop spectrum works as well for symmetric spectra of simplicial sets as long as they are levelwise Kan complexes.

Example 2.24 (Mapping spaces). There is a whole space, respectively simplicial set, of morphisms between two symmetric spectra. For symmetric spectra $X$ and $Y$ of topological spaces, every morphism from $f : X \longrightarrow Y$ consists of a family of based continuous maps $\{f_n : X_n \longrightarrow Y_n\}_{n \geq 0}$ which satisfy some conditions. So the set of morphisms from $X$ to $Y$ is a subset of the product of mapping spaces $\prod_{n \geq 0} \mathcal{T}(X_n, Y_n)$ and we give it the subspace topology of the (compactly generated) product topology. We denote this mapping space by $\mathrm{map}(X, Y)$.

Now suppose that $X$ and $Y$ are symmetric spectra of simplicial sets. Then the mapping space $\mathrm{map}(X, Y)$ is the simplicial set whose $n$-simplices are the spectrum morphisms from $\Delta[n]^+ \wedge X$ to $Y$. For a monotone map $\alpha : [n] \longrightarrow [m]$ in the simplicial category $\Delta$, the map $\alpha^* : \mathrm{map}(X, Y)_m \longrightarrow \mathrm{map}(X, Y)_n$ is given by precomposition with $\alpha_* \wedge \mathrm{Id}_X : \Delta[n]^+ \wedge X \longrightarrow \Delta[m]^+ \wedge X$. The morphism space has a natural basepoint, namely the trivial map from $\Delta[0]^+ \wedge X$ to $Y$. We can, and will, identify the vertices of $\mathrm{map}(X,Y)$ with the morphisms from $X$ to $Y$ using the natural isomorphism $\Delta[0]^+ \wedge X \cong X$.

The topological and simplicial mapping spaces are related by various adjunctions. We list some of these. For a simplicial spectrum $X$ and a topological spectrum $Y$ there is an isomorphism of simplicial sets

$$\mathrm{map}(X, \mathcal{S}(Y)) \cong \mathcal{S}(\mathrm{map}(|X|, Y))$$

which on vertices specializes to the adjunction between singular complex and geometric realization.

Furthermore, for a pointed space $K$ and topological symmetric spectra $X$ and $Y$ we have adjunction homeomorphisms

$$\mathcal{T}(K, \mathrm{map}(X,Y)) \cong \mathcal{T}(K \wedge X, Y) \cong \mathrm{map}(X, Y^K) .$$

For free symmetric spectra we have homeomorphisms

$$\mathrm{map}(F_m K, Y) \cong \mathcal{T}(K, Y_m) .$$

For $K = S^0$ this specializes to a homeomorphism $\mathrm{map}(F_m S^0, Y) \cong Y_m$. In the context of symmetric spectra of simplicial sets, the analogous isomorphisms of mapping simplicial sets hold as well.

We have associative and unital composition maps

$$\mathrm{map}(Y,Z) \wedge \mathrm{map}(X,Y) \longrightarrow \mathrm{map}(X,Z) .$$

Indeed, for symmetric spectra of topological spaces this is just the observation that composition of morphisms is continuous for the mapping space topology. For symmetric spectra of simplicial sets the composition maps are given on $n$-simplices by

$$\begin{aligned} \mathcal{S}p^\Sigma(\Delta[n]^+ \wedge Y, Z) \wedge \mathcal{S}p^\Sigma(\Delta[n]^+ \wedge X, Y) &\longrightarrow \mathcal{S}p^\Sigma(\Delta[n]^+ \wedge X, Z) \\ g \wedge f &\longmapsto g \circ (\mathrm{Id}_{\Delta[n]^+} \wedge f) \circ (\mathrm{diag}^+ \wedge \mathrm{Id}_X) \end{aligned}$$

Example 2.25 (Internal Hom spectra). Symmetric spectra have internal function objects: for symmetric spectra $X$ and $Y$ we define a symmetric spectrum $\mathrm{Hom}(X,Y)$ in level $n$ by

$$\mathrm{Hom}(X,Y)_n = \mathrm{map}(X, \mathrm{sh}^n Y)$$

with $\Sigma_n$-action induced by the action on $\mathrm{sh}^n Y$ as described in Example 2.18. The structure map $\sigma_n : \mathrm{Hom}(X,Y)_n \wedge S^1 \longrightarrow \mathrm{Hom}(X,Y)_{n+1}$ is the composite

$$\mathrm{map}(X, \mathrm{sh}^n Y) \wedge S^1 \to \mathrm{map}(X, S^1 \wedge \mathrm{sh}^n Y) \xrightarrow{\mathrm{map}(X, \lambda_{\mathrm{sh}^n Y})} \mathrm{map}(X, \mathrm{sh}^{1+n} Y) \xrightarrow{\mathrm{map}(X, \chi_{1,n})} \mathrm{map}(X, \mathrm{sh}^{n+1} Y)$$

where $\lambda_{\mathrm{sh}^n Y} : S^1 \wedge \mathrm{sh}^n Y \longrightarrow \mathrm{sh}(\mathrm{sh}^n Y)$ is the natural morphism defined in (2.19).

The internal function spectrum $\mathrm{Hom}(-,-)$ is adjoint to the internal smash product of symmetric spectra, to be discussed in Section 3. A natural isomorphism of symmetric spectra $\mathrm{Hom}(F_m S^0, Y) \cong \mathrm{sh}^m Y$ is given at level $n$ by

$$\mathrm{Hom}(F_m S^0, Y)_n = \mathrm{map}(F_m S^0, \mathrm{sh}^n Y) \cong (\mathrm{sh}^n Y)_m = Y_{n+m} \xrightarrow{\chi_{n,m}} Y_{m+n} = (\mathrm{sh}^m Y)_n .$$

In the special case $m = 0$ we have $F_0 S^0 = \mathbb{S}$, which gives a natural isomorphism of symmetric spectra $\mathrm{Hom}(\mathbb{S}, Y) \cong Y$.

Example 2.26 (Endomorphism ring spectra). For every symmetric spectrum $X$, the symmetric function spectrum $\mathrm{Hom}(X,X)$ defined in Example 2.25 has the structure of a symmetric ring spectrum which we call the *endomorphism ring spectrum* of $X$. The multiplication map $\mu_{n,m} : \mathrm{Hom}(X,X)_n \wedge \mathrm{Hom}(X,X)_m \longrightarrow \mathrm{Hom}(X,X)_{n+m}$ is defined as the composite

$$\begin{aligned} \mathrm{map}(X, \mathrm{sh}^n X) \wedge \mathrm{map}(X, \mathrm{sh}^m X) &\xrightarrow{\mathrm{sh}^m \wedge \mathrm{Id}} \mathrm{map}(\mathrm{sh}^m X, \mathrm{sh}^{m+n} X) \wedge \mathrm{map}(X, \mathrm{sh}^m X) \\ &\xrightarrow{\ \circ\ } \mathrm{map}(X, \mathrm{sh}^{m+n} X) \xrightarrow{\mathrm{map}(X, \chi_{m,n})} \mathrm{map}(X, \mathrm{sh}^{n+m} X) \end{aligned}$$

where the second map is the composition pairing of Example 2.24. While this construction always works on the pointset level, one can only expect $\mathrm{Hom}(X,X)$ to be homotopically meaningful under certain conditions on $X$. The stable model structures which we discuss later will explain which conditions are sufficient.

In much the same way as above we can define associative and unital action maps $\mathrm{Hom}(X,Z)_n \wedge \mathrm{Hom}(X,X)_m \longrightarrow \mathrm{Hom}(X,Z)_{n+m}$ and $\mathrm{Hom}(X,X)_n \wedge \mathrm{Hom}(Z,X)_m \longrightarrow \mathrm{Hom}(Z,X)_{n+m}$ for any other symmetric spectrum $Z$. This makes $\mathrm{Hom}(X,Z)$ and $\mathrm{Hom}(Z,X)$ into right respectively left modules over the endomorphism ring spectrum of $X$.

EXAMPLE 2.27 (Telescope and diagonal of a sequence). We will sometimes be confronted with a sequence of morphisms of symmetric spectra

$$X^0 \xrightarrow{f^0} X^1 \xrightarrow{f^1} X^2 \xrightarrow{f^2} \cdots \tag{2.28}$$

of which we want to take a kind of colimit in a homotopy invariant way, and such that the homotopy groups of the 'colimit' are the colimits of the homotopy groups. We describe two constructions which do this job, the mapping telescope and the diagonal.

The *mapping telescope* $\mathrm{tel}_i X^i$ of the sequence (2.28) is a classical construction for spaces which we apply levelwise to symmetric spectra. It is defined as the coequalizer of two maps of symmetric spectra

$$\bigvee_{i\geq 0} X^i \quad \rightrightarrows \quad \bigvee_{i\geq 0} [i,i+1]^+ \wedge X^i$$

Here $[i,i+1]$ denotes a copy of the unit interval (when in the context of spaces) respectively the 1-simplex $\Delta[1]$ (when in the context of simplicial sets). One of the morphisms takes $X^i$ to $\{i+1\}^+ \wedge X^i$ by the identity, the other one takes $X^i$ to $\{i+1\}^+ \wedge X^{i+1}$ by the morphism $f^i$.

The *diagonal* $\mathrm{diag}_i X^i$ of the sequence (2.28) is the symmetric spectrum given by

$$(\mathrm{diag}_i X^i)_n \;=\; X^n_n \;,$$

i.e., we take the $n$-th level of the $n$-th symmetric spectrum with its given $\Sigma_n$-action. The structure map $(\mathrm{diag}_i X^i)_n \wedge S^1 \longrightarrow (\mathrm{diag}_i X^i)_{n+1}$ is the composite around either way in the commutative square

$$\begin{tikzcd}
X^n_n \wedge S^1 \arrow[r, "\sigma^n_n"] \arrow[d, "f^n_n \wedge \mathrm{Id}"'] & X^n_{n+1} \arrow[d, "f^n_{n+1}"] \\
X^{n+1}_n \wedge S^1 \arrow[r, "\sigma^{n+1}_n"'] & X^{n+1}_{n+1}
\end{tikzcd}$$

LEMMA 2.29. *For every sequence of symmetric spectra* (2.28)*, the $k$-th homotopy group of the diagonal symmetric spectrum is naturally isomorphic to the colimit of the $k$-th homotopy groups of the spectra $X^i$, along the maps $\pi_k(f^i)$,*

$$\pi_k(\mathrm{diag}_i X^i) \;\cong\; \mathrm{colim}_i\, \pi_k(X^i)\;. \tag{2.30}$$

*For symmetric spectra of simplicial sets or symmetric spectra of non-degenerately based spaces there is a chain of two natural $\pi_*$-isomorphisms between the diagonal* $\mathrm{diag}_i X^i$ *and the mapping telescope* $\mathrm{tel}_i X^i$ *of the sequence.*

PROOF. The right hand side of (2.30) is a sequential colimit of groups which are themselves sequential colimits, and it is thus the colimit over the poset $\mathbb{N}\times\mathbb{N}$ of the functor $(n,i) \mapsto \pi_{k+n}(X^i_n)$. The left hand side $\pi_k(\mathrm{diag}_i X^i)$ equals the colimit over the diagonal terms in this system. Since the diagonal embedding $\mathbb{N} \longrightarrow \mathbb{N}\times\mathbb{N}$ is cofinal, the colimit over the diagonal terms is isomorphic to the colimit over $\mathbb{N}\times\mathbb{N}$, which proves the isomorphism (2.30).

To prove the second statement we use the 'partial telescopes' $\mathrm{tel}_{[0,n]} X^i$, the coequalizer of two maps of symmetric spectra

$$\bigvee_{i=0}^{n-1} X^i \quad \rightrightarrows \quad \bigvee_{i=0}^{n} [i,i+1]^+ \wedge X^i$$

defined as before. The spectrum $\mathrm{tel}_{[0,n]} X^i$ includes into the next spectrum $\mathrm{tel}_{[0,n+1]} X^i$ with (categorical) colimit the mapping telescope. The morphism $c_n : \mathrm{tel}_{[0,n]} X^i \longrightarrow X^n$ which projects each wedge summand

$[i, i+1]^+ \wedge X^i$ onto $X^i$ and then applies the morphism $f^{n-1} \cdots f^i : X^i \longrightarrow X^n$ is a homotopy equivalence. The commutative diagram of symmetric spectra

$$\begin{tikzcd}
\mathrm{tel}_{[0,0]} X^i \arrow[r] \arrow[d, "c_0"'] & \mathrm{tel}_{[0,1]} X^i \arrow[r] \arrow[d, "c_1"'] & \mathrm{tel}_{[0,2]} X^i \arrow[r] \arrow[d, "c_2"] & \cdots \\
X^0 \arrow[r, "f^0"'] & X^1 \arrow[r, "f^1"'] & X^2 \arrow[r, "f^2"'] & \cdots
\end{tikzcd}$$

induces a morphism

$$\mathrm{diag}_n(\mathrm{tel}_{[0,n]} X^i) \ \longrightarrow \ \mathrm{diag}_n X^n$$

on diagonals which is thus levelwise a homotopy equivalence, hence a $\pi_*$-isomorphism. On the other hand we have a morphism of symmetric spectra

$$\mathrm{diag}_n(\mathrm{tel}_{[0,n]} X^i) \ \longrightarrow \ \mathrm{tel}_i X^i$$

which is levelwise given by the inclusion of a partial telescope in the full mapping telescope. This morphism is a $\pi_*$-isomorphism by the same kind of cofinality argument as in the first part. Here we use that an (unstable) homotopy group of the mapping telescope of a sequence of simplicial sets or non-degenerately based spaces is the colimits of the sequence of homotopy groups. □

Both mapping telescope and diagonal preserve module structures. Suppose that each symmetric spectrum $X^i$ in the sequence (2.28) has the structure of right module over a symmetric ring spectrum $R$ and that all morphisms $f^i$ are $R$-linear. Then the mapping telescope is naturally an $R$-module since all constructions used to build it preserves the action by the ring spectrum. The diagonal is naturally an $R$-module as well, with action maps $(\mathrm{diag}_i X^i)_n \wedge R_m \longrightarrow (\mathrm{diag}_i X^i)_{n+m}$ equal to the composite around either way in the commutative square

$$\begin{tikzcd}
X^n_n \wedge R_m \arrow[r, "\alpha^n_{n,m}"] \arrow[d, "(f^{n+m-1}\cdots f^n)_n \wedge \mathrm{Id}"'] & X^n_{n+m} \arrow[d, "(f^{n+m-1}\cdots f^n)_{n+m}"] \\
X^{n+m}_n \wedge R_m \arrow[r, "\alpha^{n+m}_{n,m}"'] & X^{n+m}_{n+m}
\end{tikzcd}$$

Let me point out two advantages of the diagonal construction over the mapping telescope of a sequence of symmetric spectra. One advantage is that the formula (2.30) is proved formally, and it holds without any assumptions on the symmetric spectra involved. So when we work with symmetric spectra of topological spaces, no hypothesis about non-degenerate basepoints is needed. Another advantage is that the diagonal construction has nicer formal and in particular multiplicative properties, as we shall see, for example, in Examples 2.33 and 4.65.

Example 2.31 ($I$-spaces). Symmetric spectra are intimately related to the category $I$ of finite sets and injective maps. Here we denote by $I$ the category with objects the sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geq 0$ (where $\mathbf{0}$ is the empty set) and with morphisms all injective maps. In other words, $I$ is the subcategory of the category $\mathcal{F}in$ of standard finite sets (compare Remark 1.12) with only injective maps as morphisms. We denote by $\mathcal{T}^I$ the category of *$I$-spaces*, i.e., covariant functors from $I$ to the category of pointed spaces.

Given an $I$-space $T : I \longrightarrow \mathcal{T}$ and a symmetric spectrum $X$, we can form a new symmetric spectrum $T \wedge X$ by setting

$$(T \wedge X)_n \ = \ T(\mathbf{n}) \wedge X_n$$

with diagonal action of $\Sigma_n$ (which equals the monoid of endomorphism of the object $\mathbf{n}$ of $I$). The structure map is given by

$$(T \wedge X)_n \wedge S^1 \ = \ T(\mathbf{n}) \wedge X_n \wedge S^1 \ \xrightarrow{T(\iota) \wedge \sigma_n} \ T(\mathbf{n+1}) \wedge X_{n+1} \ = \ (T \wedge X)_{n+1}$$

where $\iota : \mathbf{n} \longrightarrow \mathbf{n+1}$ is the inclusion. If $K$ is a pointed space and $T$ the constant functor with values $K$, then $T \wedge X$ is equal to $K \wedge X$, i.e., this construction reduces to the pairing of Example 2.15.

Example 2.32 (Monoid ring spectra). If $R$ is a symmetric ring spectrum and $M$ a topological or simplicial monoid (depending on the kind of symmetric spectra), we can define a symmetric ring spectrum $R[M]$ by $R[M] = M^+ \wedge R$, i.e., the levelwise smash product with $M$ with a disjoint basepoint added. The unit map is the composite of the unit map of $R$ and the morphism $R \cong \{1\}^+ \wedge R \longrightarrow M^+ \wedge R$ induced by the unit of $M$. The multiplication map $\mu_{n,m}$ is given by the composite

$$(M^+ \wedge R_n) \wedge (M^+ \wedge R_m) \cong (M \times M)^+ \wedge (R_n \wedge R_m) \xrightarrow{\text{mult.}\wedge \mu_{n,m}} M^+ \wedge R_{n+m} \ .$$

If both $R$ and $M$ are commutative, then so is $R[M]$. A right module over the symmetric ring spectrum $R[M]$ amounts to the same data as an $R$-module together with a continuous (or simplicial) right action of the monoid $M$ by $R$-linear endomorphisms.

The construction of the monoid ring over $\mathbb{S}$ is left adjoint to the functor which takes a symmetric ring spectrum $R$ to the simplicial monoid $R_0$. If $R$ is semistable (see Theorem 4.44), then the homotopy groups of $R[M]$ are the $R$-homology groups of the underlying space of $M$, with the Pontryagin product as multiplication. In the special case of a *discrete* spherical monoid ring, the homotopy groups are the monoid ring, in the ordinary sense, of the homotopy groups, i.e., there is a natural isomorphism

$$\pi_* R[M] \ \cong \ (\pi_* R)[M] \ .$$

Example 2.33 (Ring spectra from multiplicative $I$-spaces). We can use the construction which pairs an $I$-space with a symmetric spectrum (see Example 2.31) to produce symmetric ring spectra which model the suspension spectra of certain infinite loop spaces such as $BO$, the classifying space of the infinite orthogonal group, even if these do not have a strictly associative multiplication. This works for infinite loop spaces which can be represented as 'monoids of $I$-spaces', as we now explain.

The symmetric monoidal sum operation restricts from the category $\mathcal{F}in$ of standard finite sets to the category $I$. Thus $I$ has a symmetric monoidal product '+' given by addition on objects and defined for morphisms $f : \mathbf{n} \longrightarrow \mathbf{n}'$ and $g : \mathbf{m} \longrightarrow \mathbf{m}'$ we define $f + g : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{n}' + \mathbf{m}'$ by

$$(f+g)(i) \ = \ \begin{cases} \ f(i) & \text{if } 1 \leq i \leq n, \text{ and} \\ g(i-n)+n' & \text{if } n+1 \leq i \leq n+m. \end{cases}$$

The product $+$ is strictly associative and has the object $\mathbf{0}$ as a strict unit. The symmetry isomorphism is the shuffle map $\chi_{n,m} : \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$.

Consider an $I$-space $T : I \longrightarrow \mathcal{T}$ with a pairing, i.e., an associative and unital natural transformation $\mu_{\mathbf{n},\mathbf{m}} : T(\mathbf{n}) \wedge T(\mathbf{m}) \longrightarrow T(\mathbf{n}+\mathbf{m})$. If $R$ is a symmetric ring spectrum, then the smash product $T \wedge R$ (see Example 2.31) becomes a symmetric ring spectrum with respect to the multiplication map

$$(T \wedge R)_n \wedge (T \wedge R)_m \ \longrightarrow \ (T \wedge R)_{n+m}$$

defined as the composite

$$T(\mathbf{n}) \wedge R_n \wedge T(\mathbf{m}) \wedge R_m \xrightarrow{\text{Id} \wedge \text{twist} \wedge \text{Id}} T(\mathbf{n}) \wedge T(\mathbf{m}) \wedge R_n \wedge R_m \xrightarrow{\mu_{\mathbf{n},\mathbf{m}} \wedge \mu_{n,m}} T(\mathbf{n}+\mathbf{m}) \wedge R_{n+m} \ .$$

If the transformation $\mu$ is commutative in the sense that the square

$$\begin{tikzcd} T(\mathbf{n}) \wedge T(\mathbf{m}) \arrow[r, "\mu_{\mathbf{n},\mathbf{m}}"] \arrow[d, "\text{twist}"'] & T(\mathbf{n}+\mathbf{m}) \arrow[d, "T(\chi_{n,m})"] \\ T(\mathbf{m}) \wedge T(\mathbf{n}) \arrow[r, "\mu_{\mathbf{m},\mathbf{n}}"'] & T(\mathbf{m}+\mathbf{n}) \end{tikzcd}$$

commutes for all $n, m \geq 0$ and if the multiplication of $R$ is commutative, then the product of $T \wedge R$ is also commutative. This construction generalizes monoid ring spectra (see Example 2.32): if $M$ is a topological (respectively simplicial) monoid, then the constant $I$-functor with values $M^+$ inherits an associative and unital product from $M$ which is commutative if $M$ is. The smash product of a ring spectrum $R$ with such a constant multiplicative functor equals the monoid ring spectrum $R[M]$.

Later we shall also discuss an internal 'tensor product' of $I$-spaces and an internal smash product of symmetric spectra and see that these constructions are intimately related.

A more interesting instance of this construction is a commutative symmetric ring spectrum which models the suspension spectrum of the space $BO^+$, the classifying space of the infinite orthogonal group with a disjoint basepoint. Here we start with the '$I$-topological group' $\mathbf{O}$, a functor from $I$ to topological groups whose value at $\mathbf{n}$ is $O(n)$, the $n$-th orthogonal group. The behavior on morphisms is determined by requiring that a permutation $\gamma \in \Sigma_n$ acts as conjugation by the permutation matrix associated to $\gamma$ and the inclusion $\iota : \mathbf{n} \longrightarrow \mathbf{n+1}$ induces

$$\iota_* \ : \ O(n) \ \longrightarrow \ O(n+1) \ , \quad A \ \longmapsto \ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \ .$$

A general injective set map $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ then induces the group homomorphism $\alpha_* : O(n) \longrightarrow O(m)$ given by

$$(\alpha_* A)_{i,j} \ = \ \begin{cases} A_{\alpha^{-1}(i),\alpha^{-1}(j)} & \text{if } i,j \in \mathrm{Im}(\alpha), \\ \quad 1 & \text{if } i=j \text{ and } i \notin \mathrm{Im}(\alpha), \\ \quad 0 & \text{if } i \neq j \text{ and } i \text{ or } j \text{ is not contained in } \mathrm{Im}(\alpha). \end{cases}$$

Orthogonal sum of matrices gives a natural transformation of group valued functors

$$\mathbf{O(n)} \times \mathbf{O(m)} \ \longrightarrow \ \mathbf{O(n+m)} \ , \quad (A,B) \ \longmapsto \ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \ .$$

This transformation is unital, associative and commutative, in a sense which by now is hopefully clear. The classifying space functor $B$ takes topological groups to pointed topological spaces and commutes with products up to unital, associative and commutative homeomorphism. So by taking classifying spaces objectwise we obtain an $I$-space $\mathbf{BO}$ with values $\mathbf{BO(n)} = BO(n)$. This $I$-space inherits a unital, associative and commutative product in the sense discussed above, but with respect to the cartesian product, as opposed to the smash product, of spaces. So if we add disjoint basepoints and perform the construction above, we obtain a symmetric spectrum $\mathbf{BO}^+ \wedge \mathbb{S}$ whose value in level $n$ is the space $BO(n)^+ \wedge S^n$. By our previous discussion, this is a commutative symmetric ring spectrum. Since the connectivity of the maps $B\iota_* : BO(n) \longrightarrow BO(n+1)$ tends to infinity with $n$, the underlying symmetric spectrum of $\mathbf{BO} \wedge \mathbb{S}$ is $\pi_*$-isomorphic to the suspension spectrum $\Sigma^\infty BO^+$ [justify].

This construction can be adapted to yield commutative symmetric ring spectra which model the suspension spectra of $BSO, BSpin, BU, BSU$ and $BSp$, which disjoint basepoints added. In each case, the respective family of classical groups fits into an '$I$-topological group' with commutative product, and from there we proceed as for the orthogonal groups. More examples of the same kind are obtained from families of discrete groups which fit into '$I$-groups' with commutative product, for example symmetric groups, alternating groups or general or special linear groups over some ring. [is commutativity of the ring needed? check $BSp$]

Example 2.34 (Matrix ring spectra). If $R$ is a symmetric ring spectrum and $k \geq 1$ we define the symmetric ring spectrum $M_k(R)$ of $k \times k$ *matrices* over $R$ by

$$M_k(R) \ = \ \mathrm{map}(k^+, k^+ \wedge R) \ .$$

Here $k^+ = \{0,1,\ldots,k\}$ with basepoint 0, and so $M_k(R)$ is a $k$-fold product of a $k$-fold coproduct (wedge) of copies of $R$. So 'elements' of $M_k(R)$ are more like matrices which in each row have at most one nonzero entry. The multiplication

$$\mu_{n,m} \ : \ \mathrm{map}(k^+, k^+ \wedge R_n) \ \wedge \ \mathrm{map}(k^+, k^+ \wedge R_m) \ \longrightarrow \ \mathrm{map}(k^+, k^+ \wedge R_{n+m})$$

sends $f \wedge g$ to the composite

$$k^+ \xrightarrow{\ g\ } k^+ \wedge R_m \xrightarrow{f \wedge R_m} k^+ \wedge R_n \wedge R_m \xrightarrow{\mu_{n,m}} R_{n+m} \ .$$

Since homotopy groups take wedges and products to direct sums, we get a natural isomorphism of graded abelian groups

$$\pi_*(M_k(R)) \;\cong\; M_k(\pi_* R) \ .$$

If $R$ semistable, then so is $M_k(R)$ and the above is an isomorphism of graded rings [prove this later].

EXAMPLE 2.35 (Inverting $m$). We consider an integer $m$ and define $\mathbb{S}[1/m]$, the *sphere spectrum with $m$ inverted* by starting from the sphere spectrum and using a map $\varphi_m : S^1 \longrightarrow S^1$ of degree $m$ as the new unit map $\iota_1$. Since the multiplication on $\mathbb{S}$ is commutative, centrality is automatic. So $\mathbb{S}[1/m]$ has the same spaces and symmetric group actions as $\mathbb{S}$, but the $n$-th unit map $\iota_n$ of $\mathbb{S}[1/m]$ is the $n$-fold smash power of $\varphi_m$, which is a self map of $S^n$ of degree $mn$. The unit maps form a morphism $\mathbb{S} \longrightarrow \mathbb{S}[1/m]$ of symmetric ring spectra which on homotopy groups induces an isomorphism

$$\pi_* \ \mathbb{S}[1/m] \;\cong\; \pi_*\mathbb{S} \otimes \mathbb{Z}[1/m] \ .$$

For $m = 0$, the homotopy groups are thus trivial and for $m = 1$ or $m = -1$ the unit morphism $\mathbb{S} \longrightarrow \mathbb{S}[1/m]$ is a $\pi_*$-isomorphism.

EXAMPLE 2.36 (Inverting homotopy elements). Let $R$ be a symmetric ring spectrum and let $x : S^1 \longrightarrow R_1$ be a central map of pointed spaces, i.e., the square

$$\begin{tikzcd}
R_n \wedge S^1 \arrow[r, "\mathrm{Id} \wedge x"] \arrow[d, "\tau"'] & R_n \wedge R_1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\chi_{n,1}"] \\
S^1 \wedge R_n \arrow[r, "x \wedge \mathrm{Id}"'] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"'] & R_{1+n}
\end{tikzcd}$$

commutes for all $n \geq 0$. We define a new symmetric ring spectrum $R[1/x]$ as follows. For $n \geq 0$ we set

$$R[1/x]_n \;=\; \mathrm{map}(S^n, R_{2n}) \ ,$$

the $n$-fold loop space of $R_{2n}$. The group $\Sigma_n$ acts on $S^n$ by coordinate permutations, on $R_{2n}$ via restriction along the diagonal embedding $\Delta : \Sigma_n \longrightarrow \Sigma_{2n}$ given by

$$\Delta(\gamma)(i) \;=\; \begin{cases} 2 \cdot \gamma((i+1)/2) - 1 & \text{for } i \text{ odd,} \\ 2 \cdot \gamma(i/2) & \text{for } i \text{ even,} \end{cases}$$

and by conjugation on the whole mapping space. The multiplication $\mu_{n,m} : R[1/x]_n \wedge R[1/x]_m \longrightarrow R[1/x]_{n+m}$ is the map

$$\begin{aligned} \mathrm{map}(S^n, R_{2n}) \wedge \mathrm{map}(S^m, R_{2m}) \;&\longrightarrow\; \mathrm{map}(S^{n+m}, R_{2(n+m)}) \\ f \wedge g \;&\longmapsto\; \mu_{2n,2m} \circ (f \wedge g) \ , \end{aligned}$$

which is associative since smashing of maps and the product of $R$ are. The multiplication map is $\Sigma_n \times \Sigma_m$-equivariant since the original multiplication maps are equivariant and since the diagonal embeddings $\Delta_n : \Sigma_n \longrightarrow \Sigma_{2n}$, $\Delta_m : \Sigma_m \longrightarrow \Sigma_{2m}$ and $\Delta_{n+m} : \Sigma_{n+m} \longrightarrow \Sigma_{2(n+m)}$ satisfy

$$\Delta_n(\gamma) \times \Delta_m(\tau) \;=\; \Delta_{n+m}(\gamma \times \tau) \ .$$

We have $R[1/x]_0 = R_0$ and the 0th unit map for $R[1/x]$ is the same as for $R$.

Next we define pointed maps $j_n : R_n \longrightarrow \mathrm{map}(S^n, R_{2n})$ as the adjoints of the maps

$$R_n \wedge S^n \xrightarrow{\mathrm{Id} \wedge x^{\wedge n}} R_n \wedge R_1^{\wedge n} \xrightarrow{\mu_{n,1,\dots,1}} R_{n+n} \xrightarrow{\chi^\times_{n,2}} R_{2n}$$

where $\chi^\times_{n,2} \in \Sigma_{n2}$ is the multiplicative shuffle defined by

$$\chi^\times_{n,2}(j) \;=\; \begin{cases} 2j - 1 & \text{for } 1 \leq j \leq n, \\ 2(j-n) & \text{for } n+1 \leq j \leq n+n. \end{cases}$$

(See Remark 1.12 for more background on the multiplicative shuffle and why to expect it here.) Since the map $\mu_{n,1,\dots,1} \circ x^{(n)} : R_n \wedge S^n \longrightarrow R_{n+n}$ is $\Sigma_n \times \Sigma_n$-equivariant and the diagonal embedding $\Delta : \Sigma_n \longrightarrow \Sigma_{2n}$

satisfies $\Delta(\gamma)\circ\chi^{\times}_{n,2} = \chi^{\times}_{n,2}(\gamma\times\gamma)$ in $\Sigma_{n+n}$; together these imply that the adjoint $j_n$ is $\Sigma_n$-equivariant. The maps $j_n$ are multiplicative in the sense of the relation $\mu_{n,m}(j_n\wedge j_m) = j_{n+m}\mu_{n,m}$; this boils down to the relation $\chi^{\times}_{n+m,2}\circ(1\times\chi_{n,m}\times 1) = \chi^{\times}_{n,2}\times\chi^{\times}_{m,2}$ in $\Sigma_{(n+m)2}$.

We define unit maps $S^n \longrightarrow R[1/x]_n$ as the composite of the unit map of $R$ with $j_n$; this finishes the definition of $R[1/x]$ which is again a symmetric ring spectrum and comes with a morphism of symmetric ring spectra [centrality of the unit of $R[1/x]$]

$$j \;:\; R \;\longrightarrow\; R[1/x]\ .$$

Note that the map $x$ does not enter in the definition of the spaces $R[1/x]_n$, and it is not used in defining the multiplication of $R[1/x]$, but it enters in the definition of the unit map of the ring spectrum $R[1/x]$. Thus the colimit systems which define the homotopy groups of $R$ and $R[1/x]$ consist of the same groups, but the effect of $x$ is twisted into the morphisms in the sequence, and so the homotopy groups of $R$ and $R[1/x]$ are potentially different. We show in Proposition 4.67 below that if $R$ is semistable, then so is $R[1/x]$ and the effect of the morphism $j : R \longrightarrow R[1/x]$ on the graded rings of homotopy groups is precisely inverting the class in $\pi_0 R$ represented by the map $x$.

Example 2.37 (Adjoining roots of unity). As an application of the localization construction of Example 2.36 we construct a commutative symmetric ring spectrum which models the 'Gaussian integers over $\mathbb{S}$' with 2 inverted. To construct it, we start with the spherical group ring $\mathbb{S}[C_4]$ of the cyclic group of order 4, a commutative symmetric ring spectrum as in Example 2.32. We invert the element

$$1-t^2 \;\in\; \mathbb{Z}[C_4] \;=\; \pi_0\,\mathbb{S}[C_4]$$

where $t\in C_4$ is a generator, and define

$$\mathbb{S}[1/2,i] \;=\; \mathbb{S}[C_4][1/(1-t^2)]\ .$$

In more detail, the space $\mathbb{S}[C_4]_1 = C_4^+\wedge S^1$ is a wedge of 4 circles and the map from $\pi_1\mathbb{S}[C_4]_1$ to the stable group $\pi_0\,\mathbb{S}[C_4]$ is surjective. So $1-t^2\in\pi_0\,\mathbb{S}[C_4]$ can be represented by a based map $x : S^1\longrightarrow\mathbb{S}[C_4]_1$ to which we apply Example 2.36. The monoid ring spectrum $\mathbb{S}[C_4]$ is commutative and semistable, and so Corollary 4.69 below shows that the ring of homotopy groups of $\mathbb{S}[C_4][1/(1-t^2)]$ is obtained from the ring $\pi_*\mathbb{S}[C_4]$ by inverting the class $1-t^2$ in $\pi_0$.

Because $(1+t^2)(1-t^2)=0$ in the group ring $\mathbb{Z}[C_4]$, inverting $1-t^2$ forces $1+t^2=0$, so $t$ becomes a square root of $-1$. Since $(1-t^2)^2 = 2(1-t^2)$, inverting $1-t^2$ also inverts 2, and in fact

$$\pi_0\,\mathbb{S}[1/2,i] \;=\; \mathbb{Z}[C_4][1/(1-t^2)] \;\cong\; \mathbb{Z}\,[1/2,i]\ .$$

The ring spectrum $\mathbb{S}[1/2,i]$ is $\pi_*$-isomorphic as a symmetric spectrum to a wedge of 2 copies of $\mathbb{S}[1/2]$, and thus deserves to be called the 'Gaussian integers over $\mathbb{S}$' with 2 inverted. Moreover, $\mathbb{S}[1/2,i]$ is a Moore spectrum for the ring $\mathbb{Z}[1/2,i]$, i.e., its integral homology is concentrated in dimension zero.

If $p$ is a prime number and $n\geq 1$, we can similarly adjoin a primitive $p^n$-th root of unity to the sphere spectrum, provided we are also willing to invert $p$ in the homotopy groups. We first form the monoid ring spectrum $\mathbb{S}[C_{p^n}]$ of the cyclic group of order $p^n$, let $t\in C_{p^n}$ denote a generator and invert the element $f = p - (t^{q(p-1)} + t^{q(p-2)} + \cdots + t^q + 1)$ in $\mathbb{Z}[C_{p^n}] = \pi_0\mathbb{S}[C_{p^n}]$, where $q = p^{n-1}$. This defines

$$\mathbb{S}[1/p,\zeta] \;=\; \mathbb{S}[C_{p^n}][1/f]\ .$$

We have $f^2 = pf$, so inverting $f$ also inverts the prime $p$ and forces the expression $p-f$ to become 0 in the localized ring. If we let $\zeta$ denote the image of $t$ in the localized ring, then the latter says that $\zeta$ is a root of the cyclotomic polynomial, i.e.,

$$\zeta^{q(p-1)} + \zeta^{q(p-2)} + \cdots + \zeta^q + 1 \;=0$$

where again $q = p^{n-1}$. In fact we have $\mathbb{Z}[C_{p^n}][1/f] = \mathbb{Z}[1/p,\zeta]$ where $\zeta$ is a primitive $p^n$-th root of unity; moreover the commutative symmetric ring spectrum $\mathbb{S}[1/p,\zeta]$ is a Moore spectrum for the ring $\mathbb{Z}[1/p,\zeta]$.

We can do the same constructions starting with any semistable commutative symmetric ring spectrum $R$ instead of the sphere spectrum, yielding a new commutative symmetric ring spectrum $R[1/p,\zeta]$. If $p$ is already invertible and the cyclotomic polynomial above is irreducible in $\pi_0 R$, then this adjoins a primitive $p^n$-th root of unity to the homotopy ring of $R$.

These examples are a special case of a much more general phenomenon: every number ring can be 'lifted' to an extension of the sphere spectrum by a commutative symmetric ring spectrum, provided we also invert the ramified primes. However, the only proofs of this general fact that I know use obstruction theory, and so we cannot give a construction which is as explicit and simple as the one above for adjoining roots of unity.

### 2.3. Restrictions from other kinds of spectra.

EXAMPLE 2.38 (Continuous functors). By a *continuous functor* we mean a functor $F : \mathcal{T} \longrightarrow \mathcal{T}$ from the category of pointed spaces to itself which is pointed in that it takes one-point spaces to one-point spaces and continuous in the sense that for all pointed spaces $K$ and $L$ the map

$$F \ : \ \mathcal{T}(K,L) \ \longrightarrow \ \mathcal{T}(F(K),F(L))$$

is continuous with respect to the compact open topology on the mapping spaces. The (continuous !) map

$$L \xrightarrow{l \ \mapsto (k \mapsto k\wedge l)} \mathcal{T}(K, K\wedge L) \xrightarrow{\ F\ } \mathcal{T}(F(K), F(K\wedge L)) \ .$$

then has an adjoint

$$F(K)\wedge L \ \longrightarrow \ F(K\wedge L)$$

which we call the *assembly map.* The assembly map is natural in $K$ and $L$, it is unital in the sense that the composite

$$F(K) \cong F(K)\wedge S^0 \xrightarrow{\text{assembly}} F(K\wedge S^0) \cong F(K)$$

is the identity and it is associative in the sense that the diagram

$$\begin{tikzcd}
(F(K)\wedge L)\wedge M \arrow[r, "\text{ass.}\wedge\mathrm{Id}"] \arrow[d, "\cong"'] & F(K\wedge L)\wedge M \arrow[r, "\text{ass.}"] & F((K\wedge L)\wedge M) \arrow[d, "F(\cong)"] \\
F(K)\wedge (L\wedge M) \arrow[rr, "\text{assembly}"'] & & F(K\wedge (L\wedge M))
\end{tikzcd}$$

commutes for all $K, L$ and $M$, where the vertical maps are associativity isomorphisms for the smash product.

As usual, there is also a simplicial version. A *simplicial functor* is an enriched, pointed functor $F$ : $\mathbf{sset}_* \longrightarrow \mathbf{sset}_*$ from the category of pointed simplicial sets to itself. So $F$ assigns to each pointed simplicial set $K$ a pointed simplicial set $F(K)$ and to each pair $K, L$ of pointed simplicial sets a morphism of pointed simplicial sets

$$F \ : \ \mathrm{map}(K,L) \ \longrightarrow \ \mathrm{map}(F(K),F(L))$$

which is associative and unital and such that $F(*) \cong *$. The restriction of $F$ to vertices is then a functor in the usual sense. The same kind of adjunctions as for continuous functors provides a simplicial functor with an *assembly map* $F(K)\wedge L \longrightarrow F(K\wedge L)$, again unital and associative.

To every continuous (respectively simplicial) functor $F$ we can associate a symmetric spectrum of spaces (respectively of simplicial sets) $F(\mathbb{S})$ by

$$F(\mathbb{S})_n \ = \ F(S^n)$$

where $\Sigma_n$ permutes the coordinates of $S^n$. The structure map $\sigma_n : F(S^n)\wedge S^1 \longrightarrow F(S^{n+1})$ is an instance of the assembly map. More generally, we can evaluate a continuous (simplicial) functor $F$ on a symmetric spectrum $X$ and get a new symmetric spectrum $F(X)$ by defining $F(X)_n = F(X_n)$ with structure map the composite

$$F(X_n)\wedge S^1 \xrightarrow{\text{assembly}} F(X_n\wedge S^1) \xrightarrow{F(\sigma_n)} F(X_{n+1}) \ .$$

Some of the symmetric spectra which we described are restrictions of continuous functors to spheres, for example suspension spectra and Eilenberg-Mac Lane spectra. Free symmetric spectra $F_m K$ or semifree symmetric spectra $G_m K$ do not arise this way (unless $m = 0$ or $K = *$) and cobordism spectra like $MO$ and $MU$ don't either.

A consequence of the formal properties of the assembly map is that the structure of a triple on a continuous or simplicial functor $T$ yields a multiplication on the symmetric spectrum $T(\mathbb{S})$. Indeed, using the assembly map twice and the triple structure map produces multiplication maps

$$T(K) \wedge T(L) \longrightarrow T(K \wedge T(L)) \longrightarrow T(T(K \wedge L)) \longrightarrow T(K \wedge L) ;$$

here $K$ and $L$ are pointed spaces. If we apply this to spheres, we get $\Sigma_p \times \Sigma_q$-equivariant maps

$$T(S^p) \wedge T(S^q) \longrightarrow T(S^{p+q})$$

which provide the multiplication. The unit maps come from the natural transformation $\mathrm{Id} \longrightarrow T$ by evaluating on spheres. Here are some examples.

- The identity triple gives the sphere spectrum as a symmetric ring spectrum.
- Let $Gr$ be the reduced free group triple, i.e., it sends a pointed set $K$ to the free group generated by $K$ modulo the normal subgroup generated by the basepoint. Since $Gr(S^n)$ is weakly equivalent to $\Omega S^{n+1}$, which in the stable range is equivalent to $S^n$, the unit maps form a $\pi_*$-isomorphism $\mathbb{S} \longrightarrow Gr(\mathbb{S})$. The same conclusion would hold with the free reduced monoid functor, also known as the 'James construction' $J$, since $J(S^n)$ is also weakly equivalence to $\Omega S^{n+1}$ as soon as $n \geq 1$.
- Let $M$ be a topological monoid and consider the pointed continuous functor $K \mapsto M^+ \wedge K$. The multiplication and unit of $M$ make this into a triple whose algebras are pointed sets with left $M$-action. The associated symmetric ring spectrum is the spherical monoid ring $\mathbb{S}[M]$.
- Let $A$ be a ring and consider the free reduced $A$-module triple $\tilde{A}[K] = A[K]/A[*]$. Then $\tilde{A}[\mathbb{S}] = HA$, the Eilenberg-Mac Lane ring spectrum. We shall see later [ref] that for every symmetric spectrum of simplicial sets $X$ the symmetric spectrum $\tilde{A}[X]$ is $\pi_*$-isomorphic to the smash product $HA \wedge X$.
- Let $B$ be a commutative ring and consider the triple $X \mapsto I(\tilde{B}(X))$, the augmentation ideal of the reduced polynomial algebra over $B$, generated by the pointed set $X$. The algebras over this triple are non-unital commutative $B$-algebras, or augmented commutative $B$-algebras (which are equivalent categories). The ring spectrum associated to this triple is denoted $DB$, and it is closely related to topological André-Quillen homology for commutative $B$-algebras. The ring spectrum $DB$ is rationally equivalent to the Eilenberg-Mac Lane ring spectrum $HB$, but $DB$ has torsion in higher homotopy groups.

More generally, if we evaluate a triple $T$ on a symmetric ring spectrum $R$, then the resulting spectrum $T(R)$ is naturally a ring spectrum with multiplication maps

$$T(R_n) \wedge T(R_m) \to T(R_n \wedge R_m) \xrightarrow{T(\mu_{n,m})} T(R_{n+m}) .$$

Example 2.39 ($\Gamma$-spaces). Many continuous or simplicial functors arise from so called $\Gamma$-*spaces*, and then the associated symmetric spectra have special properties. The category $\Gamma$ is a skeletal category of the category of finite pointed sets: there is one object $n^+ = \{0, 1, \ldots, n\}$ for every non-negative integer $n$, and morphisms are the maps of sets which send 0 to 0. ($\Gamma$ is really equivalent to the opposite of Segal's category $\Gamma$, cf. [**55**]). A $\Gamma$-*space* is a covariant functor from $\Gamma$ to the category of spaces or simplicial sets taking $0^+$ to a one point space (simplicial set). A morphism of $\Gamma$-spaces is a natural transformation of functors. We follow the established terminology to speak of $\Gamma$-*spaces* even if the values are simplicial sets.

A $\Gamma$-space $X$ can be extended to a continuous (respectively simplicial, depending on the context) functor by a coend construction. If $X$ is a $\Gamma$-space and $K$ a pointed space or simplicial set, the value of the extended functor on $K$ is given by

$$\int^{n^+ \in \Gamma} K^n \wedge X(n^+) ,$$

where we use that $K^n = \mathrm{map}(n^+, K)$ is contravariantly functorial in $n^+$. We will not distinguish notationally between the original $\Gamma$-space and its extension. The extended functor is continuous respectively simplicial.

In the simplicial context, the extension of a $\Gamma$-space admits the following different (but naturally isomorphic) description. First, $X$ can be prolonged, by direct limit, to a functor from the category of pointed sets, not necessarily finite, to pointed simplicial sets. Then if $K$ is a pointed simplicial set we get a bisimplicial set $[k] \mapsto X(K_k)$ by evaluating the (prolonged) $\Gamma$-space degreewise. The simplicial set $X(K)$ defined by the coend above is naturally isomorphic to the diagonal of this bisimplicial set.

Symmetric spectra which arise from $\Gamma$-spaces have special properties. Here we restrict to $\Gamma$-spaces of simplicial sets, where things are easier to state. First, every simplicial functor which arises from a $\Gamma$-space $X$ preserves weak equivalences of simplicial sets, see [**11**, Prop. 4.9]. So if $f : A \longrightarrow B$ is a level equivalence of symmetric spectra of simplicial sets, then $X(f) : X(A) \longrightarrow X(B)$ is again a level equivalence. We shall see later that $X(-)$ also preserves $\pi_*$-isomorphisms and stable equivalences [ref]. Another special property is that symmetric spectra of the form $X(\mathbb{S})$ for $\Gamma$-spaces of simplicial sets $X$ are connective and the colimit systems for the stable homotopy groups stabilize in a uniform way. This is because for every $\Gamma$-space $X$, the simplicial set $X(S^n)$ is always $(n-1)$-connected [**11**] and the structure map $X(S^n) \wedge S^1 \longrightarrow X(S^{n+1})$ is $2n$-connected [**34**, prop. 5.21]. Moreover, up to $\pi_*$-isomorphisms, $\Gamma$-spaces model all connective spectra (see Theorem 5.8 of [**11**] [also reference to [**55**]?])

A $\Gamma$-space $X$ is called *special* if the map $X((k+l)^+) \longrightarrow X(k^+) \times X(l^+)$ induced by the projections from $(k+l)^+ \cong k^+ \vee l^+$ to $k^+$ and $l^+$ is a weak equivalence for all $k$ and $l$. In this case, the weak map

$$X(1^+) \times X(1^+) \xleftarrow{\sim} X(2^+) \xrightarrow{X(\nabla)} X(1^+)$$

induces an abelian monoid structure on $\pi_0(X(1^+))$. Here $\nabla : 2^+ \longrightarrow 1^+$ is defined by $\nabla(1) = 1 = \nabla(2)$. The $\Gamma$-space $X$ is called *very special* if it is special and the monoid $\pi_0(X(1^+))$ is a group. By Segal's theorem ([**55**, Prop. 1.4] or [**11**, Thm. 4.2]), the spectrum $X(\mathbb{S})$ associated to a special $\Gamma$-space $X$ by evaluation on spheres is a positive $\Omega$-spectrum.

If $X$ is very special, then $X(\mathbb{S})$ is even an $\Omega$-spectrum (i.e., from the 0th level on). In particular, the homotopy groups of a very special $\Gamma$-space $X$ are naturally isomorphic to the homotopy groups of the simplicial set $X(1^+)$.

Example 2.40 (Orthogonal spectra). An *orthogonal spectrum* consists of the following data:

- a sequence of pointed spaces $X_n$ for $n \geq 0$
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on $X_n$ for each $n \geq 0$
- based maps $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ for $n \geq 0$.

This data is subject to the following condition: for all $n, m \geq 0$, the iterated structure map

$$\sigma^m \ : \ X_n \wedge S^m \ \longrightarrow \ X_{n+m}$$

is $O(n) \times O(m)$-equivariant. The orthogonal group acts on $S^m$ since this is the one-point compactification of $\mathbb{R}^n$ and $O(n) \times O(m)$ acts on the target by restriction, along orthogonal sum, of the $O(n+m)$-action.

A morphism $f : X \longrightarrow Y$ of orthogonal spectra consists of $O(n)$-equivariant based maps $f_n : X_n \longrightarrow Y_n$ for $n \geq 0$, which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge \mathrm{Id}_{S^1})$ for all $n \geq 0$.

An *orthogonal ring spectrum* $R$ consists of the following data:

- a sequence of pointed spaces $R_n$ for $n \geq 0$
- a base-point preserving continuous left action of the orthogonal group $O(n)$ on $R_n$ for each $n \geq 0$
- $O(n) \times O(m)$-equivariant *multiplication maps* $\mu_{n,m} : R_n \wedge R_m \longrightarrow R_{n+m}$ for $n, m \geq 0$, and
- $O(n)$-equivariant *unit maps* $\iota_n : S^n \longrightarrow R_n$ for all $n \geq 0$.

This data is subject to the same associativity and unit conditions as a symmetric ring spectrum (see Definition 1.3) and a centrality condition for every unit map $\iota_n$. In the unit condition, permutations such as $\chi_{n,m} \in \Sigma_{n+m}$ have to be interpreted as permutation matrices in $O(n+m)$. An orthogonal ring spectrum $R$ is *commutative* if for all $n, m \geq 0$ the relation $\chi_{n,m} \circ \mu_{n,m} = \mu_{m,n} \circ \text{twist}$ holds as maps $R_n \wedge R_m \longrightarrow R_{m+n}$.

A *morphism* $f : R \longrightarrow S$ of orthogonal ring spectra consists of $O(n)$-equivariant based maps $f_n : R_n \longrightarrow S_n$ for $n \geq 0$, which are compatible with the multiplication and unit maps (in the same sense as for symmetric ring spectra).

Orthogonal spectra are 'symmetric spectra with extra symmetry' in the sense that every orthogonal spectrum $X$ has an *underlying symmetric spectrum* $UX$. Here $(UX)_n = X_n$ and the symmetric group acts by restriction along the monomorphism $\Sigma_n \longrightarrow O(n)$ given by permutation matrices. The structure maps of $UX$ are the structure maps of $X$. Many spectra that we have introduced above have this extra symmetry, i.e., they are underlying orthogonal spectra. Examples are the sphere spectrum, suspension spectra or more generally any symmetric spectrum which is obtained from a continuous functor by evaluation on spheres. The various Thom spectra such as $MO$ and $MU$ arise from orthogonal spectra by forgetting symmetry, but they do not extend to continuous functors.

There is a more natural notion where we use vector spaces with inner product to index the spaces in an orthogonal spectrum. A *coordinate free orthogonal spectrum* $X$ consists of the following data:

- a pointed space $X(V)$ for each *inner product space* $V$, i.e., finite dimensional real vector space with a euclidian scalar product,
- a base-point preserving continuous left action of the orthogonal group $O(V)$ of $V$ on $X(V)$, for each inner product space $V$,
- a structure map $\sigma_{V,W} : X(V) \wedge S^W \longrightarrow X(V \oplus W)$ for each pair of inner product spaces $V$ and $W$ which is $O(V) \times O(W)$-equivariant. Here $S^W$ is the one-point compactification of $W$ on which the group $O(W)$ acts by extension of the action on $W$, fixing the basepoint at infinity. The group $O(V) \times O(W)$ acts on the target by restriction of the $O(V \oplus W)$-action.

This data should satisfy two conditions: the composite

$$X(V) \cong X(V) \wedge S^0 \xrightarrow{\sigma_{V,0}} X(V \oplus 0) \cong X(V)$$

should be the identity [isometries $V \cong W$ should also act] and the square

$$\begin{tikzcd}
X(V) \wedge S^W \wedge S^Z \arrow[r, "\mathrm{Id} \wedge \mu_{W,Z}"] \arrow[d, "\sigma_{V,W} \wedge \mathrm{Id}"'] & X(V) \wedge S^{W \oplus Z} \arrow[d, "\sigma_{V,W \oplus Z}"] \\
X(V \oplus W) \wedge S^Z \arrow[r, "\sigma_{V \oplus W,Z}"'] & X(V \oplus W \oplus Z)
\end{tikzcd}$$

commutes for all inner product spaces $V, W$ and $Z$.

A *morphism* $f : X \longrightarrow Y$ of coordinate free orthogonal spectra consists of $O(V)$-equivariant pointed maps $f(V) : X(V) \longrightarrow Y(V)$ for all $V$ which are compatible with the structure maps in the sense that $f(V \oplus W) \circ \sigma_{V,W} = \sigma_{V,W} \circ (f(V) \wedge \mathrm{Id})$ for all $V$ and $W$.

A coordinate free orthogonal spectrum $X$ gives rise to a coordinate free symmetric spectrum $UX$ (see Remark 1.11) by forgetting symmetry. For a finite set $A$ the space $(UX)_A$ is $X(\mathbb{R}^A)$, the value of $X$ at the inner product space $\mathbb{R}^A$ which has $A$ as orthonormal basis. [define the structure maps $\alpha_* : (UX)_A \wedge S^{B-\alpha(A)} \longrightarrow (UX)_B$]

Example 2.41 (Unitary spectra). Unitary spectra are the complex analogues of orthogonal spectra, and they again come in a coordinatized and a coordinate free flavor. A coordinate free unitary spectrum assigns a pointed spaces $X(V)$ for each *complex* inner product space $V$, i.e., finite dimensional complex vector space with a hermitian metric, together with a base-point preserving continuous left action of the unitary group $U(V)$ of $V$ on $X(V)$, and $U(V) \times U(W)$-equivariant structure maps $\sigma_{V,W} : X(V) \wedge S^W \longrightarrow X(V \oplus W)$ which are associative and unital.

Given a unitary spectrum $X$ we can produce an orthogonal spectrum $\Phi(X)$ as follows. We set

$$\Phi(X)(V) \;=\; \mathrm{map}(S^{iV}, X(\mathbb{C} \otimes_{\mathbb{R}} V)) \;,$$

where $i \in \mathbb{C}$ is a square root of $-1$ and $S^{iV}$ is the one-point compactification of the imaginary part of the complex inner product space $\mathbb{C} \otimes_{\mathbb{R}} V$. The orthogonal group acts on $S^{iV}$, on $X(\mathbb{C} \otimes_{\mathbb{R}} V)$ via the complexification map $O(V) \longrightarrow U(\mathbb{C} \otimes_{\mathbb{R}} V)$ and on the mapping space by conjugation. The structure map $\sigma_{V,W} : \Phi(X)(V) \wedge S^W \longrightarrow \Phi(X)(V \oplus W)$ is adjoint to the map

$$\begin{aligned}
\mathrm{map}(S^{iV}, X(\mathbb{C} \otimes_{\mathbb{R}} V)) \wedge S^W \wedge S^{i(V \oplus W)} \;&\cong\; \mathrm{map}(S^{iV}, X(\mathbb{C} \otimes_{\mathbb{R}} V)) \wedge S^{iV} \wedge S^{\mathbb{C} \otimes_{\mathbb{R}} W} \\
&\xrightarrow{\mathrm{eval} \wedge \mathrm{Id}} X(\mathbb{C} \otimes_{\mathbb{R}} V) \wedge S^{\mathbb{C} \otimes_{\mathbb{R}} W} \xrightarrow{\sigma_{\mathbb{C} \otimes V, \mathbb{C} \otimes W}} X(\mathbb{C} \otimes_{\mathbb{R}} (V \oplus W))
\end{aligned}$$

where we have made identifications such as $S^W \wedge S^{iW} \cong S^{W\oplus iW} \cong S^{\mathbb{C}\otimes_{\mathbb{R}} W}$ and $\mathbb{C}\otimes_{\mathbb{R}} V \oplus \mathbb{C}\otimes_{\mathbb{R}} W \cong \mathbb{C}\otimes_{\mathbb{R}} (V\oplus W)$.

The functor $\Phi : \mathcal{S}p^U \longrightarrow \mathcal{S}p^O$ turns unitary ring spectra into orthogonal ring spectra. An example of this is the complex cobordism spectrum $MU$ of Example 2.9 which arises naturally as a unitary ring spectrum, made into an orthogonal spectrum via the functor $\Phi$. More precisely, the symmetric sequence denoted $\overline{MU}$ in Example 2.9 comes from a unitary spectrum with $V$th space

$$\overline{MU}(V) \ = \ EU(V)^+ \wedge_{U(V)} S^V \ ,$$

the Thom space of the vector bundle over $BU(V)$ with total space $EU(V)\times_{U(V)} V$.

Example 2.42 (Periodic complex cobordism). We define the *periodic complex cobordism* spectrum $MUP$, a unitary spectrum, as follows. For a complex inner product space $V$ we consider the 'full Grassmannian' of $V\oplus V$. A point in this 'full Grassmannian' is any complex subvector space of $V\oplus V$, and this space is topologized as the disjoint union of the Grassmannians of $k$-dimensional subspaces of $V\oplus V$ for $k=0,\dots,2\dim(V)$. Over the 'full Grassmannian' sits a tautological hermitian vector bundle (of non-constant rank!): the total space of this bundle consist of pairs $(X,x)$ where $X$ is a complex subvector space of $V\oplus V$ and $x\in X$. We define $(MUP)(V)$ as the Thom space of this tautological vector bundle, i.e., the quotient space of the unit disc bundle by the sphere bundle. The multiplication

$$(MUP)(V)\wedge (MUP)(W) \ \longrightarrow \ (MUP)(V\oplus W)$$

sends $(X,x)\wedge(Y,y)$ to $(X+Y,(x,y))$ where $X+Y$ is the image of $X\oplus Y$ under the isometry $\mathrm{Id}\wedge\tau\wedge\mathrm{Id}$ : $(V\oplus V)\oplus(W\oplus W)\cong(V\oplus W)\oplus(V\oplus W)$. The unit map $S^V\longrightarrow (MUP)(V)$ sends $x\in V$ to $(\Delta(V),(v,v))$ where $\Delta(V)$ is the diagonal copy of $V$ in $V\oplus V$. [explain that $MUP$ is a wedge of suspended copies of $MU$, i.e., there is a $\pi_*$-isomorphism $\bigvee_{k\in\mathbb{Z}} S^{2k}\wedge MU \longrightarrow MUP$; real version $\bigvee_{k\in\mathbb{Z}} S^k\wedge MO\simeq MOP$]

Example 2.43 ($S$-modules). We describe a functor $\Phi : \mathcal{M}_S \longrightarrow \mathcal{S}p^\Sigma$ from the category of $S$-modules in the sense of Elmendorf, Kriz, Mandell and May [**19**] to the category of symmetric spectra which preserves homotopy groups and multiplicative structures. For this we need the $S$-module $S\wedge_{\mathscr{L}}\mathbb{L}S^{-1} = S\wedge_{\mathscr{L}}\mathbb{L}\Sigma_1^\infty S^0$ defined in [**19**, II 1.7], which we abbreviate to $S_c^{-1}$. What matters is not the precise form of $S_c^{-1}$, but that it is a cofibrant desuspension of the sphere $S$-module, i.e., it comes with a weak equivalence $S_c^{-1}\wedge S^1\longrightarrow S$, where $S^1$ denotes the circle. For $n>0$ we define $S_c^{-n}$ to be the $n$-fold smash power of the $S$-module $S_c^{-1}$, endowed with the permutation action of the symmetric group on $n$ letters. We set $S_c^0=S$, the unit of the smash product; here the notation is slightly misleading since $S_c^0$ is *not* cofibrant. The functor $\Phi$ is then given by

$$\Phi(X)_n \ = \ \mathcal{M}_S(S_c^{-n},X)$$

where the right hand side is the topological mapping space in the category of $S$-modules. The symmetric group acts on the mapping space through the permutation action of the source. The desuspension map $S_c^{-1}\wedge S^1\longrightarrow S$ induces a map

$$\mathcal{M}_S(S_c^{-n},X) \ \longrightarrow \ \mathcal{M}_S(S_c^{-(n+1)}\wedge S^1,X) \ \cong \mathcal{T}(S^1,\mathcal{M}_S(S_c^{-(n+1)},X))$$

whose adjoint

$$\mathcal{M}_S(S_c^{-n},X)\wedge S^1 \ \longrightarrow \mathcal{M}_S(S_c^{-(n+1)},X)$$

makes $\Phi(X)$ into a symmetric spectrum. For $n\geq 1$, the $S$-module $S_c^{-n}$ is a cofibrant model of the (-$n$)-sphere spectrum. So the functor $\Phi$ takes weak equivalences of $S$-modules to maps which are level equivalences above level 0, and the $i$-th homotopy group of the space $\Phi(X)_n$ is isomorphic to the $(i-n)$-th homotopy group of the $S$-module $X$ by [**19**, II 1.8]. In particular there is a natural isomorphism of stable homotopy groups $\pi_*\Phi(X)\cong\pi_*X$.

If $R$ is an $S$-algebra with multiplication $\mu : R\wedge R\longrightarrow R$ and unit $i : S\longrightarrow R$, then $\Phi(R)$ becomes a symmetric ring spectrum with multiplication maps

$$\mathcal{M}_S(S_c^{-m},R) \ \wedge \ \mathcal{M}_S(S_c^{-n},R) \ \xrightarrow{\ \wedge\ } \ \mathcal{M}_S(S_c^{-(m+n)},R\wedge R) \ \xrightarrow{\ \mu\ } \mathcal{M}_S(S_c^{-(m+n)},R)\ .$$

The unit maps $i_0 : S_0 \longrightarrow \Phi(S)_0 = \mathcal{M}_S(S, R)$ is the unit $i$ and the unit map $S^1 \longrightarrow \Phi(S)_1 = \mathcal{M}_S(S_c^{-1}, R)$ is adjoint to the composite $S_c^{-1} \wedge S^1 \longrightarrow S \xrightarrow{i} R$. If $R$ is a commutative $S$-algebra, then $\Phi(R)$ is a commutative symmetric ring spectrum.

Different kinds of (categories of) spectra yield symmetric spectra:

$$\begin{tikzcd}
\Gamma\text{-spaces} \arrow[r] & \text{continuous functors} \arrow[r] \arrow[dr] & \text{unitary spectra} \arrow[d, "\Phi"] & \\
 & \mathcal{M}_S \arrow[r] & \text{orthogonal spectra} \arrow[r] & \text{symmetric spectra}
\end{tikzcd}$$

The triangle only commute up to natural $\pi_*$-isomorphism: for every continuous functor $F$ there is a natural $\pi_*$-isomorphism whose $V$th term is the map $F(S^V) \longrightarrow \text{map}(S^{iV}, F(S^{\mathbb{C}\otimes_{\mathbb{R}}V}))$ adjoint to the assembly map $F(S^V) \wedge S^{iV} \longrightarrow F(S^{\mathbb{C}\otimes_{\mathbb{R}}V})$. We did not say how to make an orthogonal spectrum from an $S$-module; this construction can be found in [**38**].

## 3. Smash product

One of the main features which distinguishes *symmetric* spectra from the more classical spectra without symmetric group actions is the internal smash product. The smash product of symmetric spectra is very much like the tensor product of modules over a commutative ring. To stress that analogy, we recall three different ways to look at the classical tensor product and then give analogies involving the smash product of symmetric spectra.

In the following, $R$ is a commutative ring and $M, N$ and $W$ are right $R$-modules.

**(A) Tensor product via bilinear maps.** A bilinear map from $M$ and $N$ to another right $R$-module $W$ is a map $b : M \times N \longrightarrow W$ such that for each $m \in M$ the map $b(m, -) : N \longrightarrow W$ is $R$-linear and for each $n \in N$ the map $b(-, n) : M \longrightarrow W$ is $R$-linear. The tensor product $M \otimes_R N$ is the universal example of a right $R$-module together with a bilinear map from $M \times N$. In other words, there is a specified bilinear map $i : M \times N \longrightarrow M \otimes_R N$ such that for every $R$-module $W$ the map

$$\text{Hom}_R(M \otimes_R N, W) \;\longrightarrow\; \text{Bilin}_R(M \times N, W)\;, \quad f \;\mapsto\; f \circ i$$

is bijective. As usual, the universal property characterizes the pair $(M \otimes_R N, i)$ uniquely up to preferred isomorphism.

**(B) Tensor product as an adjoint to internal Hom.** The category of right $R$-modules has 'internal Hom-objects': the set $\text{Hom}_R(N, W)$ of $R$-linear maps between two right $R$-modules $N$ and $W$ is naturally an $R$-module by pointwise addition and scalar multiplication. For fixed right $R$-modules $M$ and $N$, the functor $\text{Hom}_R(M, \text{Hom}_R(N, -)) : \text{mod-}R \longrightarrow \text{mod-}R$ is representable and tensor product $M \otimes_R N$ can be defined as a representing $R$-module. This point of view is closely related to the first approach since the $R$-modules $\text{Hom}_R(M, \text{Hom}_R(N, W))$ and $\text{Bilin}_R(M \times N, W)$ are naturally isomorphic.

**(C) Tensor product as a construction.** Often the tensor product $M \otimes_R N$ is introduced as a specific construction, usually the following: $M \otimes_R N$ is the free $R$-module generated by symbols of the form $m \otimes n$ for all $m \in M$ and $n \in N$ subject to the following set of relations

- $(m + m') \otimes n \;=\; m \otimes n + m' \otimes n\;, \quad m \otimes (n + n') \;=\; m \otimes n + m \otimes n'$
- $(mr) \otimes n \;=\; (m \otimes n) \cdot r \;=\; m \otimes (nr)$

for all $m, m' \in M$, $n, n' \in N$ and $r \in R$. Since this is a minimal set of relations which make the map $M \times N \longrightarrow M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$ into a bilinear map, the tensor product is constructed as to have the universal property (A).

Now we introduce the smash product of symmetric spectra in three ways, analogous to the ones above.

**(A) Smash product via bilinear maps.** We define a *bimorphism* $b : (X, Y) \longrightarrow Z$ from a pair of symmetric spectra $(X, Y)$ to a symmetric spectrum $Z$ as a collection of $\Sigma_p \times \Sigma_q$-equivariant maps of pointed spaces or simplicial sets, depending on the context,

$$b_{p,q} \ : \ X_p \wedge Y_q \ \longrightarrow \ Z_{p+q}$$

for $p, q \geq 0$, such that the diagram

$$\begin{tikzcd}
 & X_p \wedge Y_q \wedge S^1 \arrow[r, "\mathrm{Id} \wedge \mathrm{twist}"] \arrow[dl, "\mathrm{Id} \wedge \sigma_q"'] \arrow[d, "b_{p,q} \wedge \mathrm{Id}"] & X_p \wedge \ S^1 \wedge Y_q \arrow[d, "\sigma_p \wedge \mathrm{Id}"] \\
X_p \wedge Y_{q+1} \arrow[dr, "b_{p,q+1}"'] & Z_{p+q} \wedge S^1 \arrow[d, "\sigma_{p+q}"] & X_{p+1} \wedge Y_q \arrow[d, "b_{p+1,q}"] \\
 & Z_{p+q+1} & Z_{p+1+q} \arrow[l, "1 \times \chi_{1,q}"]
\end{tikzcd} \tag{3.1}$$

commutes for all $p, q \geq 0$. In Exercise 5.7 we give a justification for calling this notion 'bimorphism'.

The smash product $X \wedge Y$ can now we introduced as the universal example of a symmetric spectrum with a bimorphism from $X$ and $Y$. More precisely, we will show in (C) below that for every pair of symmetric spectra $(X, Y)$ the functor which assign to $Z \in \mathcal{S}p^\Sigma$ the set of bimorphism from $(X, Y)$ to $Z$ is representable. A smash product of $X$ and $Y$ is then a representing object, i.e., a pair consisting of a symmetric spectrum $X \wedge Y$ and a bimorphism $\iota : (X, Y) \longrightarrow X \wedge Y$ such that for every symmetric spectrum $Z$ the map

$$\mathcal{S}p^\Sigma(X \wedge Y, Z) \ \longrightarrow \ \mathrm{Bimor}((X, Y), Z) \ , \quad f \longmapsto fi = \{f_{p+q} \circ i_{p,q}\}_{p,q} \tag{3.2}$$

is bijective. Very often only the object $X \wedge Y$ will be referred to as the smash product, but one should keep in mind that it comes equipped with a specific, universal bimorphism. We will often refer to the bijection (3.2) as the *universal property* of the smash product of symmetric spectra.

**(B) Smash product as an adjoint to internal Hom.** In Example 2.25 we introduced 'internal Hom objects' in the category of symmetric spectra. For every pair of symmetric spectra $(X, Y)$ we defined another symmetric spectrum $\mathrm{Hom}(X, Y)$ such that the morphism from $X$ to $Y$ are (in natural bijection to) the vertices of the 0th level of $\mathrm{Hom}(X, Y)$. We claim that for fixed symmetric spectra $X$ and $Y$, the functor $\mathrm{Hom}(X, \mathrm{Hom}(Y, -)) : \mathcal{S}p^\Sigma \longrightarrow \mathcal{S}p^\Sigma$ is representable. The smash product $X \wedge Y$ can then be defined as a representing symmetric spectrum. This point of view can be reduced to perspective (A) since the sets $\mathcal{S}p^\Sigma(X, \mathrm{Hom}(Y, Z))$ and $\mathrm{Bimor}((X, Y), Z)$ are in natural bijection (see Exercise 5.7). In particular, since the functor $\mathrm{Bimor}((X, Y), -)$ is representable, so is the functor $\mathcal{S}p^\Sigma(X, \mathrm{Hom}(Y, -))$. [extend this to an isomorphism of spectra $\mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \cong \underline{\mathrm{Bimor}}((X, Y), Z)$]

**(C) Smash product as a construction.** Now we construct a symmetric spectrum $X \wedge Y$ from two given symmetric spectra $X$ and $Y$. We want $X \wedge Y$ to be the universal recipient of a bimorphism from $(X, Y)$, and this pretty much tells us what we have to do. For $n \geq 0$ we define the $n$th level $(X \wedge Y)_n$ as the coequalizer, in the category of pointed $\Sigma_n$-spaces or pointed $\Sigma_n$-simplicial sets (depending on the context), of two maps

$$\alpha_X, \, \alpha_Y \ : \ \bigvee_{p+1+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_q} X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q \ .$$

The wedges run over all non-negative values of $p$ and $q$ which satisfy the indicated relations. The map $\alpha_X$ takes the wedge summand indexed by $(p, 1, q)$ to the wedge summand indexed by $(p + 1, q)$ using the map

$$\sigma_p^X \wedge \mathrm{Id} \ : \ X_p \wedge S^1 \wedge Y_q \ \longrightarrow \ X_{p+1} \wedge Y_q$$

and inducing up. The other map $\alpha_Y$ takes the wedge summand indexed by $(p,1,q)$ to the wedge summand indexed by $(p,1+q)$ using the composite

$$X_p \wedge S^1 \wedge Y_q \xrightarrow{\mathrm{Id}\wedge\mathrm{twist}} X_p \wedge Y_q \wedge S^1 \xrightarrow{\mathrm{Id}\wedge\sigma_q^Y} X_p \wedge Y_{q+1} \xrightarrow{\mathrm{Id}\wedge\chi_{q,1}} X_p \wedge Y_{1+q}$$

and inducing up.

The structure map $(X\wedge Y)_n \wedge S^1 \longrightarrow (X\wedge Y)_{n+1}$ is induced on coequalizers by the wedge of the maps

$$\Sigma_n^+ \wedge_{\Sigma_p\times\Sigma_q} X_p \wedge Y_q \wedge S^1 \ \longrightarrow\ \Sigma_{n+1}^+ \wedge_{\Sigma_p\times\Sigma_{q+1}} X_p \wedge Y_{q+1}$$

induced from $\mathrm{Id}\wedge\sigma_q^Y : X_p\wedge Y_q\wedge S^1 \longrightarrow X_p\wedge Y_{q+1}$. One should check that this indeed passes to a well-defined map on coequalizers. Equivalently we could have defined the structure map by moving the circle past $Y_q$, using the structure map of $X$ (instead of that of $Y$) and then shuffling back with the permutation $\chi_{1,q}$; the definition of $(X\wedge Y)_{n+1}$ as a coequalizer precisely ensures that these two possible structure maps coincide, and that the collection of maps

$$X_p \wedge Y_q \xrightarrow{x\wedge y\mapsto 1\wedge x\wedge y} \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p\times\Sigma_q} X_p \wedge Y_q \xrightarrow{\text{projection}} (X\wedge Y)_{p+q}$$

form a bimorphism – and in fact a universal one.

The smash product $X\wedge Y$ is a functor in both variables. This is fairly evident from the construction (C), but it can also be deduced from the universal property (A) or the adjunction (B) as follows. If we use the universal property (A) the contravariant functoriality of the set $\mathrm{Bimor}((X,Y),Z)$ in $X$ and $Y$ turns into functoriality of the representing objects. In more detail, if $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ are morphisms of symmetric spectra, then the collection of pointed maps

$$\left\{ X_p \wedge\ Y_q \xrightarrow{f_p\wedge g_q} X'_p \wedge\ Y'_q \xrightarrow{i'_{p,q}} (X'\wedge Y')_{p+q} \right\}_{p,q\geq 0}$$

forms a bimorphism $(X,Y) \longrightarrow X'\wedge Y'$. So there is a unique morphism of symmetric spectra $f\wedge g : X\wedge Y \longrightarrow X'\wedge Y'$ such that $(f\wedge g)_{p+q}\circ i_{p,q} = i'_{p,q}\circ(f_p\wedge g_q)$ for all $p,q\geq 0$. The uniqueness part of the universal property implies that this is compatible with identities and composition in both variables.

If we define the smash product as a representing object for the functor $\mathrm{Hom}(X,\mathrm{Hom}(Y,-))$, then functoriality in $X$ and $Y$ follows from functoriality of the latter functor in $X$ and $Y$.

Now that we have constructed a smash product functor we can investigate its formal and homotopical properties. The formal properties will be discussed in the rest of this section, but we postpone the homotopical analysis until Section 5 of Chapter II.

The first thing to show is that the smash product is symmetric monoidal. Since 'symmetric monoidal' is extra data, and not a property, we are obliged to construct associativity isomorphisms

$$\alpha_{X,Y,Z}\ :\ (X\wedge Y)\wedge Z \ \longrightarrow\ X\wedge(Y\wedge Z)\ ,$$

symmetry isomorphisms

$$\tau_{X,Y}\ :\ X\wedge Y \ \longrightarrow\ Y\wedge X$$

and right unit isomorphisms

$$r_X\ :\ X\wedge\mathbb{S} \ \longrightarrow\ X$$

which satisfy a certain list of coherence conditions. We then define left unit isomorphisms $l_X : \mathbb{S}\wedge X \longrightarrow X$ as the composite of the symmetry isomorphism $\tau_{\mathbb{S},X}$ and the right unit $r_X$.

**First construction.** We can obtain all the isomorphisms of the symmetric monoidal structure just from the universal property. So suppose that for each pair of symmetric spectra $(X,Y)$ a smash product $X\wedge Y$ and a universal bimorphism $i = \{i_{p,q}\} : (X,Y) \longrightarrow X\wedge Y$ have been chosen.

For construction the associativity isomorphism we notice that the family

$$\left\{X_p \wedge\ Y_q \wedge Z_r \xrightarrow{i_{p,q}\wedge \mathrm{Id}} (X\wedge Y)_{p+q}\wedge Z_r \xrightarrow{i_{p+q,r}} ((X\wedge Y)\wedge Z)_{p+q+r}\right\}_{p,q,r\geq 0}$$

and the family

$$\left\{X_p \wedge\ Y_q \wedge Z_r \xrightarrow{\mathrm{Id}\wedge i_{q,r}} X_p\wedge (Y\wedge Z)_{q+r} \xrightarrow{i_{p,q+r}} (X\wedge (Y\wedge Z))_{p+q+r}\right\}_{p,q,r\geq 0}$$

both have the universal property of a *tri*morphism (whose definition is hopefully clear) out of $X$, $Y$ and $Z$. The uniqueness of representing objects gives a unique isomorphism of symmetric spectra

$$\alpha_{X,Y,Z} \ : \ (X\wedge Y)\wedge Z \ \cong X\wedge (Y\wedge Z)$$

such that $(\alpha_{X,Y,Z})_{p,q,r}\circ i_{p+q,r}\circ(i_{p+q}\wedge \mathrm{Id}) = i_{p,q+r}\circ(\mathrm{Id}\wedge i_{q,r})$.

The symmetry isomorphism $\tau_{X,Y} : X\wedge Y \longrightarrow Y\wedge X$ corresponds to the bimorphism

$$\left\{X_p\wedge\ Y_q \xrightarrow{\text{twist}} Y_q\wedge\ X_p \xrightarrow{\iota_{q,p}} (Y\wedge X)_{q+p} \xrightarrow{\chi_{q,p}} (Y\wedge X)_{p+q}\right\}_{p,q\geq 0} . \tag{3.3}$$

The block permutation $\chi_{q,p}$ is crucial here: without it the diagram (3.1) would not commute and we would not have a bimorphism. If we restrict the composite $\tau_{Y,X}\circ\tau_{X,Y}$ in level $p+q$ along the map $i_{p,q} : X_p\wedge Y_q \longrightarrow (X\wedge Y)_{p+q}$ we get $i_{p,q}$ again. Thus $\tau_{Y,X}\circ\tau_{X,Y} = \mathrm{Id}_{X\wedge Y}$ and $\tau_{Y,X}$ is inverse to $\tau_{X,Y}$.

In much the same spirit, the universal properties can be used to provide a right unit isomorphism. Because of the commuting left part of the diagram (3.1) a bimorphism $b : (X,\mathbb{S}) \longrightarrow Z$ is completely determined by the components $b_{p,0} : X_p\wedge S^0 \longrightarrow Z_p$, which constitute a morphism $b_{\bullet,0} : X \longrightarrow Z$; moreover, every morphism from $X$ to $Z$ arises in this way from a unique bimorphism out of $(X,\mathbb{S})$. Hence the morphism $r_X : X\wedge\mathbb{S}\longrightarrow X$ corresponding to the bimorphism consisting of the iterated structure maps $\sigma^m : X_n\wedge S^m \longrightarrow X_{n+m}$ is an isomorphism of symmetric spectra.

**Second construction.** The coherence isomorphisms can also be obtained from the construction of the smash product in (C) above, as opposed to the universal property. In level $n$ the spectra $(X\wedge Y)\wedge Z$ and $X\wedge(Y\wedge Z)$ are quotients of the spaces

$$\bigvee_{p+q+r=n} \Sigma_n^+\wedge_{\Sigma_{p+q}\times\Sigma_r}\left(\Sigma_{p+q}^+\wedge_{\Sigma_p\times\Sigma_q} X_p\wedge Y_q\right)\wedge Z_r$$

respectively

$$\bigvee_{p+q+r=n} \Sigma_n^+\wedge_{\Sigma_p\times\Sigma_{q+r}} X_p\wedge\left(\Sigma_{q+r}^+\wedge_{\Sigma_q\times\Sigma_r} Y_q\wedge Z_r\right) .$$

The wedges run over all non-negative values of $p,q$ and $r$ which sum up to $n$. We get a well-defined maps between these two wedges by wedging over the maps

$$\begin{array}{ccc}
\Sigma_n^+\wedge_{\Sigma_{p+q}\times\Sigma_r}\left(\Sigma_{p+q}^+\wedge_{\Sigma_p\times\Sigma_q} X_p\wedge Y_q\right)\wedge Z_r & \longleftrightarrow & \Sigma_n^+\wedge_{\Sigma_p\times\Sigma_{q+r}} X_p\wedge\left(\Sigma_{q+r}^+\wedge_{\Sigma_q\times\Sigma_r} Y_q\wedge Z_r\right)\\
\sigma\wedge((\tau\wedge x\wedge y)\wedge z) & \longmapsto & (\sigma(\tau\times 1))\wedge(x\wedge(1\wedge y\wedge z))\\
\sigma(1\times\gamma)\wedge((1\wedge x\wedge y)\wedge z) & \longleftarrow & \sigma\wedge(x\wedge(\gamma\wedge y\wedge z))
\end{array}$$

where $\sigma\in\Sigma_n$, $\tau\in\Sigma_{p+q}$, $\gamma\in\Sigma_{q+r}$, $x\in X_p$, $y\in Y_q$ and $z\in Z_r$.

The symmetry isomorphism $\tau_{X,Y} : X\wedge X\longrightarrow Y\wedge X$ is obtained by wedging over the maps

$$\begin{array}{ccc}
\Sigma_n^+\wedge_{\Sigma_p\times\Sigma_q} X_p\wedge Y_q & \longrightarrow & \Sigma_n^+\wedge_{\Sigma_q\times\Sigma_p} Y_q\wedge X_p\\
\sigma\wedge x\wedge y & \longmapsto & (\sigma\chi_{p,q})\wedge y\wedge x
\end{array}$$

where $\sigma\in\Sigma_n$, $x\in X_p$ and $y\in Y_q$ and passing to quotient spaces. The shuffle permutation $\chi_{p,q}$ is needed to make this map well-defined on quotients.

THEOREM 3.4. *The associativity, symmetry and unit isomorphisms make the smash product of symmetric spectra into a symmetric monoidal product with unit object the sphere spectrum $\mathbb{S}$. This product is* closed *symmetric monoidal in the sense that the smash product is adjoint to the internal Hom spectrum, i.e., there is an adjunction isomorphism*

$$\mathrm{Hom}(X \wedge Y, Z) \;\cong\; \mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \ .$$

PROOF. We have to verify that several coherence diagrams commute. We start with the pentagon condition for associativity. Given a fourth symmetric spectrum $W$ we consider the pentagon

$$\begin{tikzcd}[column sep=small] & ((W \wedge X) \wedge Y) \wedge Z \arrow[dl, "\alpha_{W,X,Y} \wedge \mathrm{Id}"'] \arrow[dr, "\alpha_{W \wedge X, Y, Z}"] & \\ (W \wedge (X \wedge Y)) \wedge Z \arrow[d, "\alpha_{W, X \wedge Y, Z}"'] & & (W \wedge X) \wedge (Y \wedge Z) \arrow[d, "\alpha_{W,X,Y\wedge Z}"] \\ W \wedge ((X \wedge Y) \wedge Z) \arrow[rr, "\mathrm{Id} \wedge \alpha_{X,Y,Z}"'] & & W \wedge (X \wedge (Y \wedge Z))) \end{tikzcd}$$

If we evaluate either composite at level $o + p + q + r$ and precompose with

$$\begin{aligned} W_o \wedge X_p \wedge Y_q \wedge Z_r &\xrightarrow{i_{o,p} \wedge \mathrm{Id} \wedge \mathrm{Id}} (W \wedge X)_{o+p} \wedge Y_q \wedge Z_r \\ &\xrightarrow{i_{o+p,q} \wedge \mathrm{Id}} ((W \wedge X) \wedge Y)_{o+p+q} \wedge Z_r \xrightarrow{i_{o+p+q,r}} (((W \wedge X) \wedge Y) \wedge Z)_{o+p+q+r} \end{aligned}$$

then both ways around the pentagon yield the composite

$$\begin{aligned} W_o \wedge X_p \wedge Y_q \wedge Z_r &\xrightarrow{\mathrm{Id} \wedge \mathrm{Id} \wedge i_{q,r}} W_o \wedge X_p \wedge (Y \wedge Z)_{q+r} \\ &\xrightarrow{\mathrm{Id} \wedge i_{p,q+r}} W_o \wedge (X \wedge (Y \wedge Z))_{p+q+r} \xrightarrow{i_{o,p+q+r}} (W \wedge (X \wedge (Y \wedge Z)))_{o+p+q+r} \ . \end{aligned}$$

So the uniqueness part of the universal property shows that the pentagon commutes.

Coherence between associativity and symmetry isomorphisms means that the two composites from the upper left to the lower right corner of the diagram

$$\begin{tikzcd} (X \wedge Y) \wedge Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "\tau_{X,Y} \wedge \mathrm{Id}"'] & X \wedge (Y \wedge Z) \arrow[r, "\tau_{X, Y \wedge Z}"] & (Y \wedge Z) \wedge X \arrow[d, "\alpha_{Y,Z,X}"] \\ (Y \wedge X) \wedge Z \arrow[r, "\alpha_{Y,X,Z}"'] & Y \wedge (X \wedge Z) \arrow[r, "\mathrm{Id} \wedge \tau_{X,Z}"'] & Y \wedge (Z \wedge X) \end{tikzcd}$$

should be equal, and the same kind of argument as for the pentagon relation for associativity works.

It remains to check the coherence conditions relating associativity and symmetry isomorphisms to the unit morphisms. We define the left unit isomorphism $l_X : \mathbb{S} \wedge X \cong X$ as the composite $l_X = r_X \tau_{\mathbb{S},X}$ of the right unit and the symmetry isomorphism. Then the unit isomorphism are compatible with symmetry, and furthermore we have $l_{\mathbb{S}} = r_{\mathbb{S}} : \mathbb{S} \wedge \mathbb{S} \longrightarrow \mathbb{S}$ since both arise from the bimorphism $S^n \wedge S^m \xrightarrow{\cong} S^{n+m}$ made up from the canonical isomorphisms. Finally, the triangle

$$\begin{tikzcd} (X \wedge \mathbb{S}) \wedge Y \arrow[rr, "\alpha_{X,\mathbb{S},Y}"] \arrow[dr, "r_X \wedge \mathrm{Id}"'] & & X \wedge (\mathbb{S} \wedge Y) \arrow[dl, "\mathrm{Id} \wedge l_Y"] \\ & X \wedge Y & \end{tikzcd}$$

should commute, which is true since it holds after restriction with the maps

$$(X_p \wedge S^q) \wedge Y_r \xrightarrow{i_{p,q} \wedge \mathrm{Id}} (X \wedge \mathbb{S})_{p+q} \wedge Y_r \xrightarrow{i_{p+q,r}} ((X \wedge \mathbb{S}) \wedge Y)_{p+q+r} \ .$$

$\square$

Now we identify the smash products of certain kinds of symmetric spectra and relate it by natural maps to other constructions. We start by describing the smash product with a semifree spectrum. In Example 2.20 we introduced the twisted smash product $L \rhd_m X$ of a pointed $\Sigma_m$-space $L$ and a symmetric spectrum $X$.

Let $X$ be a symmetric spectrum and $L$ be a pointed $\Sigma_m$-space (or $\Sigma_m$-simplicial set) for some $m \geq 0$. The twisted smash product $L \rhd_m X$ was defined in Example 2.20 and consists of a point in levels smaller than $m$ is given in general by

$$(L \rhd_m X)_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n \ .$$

In order to link $L \rhd_m X$ to $G_m L \wedge X$ we note that as $n$ varies, the $(m, n)$-components

$$L \wedge X_n \ = \ (G_m L)_m \wedge X_n \ \xrightarrow{\iota_{m,n}} \ (G_m L \wedge X)_{m+n} \ = \ (\mathrm{sh}^m (G_m L \wedge X))_n$$

of the universal bimorphism $\iota : (G_m L, X) \longrightarrow G_m L \wedge X$ in fact define a morphism of $\Sigma_m$-symmetric spectra $\bar{b} : L \wedge X \longrightarrow \mathrm{sh}^m (G_m L \wedge X)$. By the adjunction (2.21) this morphism corresponds to a morphism of symmetric spectra $b : L \rhd_m X \longrightarrow G_m L \wedge X$.

Proposition 3.5. *Let $L$ be a pointed $\Sigma_m$-space (or $\Sigma_m$-simplicial set) for some $m \geq 0$ and $X$ a symmetric spectrum. Then the morphism of symmetric spectra*

$$b \ : \ L \rhd_m X \ \longrightarrow \ G_m L \wedge X$$

*is a natural isomorphism.*

Proof. In (2.21) we constructed a natural bijection

$$\mathcal{S}p^\Sigma (L \rhd_m X, Z) \ \cong \ \Sigma_m\text{-}\mathcal{S}p^\Sigma (L \wedge X, \mathrm{sh}^m Z) \ .$$

The adjunctions between the smash product and mapping spaces, the definition of the homomorphism spectrum and the fact that $G_m$ is adjoint to evaluation at level $m$ yield a natural bijection

$$\begin{aligned} \Sigma_m\text{-}\mathcal{S}p^\Sigma (L \wedge X, \mathrm{sh}^m Z)) \ &\cong \ \Sigma_m\text{-}\,\mathbf{sset}_*(L, \mathrm{map}(X, \mathrm{sh}^m Z)) \\ &= \ \Sigma_m\text{-}\,\mathbf{sset}_*(L, \mathrm{Hom}(X, Z)_m) \ \cong \ \mathcal{S}p^\Sigma (G_m L, \mathrm{Hom}(X, Z)) \ . \end{aligned}$$

Combining all these isomorphisms gives a representation of the functor $\mathcal{S}p^\Sigma (G_m L, \mathrm{Hom}(X, -))$ by the symmetric spectrum $L \rhd_m X$. Since the smash product $G_m L \wedge X$ represents the same functor, we get a preferred isomorphism $L \rhd_m X \cong G_m L \wedge X$, which in fact equals the morphism $b$. $\square$

We specialize the previous proposition in several steps. The special case $m = 0$ provides a natural isomorphism

$$K \wedge X \ = \ K \rhd_0 X \ \cong \ (\Sigma^\infty K) \wedge X$$

for pointed spaces (or simplicial sets) $K$ and symmetric spectra $X$. We can also consider a $\Sigma_m$-space $L$ and a $\Sigma_n$-space $L'$. If we spell out all definitions we see that $L \rhd_m (G_n L')$ is isomorphic to the semifree symmetric spectrum $G_{m+n}(\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L')$. So Proposition 3.5 specializes to a natural isomorphism

$$G_{m+n}(\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L') \ \cong \ G_m L \wedge G_n L' \ . \tag{3.6}$$

The isomorphism is adjoint to the $\Sigma_{m+n}$-equivariant map

$$\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge L' \ \longrightarrow \ (G_m L \wedge G_n L')_{m+n}$$

which in turn is adjoint to the $\Sigma_m \times \Sigma_n$-equivariant map

$$L \wedge L' = (G_m L)_m \wedge (G_n L')_n \ \xrightarrow{i_{m,n}} \ (G_m L \wedge G_n L')_{m+n}$$

given by the universal bimorphism. So the isomorphism (3.6) rephrases the fact that a bimorphism from $(G_m L, G_n L')$ to $Z$ is uniquely determined by its $(m, n)$-component, which can be any $\Sigma_m \times \Sigma_n$-equivariant map $L \wedge L' \longrightarrow Z_{m+n}$. The isomorphism (3.6), and the ones which follow below, are suitably associative, commutative and unital.

As a special case we can consider smash products of free symmetric spectra. If $K$ and $K'$ are pointed spaces or simplicial set then we have $F_m K = G_m(\Sigma_m^+ \wedge K)$ and $F_n K' = G_n(\Sigma_n^+ \wedge K')$, so the isomorphism (3.6) specializes to an associative, commutative and unital isomorphism

$$F_{m+n}(K \wedge K') \;\cong\; F_m K \wedge F_n K' \;.$$

As the even more special case for $m = n = 0$ we obtain a natural isomorphism of suspension spectra

$$(\Sigma^\infty K) \,\wedge\, (\Sigma^\infty L) \;\cong\; \Sigma^\infty(K \wedge L)$$

for all pairs of pointed spaces (or pointed simplicial sets) $K$ and $L$.

The tensor product of $I$-spaces is taken to the smash product by the smash product paring of Example 2.31. More precisely, for $I$-spaces $T, T'$ and symmetric spectra $X$ and $X'$ there is a natural and coherent isomorphism

$$(T \wedge X) \wedge (T' \wedge X') \;\cong\; (T \wedge T') \wedge (X \wedge X') \;.$$

[elaborate] Note that here the various smash product signs have three different meanings: three smash products are pairings between an $I$-space and a symmetric spectrum, two smash products are internal smash products of symmetric spectra and one is the internal smash product of $I$-spaces.

The bimorphism

$$(\mathrm{sh}\, X)_p \wedge Y_q \;=\; X_{1+p} \wedge Y_q \xrightarrow{\;i_{1+p,q}\;} (X \wedge Y)_{1+p+q} \;=\; \mathrm{sh}(X \wedge Y)_{p+q}$$

corresponds to a natural homomorphism of symmetric spectra

$$\xi_{X,Y} \;:\; (\mathrm{sh}\, X) \wedge Y \;\longrightarrow\; \mathrm{sh}(X \wedge Y) \;.$$

The homomorphism $\xi_{X,Y}$ is compatible with the unit and associativity isomorphisms in the sense that the following diagrams commute

$$\begin{tikzcd}
((\mathrm{sh}\, X) \wedge Y) \wedge Z \arrow[r, "\alpha_{\mathrm{sh}\, X,Y,Z}"] \arrow[d, "\xi_{X,Y} \wedge \mathrm{Id}"'] & (\mathrm{sh}\, X) \wedge (Y \wedge Z) \arrow[dd, "\xi_{X,Y \wedge Z}"] \\
\mathrm{sh}(X \wedge Y) \wedge Z \arrow[d, "\xi_{X \wedge Y, Z}"'] & \\
\mathrm{sh}((X \wedge Y) \wedge Z) \arrow[r, "\mathrm{sh}(\alpha_{X,Y,Z})"'] & \mathrm{sh}(X \wedge (Y \wedge Z))
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathrm{sh}\, X) \wedge \mathbb{S} \arrow[dr, "r_{\mathrm{sh}\, X}"] \arrow[dd, "\xi_{X,\mathbb{S}}"'] & \\
& \mathrm{sh}\, X \\
\mathrm{sh}(X \wedge \mathbb{S}) \arrow[ur, "\mathrm{sh}(r_X)"'] &
\end{tikzcd}$$

Moreover, the map $\xi_{\mathbb{S},Y}$ 'is' the morphism $\lambda_Y : S^1 \wedge Y \longrightarrow \mathrm{sh}\, Y$ is the following sense: we have $\mathrm{sh}\, \mathbb{S} = \Sigma^\infty S^1$ and so $(\mathrm{sh}\, \mathbb{S}) \wedge Y \cong S^1 \wedge Y$. with this identification, the following square commutes

$$\begin{tikzcd}
(\mathrm{sh}\, \mathbb{S}) \wedge Y \arrow[r, "\xi_{\mathbb{S},Y}"] \arrow[d, "\cong"'] & \mathrm{sh}(\mathbb{S} \wedge Y) \arrow[d, "\mathrm{sh}(l_Y)"] \\
S^1 \wedge Y \arrow[r, "\lambda_Y"'] & \mathrm{sh}\, Y
\end{tikzcd}$$

There are natural composition morphisms

$$\circ \;:\; \mathrm{Hom}(Y,Z) \,\wedge\, \mathrm{Hom}(X,Y) \;\longrightarrow\; \mathrm{Hom}(X,Z)$$

which are associative and unital with respect to a unit map $\mathbb{S} \longrightarrow \mathrm{Hom}(X,X)$ adjoint to the identity of $X$ (which is a vertex in level 0 of the spectrum $\mathrm{Hom}(X,X)$). The composition morphism is obtained, by the universal property of the smash product, from the bimorphism consisting of the maps

$$\begin{aligned}
\mathrm{map}(Y, \mathrm{sh}^n Z) \wedge \mathrm{map}(X, \mathrm{sh}^m Y) &\xrightarrow{\;\mathrm{sh}^m \wedge \mathrm{Id}\;} \mathrm{map}(\mathrm{sh}^m Y, \mathrm{sh}^{m+n} Z) \wedge \mathrm{map}(X, \mathrm{sh}^m Y) \\
&\xrightarrow{\;\;\circ\;\;} \mathrm{map}(X, \mathrm{sh}^{m+n} Z) \xrightarrow{\;\mathrm{map}(X,\chi_{m,n})\;} \mathrm{map}(X, \mathrm{sh}^{n+m} Z)
\end{aligned}$$

where the second map is the composition pairing of Example 2.24. If we specialize to $X = Y = Z$, we recover the multiplication of the endomorphism ring spectrum as defined in Example 2.26.

If $X$ and $Y$ are symmetric spectra we can also define natural coherent morphisms

$$X^K \wedge Y^L \longrightarrow (X \wedge Y)^{K \wedge L}$$

for pointed spaces (simplicial sets) $K$ and $L$ and morphisms

$$\begin{aligned} \mathrm{map}(A, X) \wedge \mathrm{map}(B, Y) &\longrightarrow \mathrm{map}(A \wedge B, X \wedge Y) \quad \text{and} \\ \mathrm{Hom}(A, X) \wedge \mathrm{Hom}(B, Y) &\longrightarrow \mathrm{Hom}(A \wedge B, X \wedge Y) \end{aligned}$$

for symmetric spectra $A$ and $B$.

Now we can make precise the idea that symmetric ring spectra are the same as monoid objects in the symmetric monoidal category of symmetric spectra with respect to the smash product.

Construction 3.7. Let us define an *implicit symmetric ring spectrum* as a symmetric spectrum $R$ together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$ which are associative and unital in the sense that the following diagrams commute

$$\begin{tikzcd} (R \wedge R) \wedge R \arrow[r, "\alpha_{R,R,R}"] \arrow[d, "\mu \wedge \mathrm{Id}"'] & R \wedge (R \wedge R) \arrow[r, "\mathrm{Id} \wedge \mu"] & R \wedge R \arrow[d, "\mu"] \\ R \wedge R \arrow[rr, "\mu"'] & & R \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \wedge R \arrow[r, "\iota \wedge \mathrm{Id}"] \arrow[dr, "l_R"'] & R \wedge R \arrow[d, "\mu"] & R \wedge \mathbb{S} \arrow[l, "\mathrm{Id} \wedge \iota"'] \arrow[dl, "r_R"] \\ & R & \end{tikzcd}$$

We say that the implicit symmetric ring spectrum $(R, \mu, \iota)$ is *commutative* if the multiplication is unchanged when composed with the symmetric isomorphism, i.e., if the relation $\mu \circ \tau_{R,R} = \mu$ holds.

Given an implicit symmetric ring spectrum $(R, \mu, \iota)$ we can make the collection of $\Sigma_n$-spaces $R_n$ into a symmetric ring spectrum in the sense of the original Definition 1.3 as follows. As unit maps we simply take the components of $\iota : \mathbb{S} \longrightarrow R$ in levels 0 and 1. We define the multiplication map $\mu_{n,m} : R_n \wedge R_m \longrightarrow R_{n+m}$ as the composite

$$R_n \wedge R_m \xrightarrow{i_{n,m}} (R \wedge R)_{n+m} \xrightarrow{\mu_{n+m}} R_{n+m} .$$

Then the associativity condition for $\mu$ above directly translates into the associativity condition of Definition 1.3 for the maps $\mu_{n,m}$. Evaluating the two commuting unit triangles in level 0 gives the unit condition of Definition 1.3. Spelling out the condition $\mu(\iota \wedge \mathrm{Id}) = l_R = r_R \circ \tau_{\mathbb{S},R}$ in level $1 + n$ and composing with the map $i_{1,n} : S^1 \wedge R_n \longrightarrow (\mathbb{S} \wedge R)_{1+n}$ gives the centrality condition of Definition 1.3. Finally, the condition $\mu(\mathrm{Id} \wedge \iota) = r_R$ in level $n + 1$ composed with $\iota_{n,1} : R_n \wedge S^1 \longrightarrow (R \wedge \mathbb{S})_{n+1}$ shows that $\mu_{n,1} \circ (\mathrm{Id} \wedge \iota_1)$ equals the structure map $\sigma_n : R_n \wedge S^1 \longrightarrow R_{n+1}$ of the underlying symmetric spectrum of $R$. So the conceivably different meaning of 'underlying symmetric spectrum' in the sense of Remark 1.6 (iii) in fact coincides with the underlying spectrum $R$.

Theorem 3.8. *The construction 3.7 which turns an implicit symmetric ring spectrum into a symmetric ring spectrum in sense of the original Definition 1.3 is an isomorphism between the category of implicit symmetric ring spectra and the category of symmetric ring spectra. The functor restricts to an isomorphism from the category of commutative implicit symmetric ring spectra to the category of commutative symmetric ring spectra.*

Proof. □

Now that we have carefully stated and proved Theorem 3.8 we will start to systematically blur the distinction between implicit and explicit symmetric ring spectra. Whenever convenient we use the isomorphism of categories to go back and forth between the two notions without further mentioning.

Example 3.9 (Smash product of ring spectra). Here is a construction of a new symmetric ring spectrum from old ones for which the possibility to define ring spectra 'implicitly' is crucial. If $R$ and $S$ are symmetric ring spectra, then the smash product $R \wedge S$ has a natural structure as symmetric ring spectrum as follows. The unit map is defined from the unit maps of $R$ and $S$ as the composite

$$\mathbb{S} \xleftarrow{r^{-1}_{\mathbb{S},\mathbb{S}}=l^{-1}_{\mathbb{S},\mathbb{S}}} \mathbb{S} \wedge \mathbb{S} \xrightarrow{\iota \wedge \iota} R \wedge S \ .$$

The multiplication map of $R \wedge S$ is defined from the multiplications of $R$ and $S$ as the composite

$$(R \wedge S) \wedge (R \wedge S) \xrightarrow{\mathrm{Id} \wedge \tau_{S,R} \wedge \mathrm{Id}} (R \wedge R) \wedge (S \wedge S) \xrightarrow{\mu \wedge \mu} R \wedge S \ ,$$

where we have suppressed some associativity isomorphisms. It is a good exercise to insert these associativity isomorphisms and observe how the hexagon condition for associativity and symmetry isomorphisms enters the verification that the product of $R \wedge S$ is in fact associative.

Example 3.10. Another class of examples which can only be given as implicit symmetric ring spectra are symmetric ring spectra 'freely generated' by a symmetric spectrum. These come in two flavors, an associative and a commutative (and associative) version.

Given a symmetric spectrum $X$ we define the *tensor algebra* as the symmetric spectrum

$$TX \ = \ \bigvee_{n \geq 0} \underbrace{X \wedge \cdots \wedge X}_{n}$$

with the convention that a 0-fold smash product is the unit object $\mathbb{S}$. The unit morphism $\iota : \mathbb{S} \longrightarrow TX$ is the inclusion of the wedge summand for $n = 0$. The multiplication is given by 'concatenation', i.e., the restriction of $\mu : TX \wedge TX \longrightarrow TX$ to the $(n, m)$ wedge summand is the canonical isomorphism

$$X^{\wedge n} \wedge X^{\wedge m} \xrightarrow{\cong} X^{\wedge(n+m)}$$

followed by the inclusion of the wedge summand indexed by $n+m$. In order to be completely honest here we should throw in several associativity isomorphisms; strictly speaking already the definition of $TX$ requires choices of how to associate expressions such as $X \wedge X \wedge X$ and higher smash powers. However, all of this is taken care of by the coherence conditions of the associativity (and later the symmetry) isomorphisms, and we will not belabor this point any further.

Given any symmetric ring spectrum $R$ and a morphism of symmetric spectra $f : X \longrightarrow R$ we can define a new morphism $\hat{f} : TX \longrightarrow R$ which on the $n$th wedge summand is the composite

$$X^{\wedge n} \xrightarrow{f^{\wedge n}} R^{\wedge n} \xrightarrow{\mu_n} R \ .$$

Here $\mu_n$ is the iterated multiplication map, which for $n = 0$ has to be interpreted as the unit morphism $\iota : \mathbb{S} \longrightarrow R$. This extension $\hat{f} : TX \longrightarrow R$ is in fact a homomorphism of (implicit) symmetric ring spectra. Moreover, if $g : TX \longrightarrow R$ is any homomorphism of symmetric ring spectra then $g = \hat{g}_1$ for $g_1 : X \longrightarrow R$ the restriction of $g$ to the wedge summand indexed by 1. Another way to say this is that

$$\mathrm{Hom}_{\text{ring spectra}}(TX, R) \ \longrightarrow \ \mathcal{S}p^{\Sigma}(X, R) \ , \quad g \ \mapsto g_1$$

is a natural bijection. In fact, this bijection makes the tensor algebra functor into a left adjoint of the forgetful functor from symmetric ring spectra to symmetric spectra.

The construction has a commutative variant. We define the *symmetric algebra* generated by a symmetric spectrum $X$ as

$$PX \ = \ \bigvee_{n \geq 0} (X^{\wedge n})/\Sigma_n \ .$$

Here $\Sigma_n$ permutes the smash factors [elaborate] of $X^{\wedge n}$ using the symmetry isomorphisms, and we take the quotient symmetric spectrum. This symmetric spectrum has unique unit and multiplication maps such that the quotient morphism $TX \longrightarrow PX$ becomes a homomorphism of symmetric ring spectra. So the unit morphism $\iota : \mathbb{S} \longrightarrow PX$ is again the inclusion of the wedge summand for $n = 0$ and the multiplication is the wedge of the morphisms

$$(X^{\wedge n})/\Sigma_n \wedge (X^{\wedge m})/\Sigma_m \longrightarrow (X^{\wedge(n+m)})/\Sigma_{n+m}$$

induced on quotients by $X^{\wedge n} \wedge X^{\wedge m} \cong X^{\wedge(n+m)}$.

Example 3.11. For two abelian groups $A$ and $B$, a natural morphism of symmetric spectra

$$HA \wedge HB \longrightarrow H(A \otimes B)$$

is obtained, by the universal property (3.2), from the bilinear morphism

$$\begin{aligned}(HA)_n \wedge (HB)_m \;&=\; (A \otimes \mathbb{Z}[S^n]) \wedge (B \otimes \mathbb{Z}[S^m]) \\ &\longrightarrow\; (A \otimes B) \otimes \mathbb{Z}[S^{n+m}] \;=\; (H(A \otimes B))_{n+m}\end{aligned}$$

given by

$$\left(\sum_i a_i \cdot x_i\right) \wedge \left(\sum_j b_j \cdot y_j\right) \longmapsto \sum_{i,j} (a_i \otimes b_j) \cdot (x_i \wedge y_j) \ .$$

A unit map $\mathbb{S} \longrightarrow H\mathbb{Z}$ is given by the inclusion of generators. With respect to these maps, $H$ becomes a lax symmetric monoidal functor from the category of abelian groups to the category of symmetric spectra. As a formal consequence, $H$ turns a ring $A$ into a symmetric ring spectrum with multiplication map

$$HA \wedge HA \longrightarrow H(A \otimes A) \longrightarrow HA \ .$$

This is the 'implicit' construction of an Eilenberg-Mac Lane ring spectrum whose explicit variant appeared in Example 2.7. Similarly, an $A$-module structure on $B$ gives rise to an $HA$-module structure on $HB$.

The definition of the symmetric spectrum $HA$ makes just as much sense when $A$ is a *simplicial* abelian group; thus the Eilenberg-Mac Lane functor makes simplicial rings into symmetric ring spectra, respecting possible commutativity of the multiplications.

## 4. Homotopy groups, $\mathcal{M}$-modules and semistability

As we shall explain in Section 4 of Chapter II, formally inverting the $\pi_*$-isomorphisms between symmetric spectra leaves 'too many homotopy types'. Instead, we will later introduce a strictly larger class of *stable equivalences*, defined as the morphisms which induce isomorphisms on all cohomology theories. In order to understand the relationship between $\pi_*$-isomorphisms and stable equivalences, it is useful to exploit extra algebraic structure on the homotopy groups of a symmetric spectrum. This extra structure is an action of the *injection monoid* $\mathcal{M}$, the monoid of injective self-maps of the set of natural numbers under composition. The $\mathcal{M}$-modules that come up, however, have a special property which we call *tameness*, see Definition 4.11. Tameness has strong algebraic consequences and severely restricts the kinds of $\mathcal{M}$-modules which can arise as homotopy groups of symmetric spectra.

An important class of symmetric spectra is formed by the *semistable* symmetric spectra. Within this class, stable equivalences coincide with $\pi_*$-isomorphisms, so it is very useful to recognize a given symmetric spectrum as semistable. In Theorem 4.44, we characterize the semistable symmetric spectra as those for which the $\mathcal{M}$-action on homotopy groups is trivial.

**4.1. Exact sequences of stable homotopy groups.** In this section we construct long exact sequences of homotopy groups from a morphism of symmetric spectra. The long exact homotopy sequence involving the homotopy fibre also exists unstably. However, a fundamental property of stable homotopy groups of spectra which is not satisfied by (unstable) homotopy groups of spaces is that also the mapping cone fits into a long exact homotopy sequence.

The *mapping cone* $C(f)$ of a morphism of symmetric spectra $f : X \longrightarrow Y$ is defined as

$$C(f) \;=\; ([0,1] \wedge X) \cup_f Y \ , \tag{4.1}$$

where $[0,1]$ is pointed by $0 \in [0,1]$, so that $[0,1] \wedge X$ is the cone of $X$. At level $n$, $C(f)_n$ is just the pointed mapping cone of $f_n : X_n \longrightarrow Y_n$. The mapping cone comes with an inclusion $Y \longrightarrow C(f)$ whose quotient $C(f)/Y$ is the spectrum $S^1 \wedge X$. [how do we identify $[0,1]/\{0,1\}$ with $S^1$ ?]

We define a *connecting homomorphism* $\delta : \pi_{1+k}C(f) \longrightarrow \pi_k X$ as the composite

$$\pi_{1+k}C(f) \xrightarrow{\pi_{1+k}(\text{proj})} \pi_{1+k}(S^1 \wedge X) \cong \pi_k X \ , \tag{4.2}$$

where the first map is the effect of the projection $C(f) \longrightarrow S^1 \wedge X$ on homotopy groups, and the second map is the inverse of the isomorphism $S^1 \wedge - : \pi_k X \longrightarrow \pi_{1+k}(S^1 \wedge X)$ introduced in Example 2.16.

There is construction 'dual' to the mapping cone, namely the *homotopy fibre*. Let $f : X \longrightarrow Y$ be a morphism between symmetric spectra. The homotopy fibre $F(f)$ is the symmetric spectrum

$$F(f) \ = \ * \times_Y Y^{[0,1]} \times_Y X$$

i.e., the pullback in the cartesian square

$$\begin{tikzcd} F(f) \arrow[r, "p"] \arrow[d] & X \arrow[d, "{(*,f)}"] \\ Y^{[0,1]} \arrow[r, "{(\mathrm{ev}_0,\mathrm{ev}_1)}"'] & Y \times Y \end{tikzcd} \tag{4.3}$$

Here $\mathrm{ev}_i : Y^{[0,1]} \longrightarrow Y$ for $i = 0, 1$ is the $i$th evaluation map which takes a path $\omega \in Y^{[0,1]}$ to $\omega(i)$, i.e., the start or endpoint. So levelwise a point in $F(f)$ is a pair $(\omega, x)$ where $\omega$ is a path in $Y$ starting at the basepoint and $x$ is a lift of the endpoint, i.e., $f(x) = \omega(1)$.

There are morphisms

$$\Omega Y \xrightarrow{i} F(f) \xrightarrow{p} X \xrightarrow{f} Y$$

the first two being given by

$$i(\omega) = (\omega, *) \quad \text{respectively} \quad p(\omega, x) = x \ .$$

The composite $pi$ is the trivial map and the composite $fp$ comes with a preferred null-homotopy $H : [0,1]^+ \wedge F(f) \longrightarrow Y$ [specify], i.e., $H$ starts with the constant map at the basepoint and ends with $fp$. Thus $H$ factors over a well-defined morphism

$$h \ : \ S^1 \wedge F(f) \longrightarrow Y/f(X) \ . \tag{4.4}$$

We define a *connecting homomorphism* $\delta : \pi_{1+k}Y \longrightarrow \pi_k F(f)$ as the composite

$$\pi_{1+k}Y \xrightarrow{\alpha^{-1}} \pi_k(\Omega Y) \xrightarrow{\pi_k(i)} \pi_k F(f), \tag{4.5}$$

where $\alpha : \pi_k(\Omega Y) \longrightarrow \pi_{1+k}Y$ is the isomorphism defined in Example 2.23.

The 'duality' between the two constructions manifests itself when we take morphisms into a third symmetric spectrum, in the form of a natural isomorphism

$$\mathrm{Hom}(C(f), Z) \ \cong \ F(\mathrm{Hom}(f, Z) : \mathrm{Hom}(Y, Z) \longrightarrow \mathrm{Hom}(X, Z)) \ ,$$

along with analogous isomorphisms for homotopy classes and mapping spaces.

The constructions of mapping cone and homotopy fibre make perfect sense for a morphism between symmetric spectra of simplicial sets, provided the interval $[0,1]$ is replaced by the simplicial 1-simplex $\Delta[1]$ throughout. The simplicial face operators $d_0, d_1 : \Delta[0] \longrightarrow \Delta[1]$ then take the roles of the inclusions of the two endpoints of the interval.

The following elementary lemma will be needed below to establish a long exact sequence of homotopy groups. Here, $CA = [0,1] \wedge A$ is the cone of a based space $A$.

Lemma 4.6. *Let $f : A \longrightarrow B$ be a based continuous maps between based spaces and let $p_A : CA \longrightarrow S^1 \wedge A$ and $p_B : CB \longrightarrow S^1 \wedge B$ denote the projections.* [specify an identification between $S^1$ and $[0,1]/0 \sim 1$]

(i) *The triangle*

$$\begin{tikzcd} & CA \cup_f CB \arrow[dl, "p_A \cup *"'] \arrow[dr, "* \cup p_B"] & \\ S^1 \wedge A \arrow[rr, "\tau \wedge f"'] & & S^1 \wedge B \end{tikzcd}$$

*commutes up to homotopy, where* $\tau : S^1 \longrightarrow S^1$ *is the map of degree -1 given by* $\tau(t) = 1 - t$.

(ii) *Let* $\beta \in \pi_m B$ *be a homotopy class in the kernel of* $\pi_m(i) : \pi_m B \longrightarrow \pi_m C(f)$*. Then there exists a homotopy class* $\alpha \in \pi_{1+m}(S^1 \wedge A)$ *such that* $(S^1 \wedge f)_*(\alpha) = S^1 \wedge \beta$ *in* $\pi_{1+m}(S^1 \wedge B)$.

Proof. (i) We define a based homotopy $H : CA \times [0,1] \longrightarrow S^1 \wedge B$ by the formula

$$H([s,a],t) \ = \ [2-s-t, f(a)]$$

which has to be interpreted as the basepoint if $2 - s - t \geq 1$. Another based homotopy $H' : CB \times [0,1] \longrightarrow S^1 \wedge B$ is given by the formula

$$H'([s,b],t) \ = \ [s-t, b]$$

which has to be interpreted as the basepoint if $s \leq t$. The two homotopies are compatible in the sense that

$$H([1,a],t) \ = \ [1-t, f(a)] \ = \ H'([1,f(a)],t)$$

for all $t \in [0,1]$ and $a \in A$. So $H$ and $H'$ glue together and yield a homotopy

$$(CA \cup_f CB) \times [0,1] \cong (CA \times [0,1]) \cup_{f \times \mathrm{Id}} (CB \times [0,1]) \xrightarrow{H \cup H'} S^1 \wedge B \ .$$

For $t = 0$ this homotopy starts with the map $* \cup p_B$, and it ends for $t = 1$ with the map $(\tau \wedge f) \circ (p_A \cup *)$.

(ii) Let $b : S^m \longrightarrow B$ be a representative of $\beta$ and let $H : C(S^m) \longrightarrow C(f)$ be a based nullhomotopy of the composite of $b$ with $i : B \longrightarrow C(f)$, i.e., such that $H[1,x] = i(b(x))$ for all $x \in S^m$. We collaps $1 \times S^m$ in $C(S^m)$ and the image of $B$ in $C(f)$ and get a map $\bar{H} : S^1 \wedge S^m \longrightarrow S^1 \wedge A$ induced by $H$ on the quotient spaces. We claim that the homotopy class $\alpha = [H]$ has the required property.

To prove the claim we need the homotopy equivalence $p \cup * : C(S^m) \cup_{1 \times S^m} C(S^m) \longrightarrow S^1 \wedge S^m$ which collapses the second cone. We obtain a sequence of equalities and homotopies

$$\begin{aligned} (S^1 \wedge f) \circ \bar{H} \circ (p \cup *) \ &= \ (S^1 \wedge f) \circ (p_A \cup *) \circ (H \cup C(b)) \\ &\simeq \ (\tau \wedge B) \circ (* \cup p_B) \circ (H \cup C(b)) \\ &= \ (\tau \wedge B) \circ (S^1 \wedge b) \circ (* \cup p) \\ &= \ (S^1 \wedge b) \circ (\tau \wedge S^m) \circ (* \cup p) \ \simeq \ (S^1 \wedge b) \circ (p \cup *) \end{aligned}$$

Here $H \cup C(b) : C(S^m) \cup_{1 \times S^m} C(S^m) \longrightarrow C(f) \cup_B CB \cong CA \cup_f CB$. The two homotopies result from part (i) applied to $f$ respectively the identity of $S^m$, and we used the naturality of various constructions. Since the map $p \cup *$ is a homotopy equivalence, this proves that the map $(S^1 \wedge f) \circ \bar{H}$ which represents $(S^1 \wedge f)_*(\alpha)$ is homotopic to $S^1 \wedge b$. □

Proposition 4.7. *Let* $f : X \longrightarrow Y$ *be a morphism of symmetric spectra.*

(i) *The long sequence of abelian groups*

$$\cdots \longrightarrow \pi_k X \xrightarrow{\pi_k f} \pi_k Y \xrightarrow{\pi_k(\mathrm{incl})} \pi_k C(f) \xrightarrow{\delta} \pi_{k-1} X \longrightarrow \cdots$$

*is exact.*

(ii) *In the simplicial context, suppose also that* $X$ *and* $Y$ *are levelwise Kan complexes. Then the long sequence of abelian groups*

$$\cdots \longrightarrow \pi_{1+k} X \xrightarrow{\pi_{1+k} f} \pi_{1+k} Y \xrightarrow{\delta} \pi_k F(f) \xrightarrow{\pi_k p} \pi_k X \longrightarrow \cdots$$

*is exact.*

PROOF. (i) In the simplicial context, homotopy groups are defined after geometric realization, which commutes with mapping cones. So it suffices to treat the case of symmetric spectra of spaces. We show that the sequence

$$\pi_k X \xrightarrow{\pi_k f} \pi_k Y \xrightarrow{\pi_k(\mathrm{incl})} \pi_k C(f) \xrightarrow{\pi_k(\mathrm{proj})} \pi_k(S^1 \wedge X) \xrightarrow{\pi_k(S^1\wedge f)} \pi_k(S^1 \wedge Y)$$

is exact; when we substitute definition (4.2) of the boundary map $\delta$, this becomes the exact sequence of part (i).

Exactness at $\pi_k Y$: the composite of $f : X \longrightarrow Y$ and the inclusion $Y \longrightarrow C(f)$ is nullhomotopic, so it induces the trivial map on $\pi_k$. So it remains to show that every element in the kernel of $\pi_k(\mathrm{incl}) : \pi_k Y \longrightarrow \pi_k C(f)$ is in the image of $\pi_k f$.

Let $\beta \in \pi_{k+n} Y_n$ represent an element in the kernel. By increasing $n$, if necessary, we can assume that $\mathrm{incl}_*(\beta)$ is trivial in $\pi_{k+n} C(f_n)$. By Lemma 4.6 (ii) there is a homotopy class $\alpha \in \pi_{1+k+n}(S^1 \wedge X_n)$ such that $(S^1 \wedge f_n)_*(\alpha) = S^1 \wedge \beta$. The homotopy class $\tilde{\alpha} = (-1)^{k+n} \cdot (\tau_{S^1,X_n})_*(\alpha) \in \pi_{k+n+1}(X_n \wedge S^1)$ then satisfies $(f_n \wedge S^1)_*(\tilde{\alpha}) = \beta \wedge S^1$, and thus $(\sigma_n)_*(\tilde{\alpha}) \in \pi_{k+n+1} X_{n+1}$ hits $\iota_*(\beta) \in \pi_{k+n+1} Y_{n+1}$. So the class represented by $\beta$ in the colimit $\pi_k Y$ is in the image of $\pi_k f : \pi_k X \longrightarrow \pi_k Y$.

Exactness at $\pi_k C(f)$: If we apply the previous paragraph to the inclusion $i : Y \longrightarrow C(f)$ instead of $f$, we see that the sequence

$$\pi_k Y \xrightarrow{\pi_k(i)} \pi_k C(f) \xrightarrow{\pi_k(\mathrm{incl}_i)} \pi_k C(i)$$

is exact. We claim that the collaps map

$$* \cup p \ : \ C(i) \cong CY \cup_f CX \ \longrightarrow \ S^1 \wedge X$$

is a homotopy equivalence, and thus induces an isomorphism of homotopy groups. Since the composite of the homotopy equivalence $* \cup p : C(i) \longrightarrow S^1 \wedge X$ with the inclusion of $C(f)$ equals the projection $C(f) \longrightarrow S^1 \wedge X$, we can replace the group $\pi_k C(i)$ by the isomorphic group $\pi_k(S^1 \wedge X)$ and still obtain an exact sequence.

To prove the claim we define a homotopy inverse

$$r \ : \ S^1 \wedge X \ \longrightarrow \ CY \cup_f CX$$

by the formula

$$r([s,x]) \ = \ \begin{cases} [2s,x] \quad \in CX & \text{for } 0 \leq s \leq 1/2, \text{ and} \\ [2-2s, f(x)] \in CY & \text{for } 1/2 \leq s \leq 1, \end{cases}$$

which is to be interpreted levelwise. [specify the homotopies $r(* \cup p) \simeq \mathrm{Id}$ and $(* \cup p)r \simeq \mathrm{Id}$]

Exactness at $\pi_k(S^1 \wedge Y)$: If we apply the previous paragraph to the inclusion $i : Y \longrightarrow C(f)$ instead of $f$, we see that the sequence

$$\pi_k Y \xrightarrow{\pi_k(\mathrm{incl}_f)} \pi_k C(f) \xrightarrow{\pi_k(\mathrm{incl}_i)} \pi_k(C(\mathrm{incl}))$$

is exact. We claim that the collaps map

$$C(\mathrm{proj}) \cong C(CX \cup_f Y) \cup_{\mathrm{proj}} (S^1 \wedge X) \ \longrightarrow \ S^1 \wedge Y$$

[define; give details] is a homotopy equivalence, so induces an isomorphism of homotopy groups. Moreover, the composite

$$S^1 \wedge X \xrightarrow{\mathrm{incl}} C(\mathrm{proj}) \to S^1 \wedge Y$$

is homotopic to the morphism $\tau \wedge f : S^1 \wedge X \longrightarrow S^1 \wedge Y$, whose effect on homotopy groups is the negative of $\pi_k(S^1 \wedge f)$. Since the sign has no effect on the kernel, we can replace the group $\pi_k C(\mathrm{proj})$ by the isomorphic group $\pi_k(S^1 \wedge Y)$ and still obtain an exact sequence.

(ii) Pass to the colimit from the exact sequences of unstable homotopy groups. $\square$

We draw some consequences of Proposition 4.7. For every morphism $A \longrightarrow B$ of symmetric spectra which is levelwise an h-cofibration [define] (in the topological context) respectively levelwise injective (in the simplicial context), the quotient spectrum $B/A$ is level equivalent to the mapping cone. Dually, if $f : X \longrightarrow Y$ is a morphism of symmetric spectra which is levelwise a Serre fibration of spaces respectively Kan fibration of simplicial sets, the strict fibre $F$ is level equivalent to the homotopy fibre. This gives:

Corollary 4.8. (i) *Suppose $f : A \longrightarrow B$ is a h-cofibration of symmetric spectra of topological spaces or an injective morphism of symmetric spectra of simplicial sets. Denote by $p : B \longrightarrow B/A$ the quotient map. Then there is a natural long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_k A \xrightarrow{\pi_k(f)} \pi_k B \xrightarrow{\pi_k(p)} \pi_k(B/A) \xrightarrow{\delta} \pi_{k-1} A \longrightarrow \cdots$$

*where the connecting map $\delta$ is the composite of the inverse of the isomorphism $\pi_k C(f) \longrightarrow \pi_k(B/A)$ induced by the level equivalence $C(f) \longrightarrow B/A$ which collapses the cone of $A$ and the connecting homomorphism $\pi_k C(f) \longrightarrow \pi_{k-1} A$ defined in* (4.2).

(ii) *Suppose $f : X \longrightarrow Y$ is a morphism of symmetric spectra which is levelwise a Serre fibration of spaces respectively Kan fibration of simplicial sets. Denote by $i : F \longrightarrow X$ the inclusion of the fibre over the basepoint. Then there is a natural long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_k F \xrightarrow{\pi_k(i)} \pi_k X \xrightarrow{\pi_k(f)} \pi_k Y \xrightarrow{\delta} \pi_{k-1} F \longrightarrow \cdots$$

*where the connecting map $\delta$ is the composite of the connecting homomorphism $\pi_k Y \longrightarrow \pi_{k-1} F(f)$ defined in* (4.5) *and the inverse of the isomorphism $\pi_{k-1} F(f) \longrightarrow \pi_{k-1} F$ induced by the level equivalence $F \longrightarrow F(f)$ which sends $x \in F$ to* $(\mathrm{const}_*, x)$.

(iii) *Suppose that $f : X \longrightarrow Y$ is an h-cofibration (when in the topological context) respectively levelwise injective and $X$ and $Y$ are levelwise Kan complexes (when in the simplicial context). Then the morphism $h : S^1 \wedge F(f) \longrightarrow Y/X$* (4.4) *from the suspension of the homotopy fibre to the quotient of $f$ is a $\pi_*$-isomorphism.*

Proof. (iii) Compare the two long exact sequences and use the five lemma. □

Corollary 4.9. (i) *For every family of symmetric spectra $\{X^i\}_{i \in I}$ and every integer $k$ the canonical map*

$$\bigoplus_{i \in I} \pi_k X^i \longrightarrow \pi_k \left( \bigvee_{i \in I} X^i \right)$$

*is an isomorphism of abelian groups.*

(ii) *For every* finite *indexing set $I$, every family $\{X^i\}_{i \in I}$ of symmetric spectra and every integer $k$ the canonical map*

$$\pi_k \left( \prod_{i \in I} X^i \right) \longrightarrow \prod_{i \in I} \pi_k X^i$$

*is an isomorphism of abelian groups.*

(iii) *For every* finite *family of symmetric spectra the natural morphism from the wedge to the product is a $\pi_*$-isomorphism.*

The restriction to *finite* indexing sets in part (ii) of the previous corollary is essential, compare Example 4.39.

Proof. (i) We first show the special case of two summands. If $A$ and $B$ are two symmetric spectra, then the wedge inclusion $i_A : A \longrightarrow A \vee B$ has a retration. So the associated long exact homotopy group sequence of Proposition 4.7 (i) splits into short exact sequences

$$0 \longrightarrow \pi_k A \xrightarrow{\pi_k(i_A)} \pi_k(A \vee B) \xrightarrow{\text{incl}} \pi_k(C(i_A)) \longrightarrow 0 \ .$$

The mapping cone $C(i_A)$ is isomorphic to $(CA) \vee B$ and thus homotopy equivalent to $B$. So we can replace $\pi_k(C(i_A))$ by $\pi_k B$ and conclude that $\pi_k(A \vee B)$ splits as the sum of $\pi_k A$ and $\pi_k B$, via the canonical map.

The case of a finite indexing set $I$ now follows by induction, and the general case follows since homotopy groups of symmetric spectra commute with filtered colimits [since the image of every compact space in an infinite wedge lands in a finite wedge].

(ii) Unstable homotopy groups commute with products, which for finite indexing sets are also sums, which commute with filtered colimits.

(iii) This is a direct consequence of (i) and (ii). More precisely, for finite indexing set $I$ and every integer $k$ the composite map

$$\bigoplus_{i\in I} \pi_k X^i \longrightarrow \pi_k(\bigvee_{i\in I} X^i) \ \longrightarrow\ \pi_k(\prod_{i\in I} X^i) \longrightarrow \prod_{i\in I} \pi_k X^i$$

is an isomorphism, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by parts (i) respectively (ii), hence so is the middle map. □

Remark 4.10. A wedge of $\pi_*$-isomorphisms is a $\pi_*$-isomorphism, and [justify] a wedge of stable equivalences is a stable equivalence. This is a stable phenomenon; unstably, one needs a non-degeneracy conditions. [Give an example]

**4.2. $\mathcal{M}$-action on homotopy groups.** The definition of homotopy groups does not take the symmetric group actions into account; using these actions we will now see that the homotopy groups of a symmetric spectrum have more structure.

Definition 4.11. The *injection monoid* $\mathcal{M}$ is the monoid, under composition, of injective self-maps of the set $\omega = \{1, 2, 3, \dots\}$ of natural numbers. An $\mathcal{M}$*-module* is a left module over the monoid ring $\mathbb{Z}[\mathcal{M}]$. We call an $\mathcal{M}$-module $W$ *tame* if for every element $x \in W$ there exists a number $n \geq 0$ with the following property: for every element $f \in \mathcal{M}$ which fixes the set $\mathbf{n} = \{1, \dots, n\}$ elementwise we have $fx = x$.

As we shall soon see, the homotopy groups of a symmetric spectrum have a natural tame $\mathcal{M}$-action. An example of an $\mathcal{M}$-module which is not tame is the free module of rank 1. Tameness has many algebraic consequences which we discuss in the next section. We will see in Remark 4.41 that the $\mathcal{M}$-action gives all natural operations on the homotopy groups of a symmetric spectrum; more precisely, we show that the ring of natural operations on $\pi_0 X$ is a completion of the monoid ring of $\mathcal{M}$, so that an arbitrary operation is a sum, possibly infinite, of operations by elements from $\mathcal{M}$.

Construction 4.12. We define an action of the injection monoid $\mathcal{M}$ on the homotopy groups of a symmetric spectrum $X$. We break the construction up into two steps and pass through the intermediate category of *$I$-functors*. The category $I$ has an object $\mathbf{n} = \{1, \dots, n\}$ for every non-negative integer $n$, including $\mathbf{0} = \emptyset$. Morphisms in $I$ all injective maps. An *$I$-functor* is a covariant functor from the category $I$ to the category of abelian groups.

*Step 1: from symmetric spectra to $I$-functors.* For every integer $k$ we assign an $I$-functor $\underline{\pi}_k X$ to the symmetric spectrum $X$. On objects, this $I$-functor is given by

$$(\underline{\pi}_k X)(\mathbf{n}) \ = \ \pi_{k+n} X_n$$

if $k + n \geq 2$ and $(\underline{\pi}_k X)(\mathbf{n}) = 0$ for $k + n < 2$. If $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ is an injective map and $k + n \geq 2$, then $\alpha_* : (\underline{\pi}_k X)(\mathbf{n}) \longrightarrow (\underline{\pi}_k X)(\mathbf{m})$ is given as follows. We choose a permutation $\gamma \in \Sigma_m$ such that $\gamma(i) = \alpha(i)$ for all $i = 1, \dots, n$ and set

$$\alpha_*(x) \ = \ \mathrm{sgn}(\gamma) \cdot \gamma_*(\iota_*^{m-n}(x))$$

where $\iota_* : \pi_{k+n} X_n \longrightarrow \pi_{k+n+1} X_{n+1}$ is the stabilization map (1.5).

We have to justify that this definition is independent of the choice of permutation $\gamma$. Suppose $\gamma' \in \Sigma_m$ is another permutation which agrees with $\alpha$ on $\mathbf{n}$. Then $\gamma^{-1}\gamma'$ is a permutation of $\mathbf{m}$ which fixed the numbers $1, \dots, n$, so it is of the form $\gamma^{-1}\gamma' = 1 \times \tau$ for some $\tau \in \Sigma_{m-n}$, where 1 is the unit of $\Sigma_n$. It suffices to show that for such permutations the induced action on $\pi_{k+m} X_m$ via the action on $X_m$ satisfies the relation

$$(1 \times \tau)_*(\iota_*^{m-n}(x)) \ = \ \mathrm{sgn}(\tau) \cdot (\iota_*^{m-n}(x)) \tag{4.13}$$

for all $x \in \pi_{k+n}X_n$. To justify this we let $f : S^{k+n} \longrightarrow X_n$ represent $x$. Since the iterated structure map $\sigma^{m-n} : X_n \wedge S^{m-n} \longrightarrow X_m$ is $\Sigma_n \times \Sigma_{m-n}$-equivariant, we have a commutative diagram

$$\begin{tikzcd}
S^{k+m} \arrow[r, "f\wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id}\wedge\tau"'] & X_n \wedge S^{m-n} \arrow[r, "\sigma^{m-n}"] \arrow[d, "\mathrm{Id}\wedge\tau"] & X_m \arrow[d, "1\times\tau"] \\
S^{k+m} \arrow[r, "f\wedge \mathrm{Id}"'] & X_n \wedge S^{m-n} \arrow[r, "\sigma^{m-n}"'] & X_m
\end{tikzcd}$$

The composite through the upper right corner represents $(1 \times \tau)_*(\iota_*^{m-n}(x))$. Since the effect on homotopy groups of precomposing with a coordinate permutation of the sphere is multiplication by the sign of the permutation, the composite through the lower left corner represents $\mathrm{sgn}(\tau) \cdot (\iota_*^{m-n}(x))$. This proves formula (4.13) and completes the definition of $\alpha_* : (\underline{\pi}_k X)(\mathbf{n}) \longrightarrow (\underline{\pi}_k X)(\mathbf{m})$.

The inclusion $\mathbf{n} \longrightarrow \mathbf{n+1}$ induces the map $\iota_*$ over which the colimit $\pi_k X$ is formed, so if we denote the inclusion by $\iota$, then two meanings of $\iota_*$ are consistent. We let $\mathbb{N}$ denote the subcategory of $I$ which contains all objects but only the inclusions as morphisms, and then we have

$$\pi_k X \ = \ \mathrm{colim}_{\mathbb{N}} \ \underline{\pi}_k X \ .$$

*Step 2: from $I$-functors to tame $\mathcal{M}$-modules.* The next observation is that for any $I$-functor $F$ the colimit of $F$, formed over the subcategory $\mathbb{N}$ of inclusions, has a natural left action by the injection monoid $\mathcal{M}$. Applied to the $I$-functor $\underline{\pi}_k X$ coming from a symmetric spectrum $X$, this yields the $\mathcal{M}$-action on the stable homotopy group $\pi_k X$.

We let $I_\omega$ denote the category with objects the sets $\mathbf{n}$ for $n \geq 0$ and the set $\omega$ and with all injective maps as morphisms. So $I_\omega$ contains $I$ as a full subcategory and contains one more object $\omega$ whose endomorphism monoid is $\mathcal{M}$. We will now extend an $I$-functor $F$ to a functor from the category $I_\omega$ in such a way that the value of the extension at the object $\omega$ is the colimit of $F$, formed over the subcategory $\mathbb{N}$ of inclusions. It will thus be convenient, and suggestive, to denote the colimit of $F$, formed over the subcategory $\mathbb{N}$ of inclusions, by $F(\omega)$ and not introduce new notation for the extended functor. The $\mathcal{M}$-action on the colimit of $F$ is then the action of the endomorphisms of $\omega$ in $I_\omega$ on $F(\omega)$.

So we set $F(\omega) = \mathrm{colim}_{\mathbb{N}} F$ and first define $\beta_* : F(\mathbf{n}) \longrightarrow F(\omega)$ for every injection $\beta : \mathbf{n} \longrightarrow \omega$ as follows. We set $m = \max\{\beta(\mathbf{n})\}$, denote by $\beta|_{\mathbf{n}} : \mathbf{n} \longrightarrow \mathbf{m}$ the restriction of $\beta$ and take $\beta_*(x)$ to be the class in the colimit represented by the image of $x$ under

$$(\beta|_{\mathbf{n}})_* \ : \ F(\mathbf{n}) \ \longrightarrow \ F(\mathbf{m}) \ .$$

It is straightforward to check that this is a functorial extension of $F$, i.e., for every morphism $\alpha : \mathbf{k} \longrightarrow \mathbf{n}$ in $I$ we have $(\beta\alpha)_*(x) = \beta_*(\alpha_*(x))$.

Now we let $f : \omega \longrightarrow \omega$ be an injective self-map of $\omega$, and we want to define $f_* : F(\omega) \longrightarrow F(\omega)$. If $[x] \in F(\omega)$ is an element in the colimit represented by a class $x \in F(\mathbf{n})$, then we set $f_*[x] = [(f|_{\mathbf{n}})_*(x)]$ where $f|_{\mathbf{n}} : \mathbf{n} \longrightarrow \omega$ is the restriction of $f$ and $f_* : F(\mathbf{n}) \longrightarrow F(\omega)$ was defined in the previous paragraph. Again it is straightforward to check that this definition does not depend on the representative $x$ of the class $[x]$ in the colimit and that the extension is functorial, i.e., we have $(f\alpha)_*(x) = f_*(\alpha_*(x))$ for injections $\alpha : \mathbf{n} \longrightarrow \omega$ as well as $(fg)_*[x] = f_*(g_*[x])$ when $g$ is another injective self-map of $\omega$. As an example, if we also write $\iota : \mathbf{n} \longrightarrow \omega$ for the inclusion, then we have $\iota_*(x) = [x]$ for $x \in F(\mathbf{n})$.

The definition just given is in fact the universal way to extend an $I$-functor $F$ to a functor on the category $I_\omega$, i.e., we have just constructed a left Kan extension of $F : I \longrightarrow \mathcal{A}b$ along the inclusion $I \longrightarrow I_\omega$. However, we do not need this fact, so we omit the proof.

A trivial but important observation straight from the definition is that the action of the injection monoid $\mathcal{M}$ on the colimit of any $I$-functor $F$ is tame in the sense of Definition 4.11: every element in the colimit $F(\omega)$ is represented by a class $x \in F(\mathbf{n})$ for some $n \geq 0$; then for every element $f \in \mathcal{M}$ which fixes the numbers $1, \ldots, n$, we have $f_*[x] = [x]$.

EXAMPLE 4.14. To illustrate the action of the injection monoid $\mathcal{M}$ on the homotopy groups of a symmetric spectrum $X$ we make it explicit for the injection $d : \omega \longrightarrow \omega$ given by $d(i) = i + 1$, which will also play an important role later. For every $n \geq 1$, the map $d$ and the cycle $(1, 2, \dots, n, n + 1) = \chi_{n,1}$ of $\Sigma_{n+1}$ agree on $\mathbf{n}$, so $d$ acts on $\pi_k X$ as the colimit of the system

$$\begin{tikzcd}
\pi_k X_0 \arrow[r, "\iota_*"] \arrow[d, "\iota_*"'] & \pi_{k+1} X_1 \arrow[r, "\iota_*"] \arrow[d, "-(1,2)_* \circ \iota_*"] & \pi_{k+2} X_2 \arrow[r, "\iota_*"] \arrow[d, "(1,2,3)_* \circ \iota_*"] & \cdots \arrow[r, "\iota_*"] & \pi_{k+n} X_n \arrow[r, "\iota_*"] \arrow[d, "(-1)^n (1,2,\dots,n,n+1)_* \circ \iota_*"] & {} \\
\pi_{k+1} X_1 \arrow[r, "\iota_*"'] & \pi_{k+2} X_2 \arrow[r, "\iota_*"'] & \pi_{k+3} X_3 \arrow[r, "\iota_*"'] & \cdots \arrow[r, "\iota_*"'] & \pi_{k+n+1} X_{n+1} \arrow[r, "\iota_*"'] & {}
\end{tikzcd}$$

(at least for $k \geq 0$; for negative values of $k$ only a later portion of the system makes sense).

REMARK 4.15. The stable homotopy group $\pi_k X$ of a symmetric spectrum $X$ can also be calculated from the system of *stable* as opposed to *unstable* homotopy groups of the individual spaces $X_n$. For us, the $m$th stable homotopy group $\pi^{\mathrm{s}}_m K$ of a pointed space $K$ is the colimit of the sequence of abelian groups

$$\pi_m K \xrightarrow{S^1 \wedge} \pi_{1+m}(S^1 \wedge K) \xrightarrow{S^1 \wedge} \pi_{2+m}(S^2 \wedge K) \xrightarrow{S^1 \wedge} \cdots \tag{4.16}$$

[not consistent with earlier definition as $\pi_k(\Sigma^\infty K)$] where we stabilize from the left. Smashing with the identity of $S^1$ from the right provides a stabilization map (even an isomorphism) $\pi^{\mathrm{s}}_m K \longrightarrow \pi^{\mathrm{s}}_{m+1}(K \wedge S^1)$. For a symmetric spectrum we can then define an $I$-functor $\underline{\pi}^{\mathrm{s}}_k X$ by setting $(\underline{\pi}^{\mathrm{s}}_k X)(\mathbf{n}) = \pi^{\mathrm{s}}_{k+n} X_n$ on objects (with no restriction on $k + n$) and defining the action of a morphism $\mathbf{n} \longrightarrow \mathbf{m}$ in the same way as for the $I$-functor $\underline{\pi}_k X$ of unstable homotopy groups.

The map from the initial term to the colimit of the sequence (4.16) provides a natural transformation $\pi_m K \longrightarrow \pi^{\mathrm{s}}_m K$ which is compatible with stabilization, so it defines a morphism of $I$-functors $\underline{\pi}_k X \longrightarrow \underline{\pi}^{\mathrm{s}}_k X$ for every symmetric spectrum $X$. The induced map on colimits

$$\mathrm{colim}_{\mathbb{N}}\ \underline{\pi}_k X \xrightarrow{\cong} \mathrm{colim}_{\mathbb{N}}\ \underline{\pi}^{\mathrm{s}}_k X$$

is bijective and thus an isomorphism of $\mathcal{M}$-modules.

EXAMPLE 4.17. Here is an alternative perspective on the $I$-functor $\underline{\pi}_k X$ associated to a symmetric spectrum $X$. In Example 2.31 we associated to every $I$-space $T : I \longrightarrow \mathcal{T}$ a symmetric spectrum $T \wedge \mathbb{S}$. This construction has a right adjoint $\Omega^\bullet : \mathcal{S}p^\Sigma \longrightarrow \mathcal{T}^I$ defined as follows. If $X$ is a symmetric spectrum, we set

$$(\Omega^\bullet X)(\mathbf{n}) \;=\; \mathrm{map}(S^n, X_n)$$

on objects, where the symmetric group $\Sigma_n$ acts by conjugation, i.e., $(\gamma_* f)(x) = \gamma f(\gamma^{-1} x)$ for $f : S^n \longrightarrow X_n$ and $\gamma \in \Sigma_n$.

If $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ is an injective map then $\alpha_* : \mathrm{map}(S^n, X_n) \longrightarrow \mathrm{map}(S^m, X_m)$ is given as follows. We choose a permutation $\gamma \in \Sigma_m$ such that $\gamma(i) = \alpha(i)$ for all $i = 1, \dots, n$ and set

$$\alpha_*(f) \;=\; \gamma_*(\sigma^{m-n}(f \wedge S^{m-n}))\ ,$$

i.e., we let $\gamma$ acts as just defined on the composite

$$S^m \cong S^n \wedge S^{m-n} \xrightarrow{f \wedge S^{m-n}} X_n \wedge S^{m-n} \xrightarrow{\sigma^{m-n}} X_m\ .$$

The proof of the relation (4.13) above in fact shows that this definition is independent of the choice of permutation $\gamma$. Functoriality of the assignment $\alpha \mapsto \alpha_*$ is then straightforward.

The adjunction

$$\mathcal{S}p^\Sigma(T \wedge \mathbb{S}, X) \;\cong\; \mathcal{T}^I(T, \Omega^\bullet X)$$

takes a morphism $\varphi : T \wedge \mathbb{S} \longrightarrow X$ to the natural transformation $\hat{\varphi} : T \longrightarrow \Omega^\bullet X$ whose value at the object $\mathbf{n}$ is the adjoint $T(\mathbf{n}) \longrightarrow \mathrm{map}(S^n, X_n)$ of $\varphi_n : T(\mathbf{n}) \wedge S^n \longrightarrow X_n$.

The isomorphism $\pi_k\, \mathrm{map}(S^n, X_n) \cong \pi_{k+n} X_n$ (adjoint the loop coordinates to the right) gives an isomorphism of abelian groups between $\pi_k(\Omega^\bullet X)(\mathbf{n})$ and $(\underline{\pi}_k X)(\mathbf{n})$. The $\Sigma_n$-action on the source sphere in $\mathrm{map}(S^n, X_n)$ induces the sign action on homotopy groups, so the above isomorphism is $\Sigma_n$-equivariant.

Since the stabilization map $\iota_* : \pi_{k+n}X_n \longrightarrow \pi_{k+n+1}X_{n+1}$ corresponds precisely to the effect of $\iota_*$ : $\mathrm{map}(S^n, X_n) \longrightarrow \mathrm{map}(S^{n+1}, X_{n+1})$ on $\pi_k$, we in fact have an isomorphism of $I$-functors $\pi_k(\Omega^\bullet X) \cong (\underline{\pi}_k X)$.

We now give a construction which associates to an $I$-functor with $\Sigma_m$-action $F$ (i.e., a covariant functor $F : I \longrightarrow \mathbb{Z}[\Sigma_m]$-mod) a new $I$-functor $\triangleright_m F$ and give a formula for the $\mathcal{M}$-module $(\triangleright_m F)(\omega)$. This will be relevant later when we identify the $\mathcal{M}$-action on the homotopy groups of semifree symmetric spectra.

Given $F : I \longrightarrow \mathbb{Z}[\Sigma_m]$-mod we define a new $I$-functor $\triangleright_m F$ by $(\triangleright_m F)(\mathbf{k}) = 0$ for $k < m$ and

$$(\triangleright_m F)(\mathbf{m}+\mathbf{n}) \;=\; \mathbb{Z}[\Sigma_{m+n}] \otimes_{\Sigma_m\times\Sigma_n} F(\mathbf{n}) \;. \tag{4.18}$$

We define $\triangleright_m F$ on morphisms $\alpha : \mathbf{m}+\mathbf{n} \longrightarrow \mathbf{m}+\mathbf{k}$ in $I$ as follows. We choose a permutation $\gamma \in \Sigma_{m+k}$ which agrees with $\alpha$ on $\mathbf{m}+\mathbf{n}$ and define

$$\alpha_* \;:\; (\triangleright_m F)(\mathbf{m}+\mathbf{n}) = \mathbb{Z}[\Sigma_{m+n}] \otimes_{\Sigma_m\times\Sigma_n} F(\mathbf{n}) \;\longrightarrow\; \mathbb{Z}[\Sigma_{m+k}] \otimes_{\Sigma_m\times\Sigma_k} F(\mathbf{k}) = (\triangleright_m F)(\mathbf{m}+\mathbf{k})$$

by $\alpha_*(\tau \otimes x) = \gamma(\tau \times 1_{k-n}) \otimes \iota_*(x)$ where $\iota : \mathbf{n} \longrightarrow \mathbf{k}$ is the inclusion. [functorial...]

We define a homomorphism of monoids $\times : \Sigma_m \times \mathcal{M} \longrightarrow \mathcal{M}$ by

$$(\gamma \times f)(i) \;=\; \begin{cases} \gamma(i) & \text{for } 1 \le i \le m, \text{ and} \\ f(i-m)+m & \text{for } m+1 \le i. \end{cases} \tag{4.19}$$

We denote by $\mathbb{Z}[\mathcal{M}]^{+m}$ the monoid ring of $\mathcal{M}$ with its usual left multiplication action, but with action by the monoid $\Sigma_m \times \mathcal{M}$ via restriction along the homomorphism $\times : \Sigma_m \times \mathcal{M} \longrightarrow \mathcal{M}$. Since $F$ takes values in $\Sigma_m$-modules, the colimit $F(\omega)$ not only has an action of $\mathcal{M}$, but also a compatible left action by the group $\Sigma_m$. So we can form

$$\mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m\times\mathcal{M}} F(\omega)$$

which is a left $\mathcal{M}$-module via the left multiplication action of $\mathbb{Z}[\mathcal{M}]$ on itself. The point of this construction is that it describes the colimit of the functor $\triangleright_m F$ as an $\mathcal{M}$-module:

LEMMA 4.20. *For every $I$-functor with $\Sigma_m$-action $F$ the natural map*

$$\begin{aligned} \mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m\times\mathcal{M}} F(\omega) &\longrightarrow (\triangleright_m F)(\omega) \\ f \otimes [x] &\longmapsto f \cdot [1 \otimes x] \end{aligned}$$

*is an isomorphism of $\mathcal{M}$-modules.*

PROOF. We define a morphism in the other direction by

$$(\triangleright_m F)(\omega) \;\longrightarrow\; \mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m\times\mathcal{M}} F(\omega) \;, \quad [\gamma \otimes x] \longmapsto \gamma \otimes [x]$$

where $x \in F(n)$ and $\gamma \in \Sigma_{m+n}$. The main point is to check that the formulas for both maps are actually well-defined, i.e., they indeed factor over the tensor product over $\Sigma_m \times \mathcal{M}$ respectively over $\Sigma_m \times \Sigma_n$ and are independent of the representative $x$ respectively $[\gamma \otimes x]$ in the colimit $F(\omega)$ respectively $(\triangleright_m F)(\omega)$. We omit these routine verifications, and after that it is clear that both maps are $\mathcal{M}$-linear and inverse to each other. □

**4.3. Algebraic properties of tame $\mathcal{M}$-modules.** In this section we discuss some algebraic properties of tame $\mathcal{M}$-modules. It turns out that tameness is a rather restrictive condition. We start with a result which controls a lot of the homological of the monoid ring of $\mathcal{M}$.

LEMMA 4.21. *The classifying space $B\mathcal{M}$ of the injection monoid $\mathcal{M}$ is contractible.*

PROOF. The classifying space $B\mathcal{M}$ is the geometric realization of the nerve of the category $\underline{B}\mathcal{M}$ with one object whose monoid of endomorphisms is $\mathcal{M}$. Let $t \in \mathcal{M}$ be given by $t(i) = 2i$. We define an injective endomorphism $c_t : \mathcal{M} \longrightarrow \mathcal{M}$ as follows. For $f \in \mathcal{M}$ and $i \in \omega$ we set

$$c_t(f)(i) \;=\; \begin{cases} i & \text{if } i \text{ is odd, and} \\ 2 \cdot f(i/2) & \text{if } i \text{ is even.} \end{cases}$$

Even though $t$ is not bijective, the endomorphism $c_t$ behaves like conjugation by $t$ in the sense that the formula $c_t(f) \cdot t = t \cdot f$ holds. Thus $t$ provides a natural transformation from the identity functor of $\underline{B}\mathcal{M}$ to $\underline{B}(c_t)$. On the other hand, if $s \in \mathcal{M}$ is given by $s(i) = 2i - 1$, then $c_t(f) \cdot s = s$ for all $f \in \mathcal{M}$, so $s$ provides a natural transformation from the constant functor of $\underline{B}\mathcal{M}$ with values $1 \in \mathcal{M}$ to $\underline{B}(c_t)$. Thus via the homotopies induced by $t$ and $s$, the identity of $B\mathcal{M}$ is homotopic to a constant map, so $B\mathcal{M}$ is contractible. $\square$

We introduce some useful notation and terminology. For an injective map $f : \omega \longrightarrow \omega$ we write $|f|$ for the smallest number $i \geq 0$ such that $f(i+1) \neq i+1$. So in particular, $f$ restricts to the identity on $\{1, \ldots, |f|\}$. An element $x$ of an $\mathcal{M}$-module $W$ has *filtration* $n$ if for every $f \in \mathcal{M}$ with $|f| \geq n$ we have $fx = x$. We denote by $W^{(n)}$ the subgroup of $W$ of elements of filtration $n$; for example, $W^{(0)}$ is the set of elements fixed by all $f \in \mathcal{M}$. We say that $x$ has *filtration exactly* $n$ if it lies in $W^{(n)}$ but not in $W^{(n-1)}$. By definition, an $\mathcal{M}$-module $W$ is tame if and only if every element has a finite filtration, i.e., if the groups $W^{(n)}$ exhaust $W$.

The following lemmas collect some elementary observations, first for arbitrary $\mathcal{M}$-modules and then for tame $\mathcal{M}$-modules.

LEMMA 4.22. *Let $W$ be any $\mathcal{M}$-module.*

(i) *If two elements $f$ and $g$ of $\mathcal{M}$ coincide on $\mathbf{n} = \{1, \ldots, n\}$, then $fx = gx$ for all $x \in W$ of filtration $n$.*
(ii) *For $n \geq 0$ and $f \in \mathcal{M}$ set $m = \max\{f(\mathbf{n})\}$. Then $f \cdot W^{(n)} \subseteq W^{(m)}$.*
(iii) *We denote by $d \in \mathcal{M}$ the map given by $d(i) = i + 1$. If $x \in W$ has filtration exactly $n$ with $n \geq 1$, then $dx$ has filtration exactly $n + 1$.*
(iv) *Let $V \subseteq W$ be an $\mathcal{M}$-submodule such that the action of $\mathcal{M}$ on $V$ and $W/V$ is trivial. Then the action of $\mathcal{M}$ on $W$ is also trivial.*

PROOF. (i) We can choose a bijection $\gamma \in \mathcal{M}$ which agrees with $f$ and $g$ on $\mathbf{n}$, and then $\gamma^{-1}f$ and $\gamma^{-1}g$ fix $\mathbf{n}$ elementwise. So for $x$ of filtration $n$ we have $(\gamma^{-1}f)x = x = (\gamma^{-1}g)x$. Multiplying by $\gamma$ gives $fx = gx$.

(ii) If $g \in \mathcal{M}$ satisfies $|g| \geq m$, then $gf$ and $f$ agree on $\mathbf{n}$. So for all $x \in W^{(n)}$ we have $gfx = fx$ by (i), which proves that $fx \in W^{(m)}$.

(iii) We have $d \cdot W^{(n)} \subseteq W^{(n+1)}$ by part (ii). To prove that $d$ increases the exact filtration we consider $x \in W^{(n)}$ with $n \geq 1$ and show that $dx \in W^{(n)}$ implies $x \in W^{(n-1)}$.

For $f \in \mathcal{M}$ with $|f| = n - 1$ we define $g \in \mathcal{M}$ by $g(1) = 1$ and $g(i) = f(i-1) + 1$ for $i \geq 2$. Then we have $gd = df$ and $|g| = n$. We let $h$ be the cycle $h = (f(n)+1, f(n), \ldots, 2, 1)$ so that we have $|hd| = f(n) = \max\{f(\mathbf{n})\}$. Then $fx \in W^{(f(n))}$ by part (ii) and so

$$fx \;=\; (hd)(fx) \;=\; h(g(dx)) \;=\; (hd)x \;=\; x\;.$$

Altogether this proves that $x \in W^{(n-1)}$.

(iv) Since the $\mathcal{M}$-action is trivial on $V$ and $W/V$, every $f \in \mathcal{M}$ determines an additive map $\delta_f : W/V \longrightarrow V$ such that $x - fx = \delta_f(x + V)$ for all $x \in W$. These maps satisfy $\delta_{fg}(x) = \delta_f(x) + \delta_g(x)$ and so $\delta$ is a homomorphism from the monoid $\mathcal{M}$ to the abelian group of additive maps from $W/V$ to $V$. By Lemma 4.21 the classifying space $B\mathcal{M}$ of the monoid $\mathcal{M}$ is contractible, so $H^1(B\mathcal{M}, A) = \mathrm{Hom}(\mathcal{M}, A)$ is trivial for every abelian group $A$. Thus $\delta_f = 0$ for all $f \in \mathcal{M}$, i.e., $\mathcal{M}$ acts trivially on $W$. $\square$

COROLLARY 4.23. *The assignment $\mathbf{n} \mapsto W^{(n)}$ extends to an $I$-functor $W^{(\bullet)}$ in such a way that $W \mapsto W^{(\bullet)}$ is right adjoint to the functor which assigns to an $I$-functor $F$ the $\mathcal{M}$-module $F(\omega)$. The counit of the adjunction $(W^{(\bullet)})(\omega) \longrightarrow W$ is injective with image the subgroup of elements of finite filtration, which is also the largest tame submodule of $W$. The assignment $W \mapsto (W^{(\bullet)})(\omega) = \bigcup_n W^{(n)}$ is right adjoint to the inclusion of tame $\mathcal{M}$-modules into all $\mathcal{M}$-modules.*

PROOF. To define the $I$-functor $W^{(\bullet)}$ on morphisms $\alpha : \mathbf{n} \longrightarrow \mathbf{m}$ in the category $I$ we choose any extension $\tilde{\alpha} : \omega \longrightarrow \omega$ of $\alpha$ and define $\alpha_* : W^{(n)} \longrightarrow W^{(m)}$ as the restriction of $\tilde{\alpha}\cdot : W \longrightarrow W$. This really has image in $W^{(m)}$ by part (ii) of Lemma 4.22 and is independent of the extension by (i) of that lemma. The rest is immediate. □

LEMMA 4.24. *Let $W$ be a tame $\mathcal{M}$-module.*

(i) *Every element of $\mathcal{M}$ acts injectively on $W$.*
(ii) *If the filtration of elements of $W$ is bounded, then $W$ is a trivial $\mathcal{M}$-module.*
(iii) *If the map $d$ given by $d(i) = i + 1$ acts surjectively on $W$, then $W$ is a trivial $\mathcal{M}$-module.*
(iv) *If $W$ is finitely generated as an abelian group, then $W$ is a trivial $\mathcal{M}$-module.*

PROOF. (i) Consider $f \in \mathcal{M}$ and $x \in W^{(n)}$ with $fx = 0$. Since $f$ is injective, we can choose $h \in \mathcal{M}$ with $|hf| \geq n$. Then $x = (hf)x = h(fx) = 0$, so $f$ acts injectively.

(ii) Lemma 4.22 (iii) implies that if $W = W^{(n)}$ for some $n \geq 0$, then $n = 0$, so the $\mathcal{M}$-action is trivial.

(iii) Suppose $\mathcal{M}$ does not act trivially, so that $W^{(0)} \neq W$. Let $n$ be the smallest positive integer such that $W^{(0)} \neq W^{(n)}$. Then by part (iii) of Lemma 4.22, any $x \in W^{(n)} - W^{(0)}$ is not in the image of $d$, so $d$ does not act surjectively.

(iv) The union of the nested sequence of subgroups $W^{(0)} \subseteq W^{(1)} \subseteq W^{(2)} \subseteq \cdots$ is $W$. Since finitely generated abelian groups are Noetherian, we have $W^{(n)} = W$ for all large enough $n$. By part (ii), the monoid $\mathcal{M}$ must act trivially. □

Parts (i), (iii) and (iv) of Lemma 4.24 can fail for non-tame $\mathcal{M}$-modules: we can let $f \in \mathcal{M}$ act on the abelian group $\mathbb{Z}$ as the identity if the image of $f : \omega \longrightarrow \omega$ has finite complement, and we let $f$ acts as 0 if its image has infinite complement.

EXAMPLE 4.25. We introduce some important tame $\mathcal{M}$-modules $\mathcal{P}_n$ for $n \geq 0$. The module $\mathcal{P}_n$ is the free abelian group with basis the set of ordered $n$-tuples of pairwise distinct elements of $\omega$ (or equivalently the set of injective maps from $\mathbf{n}$ to $\omega$). The monoid $\mathcal{M}$ acts from the left on this basis by componentwise evaluation, i.e., $f(x_1, \ldots, x_n) = (f(x_1), \ldots, f(x_n))$, and it acts on $\mathcal{P}_n$ by additive extension. For $n = 0$, there is only one basis element, the empty tuple, and so $\mathcal{P}_0$ is isomorphic to $\mathbb{Z}$ with trivial $\mathcal{M}$-action. For $n \geq 1$, the basis is countably infinite and the $\mathcal{M}$-action is non-trivial. The module $\mathcal{P}_n$ is tame: the filtration of a basis element $(x_1, \ldots, x_n)$ is the maximum of the components. So the filtration subgroup $\mathcal{P}_n^{(m)}$ is generated by the $n$-tuples all of whose components are less than or equal to $m$. An equivalent way of saying this is that $\mathcal{P}_n^{(m)} = \mathbb{Z}[I(\mathbf{n}, \mathbf{m})]$, the free abelian group generated by all injections from $\mathbf{n}$ to $\mathbf{m}$; in particular, $\mathcal{P}_n^{(m)}$ is trivial for $m < n$.

The module $\mathcal{P}_n$ represents the functor of taking elements of filtration $n$: for every $\mathcal{M}$-module $W$, the map

$$\mathrm{Hom}_{\mathcal{M}\text{-mod}}(\mathcal{P}_n, W) \longrightarrow W^{(n)} , \quad \varphi \mapsto \varphi(1, \ldots, n)$$

is bijective.

We end this section with some homological algebra of tame $\mathcal{M}$-modules. This will not be needed until Section 4.2 of Chapter II where we construct a spectral sequence which converges to the 'true' homotopy groups of a symmetric spectrum. The $E^2$-term of that spectral sequence consists of Tor groups over the monoid ring $\mathbb{Z}[\mathcal{M}]$.

LEMMA 4.26. (i) *Let $\mathbb{Z}[\mathcal{M}]^+$ denote the monoid ring of $\mathcal{M}$ with its usual left action, but with right action through the monomorphism $(1 \times -) : \mathcal{M} \longrightarrow \mathcal{M}$ given by $(1 \times f)(1) = 1$ and $(1 \times f)(i) = f(i-1)+1$ for $i \geq 2$. Then for every $n \geq 0$ the map*

$$\kappa \ : \ \mathcal{P}_{1+n} \longrightarrow \ \mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} \mathcal{P}_n$$

*which sends the generator $(1, \ldots, n+1)$ to the element $1 \otimes (1, \ldots, n)$ of filtration $n+1$ in $\mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} \mathcal{P}_n$ is an isomorphism of $\mathcal{M}$-modules.*

(ii) *For every $n \geq 0$ and every abelian group $A$, the groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \mathcal{P}_n \otimes A)$ vanish in positive dimensions.*

(iii) *For every $n \geq 0$ and every $\Sigma_n$-module $B$ we have a natural isomorphism*

$$\mathrm{Tor}_*^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \mathcal{P}_n \otimes_{\Sigma_n} B) \;\cong\; H_*(\Sigma_n; B) \ .$$

PROOF. (i) For any $n$-tuple $(x_1, \dots, x_n)$ of pairwise distinct natural numbers we can choose $g \in \mathcal{M}$ with $g(i) = x_i$ for $1 \leq i \leq n$. Because of

$$f \otimes (x_1, \dots, x_n) \;=\; f \otimes g(1, \dots, n) \;=\; f(1 \times g) \cdot (1 \otimes (1, \dots, n))$$

the element $1 \otimes (1, \dots, n)$ generates $\mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} \mathcal{P}_n$, so the map $\kappa$ is surjective. The map $\mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} \mathcal{P}_n \longrightarrow \mathcal{P}_{1+n}$ which sends $f \otimes (x_1, \dots, x_n)$ to $(f(1), f(x_1+1), \dots, f(x_n+1))$ is right inverse to $\kappa$ since the composite sends the generator $(1, \dots, n+1)$ to itself. So $\kappa$ is also injective.

(ii) The groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, A)$ are isomorphic to the singular homology groups with coefficients in $A$ of the classifying space $B\mathcal{M}$ of the monoid $\mathcal{M}$. This classifying space is contractible by Lemma 4.21, so the groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, A)$ vanish for $p \geq 1$, which proves the case $n = 0$.

For $n \geq 1$, the $\mathcal{M}$-modules $\mathcal{P}_{1+n} \otimes A$ and $\mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} \mathcal{P}_n \otimes A$ are isomorphic by part (i). Since the $\mathcal{M}$-bimodule $\mathbb{Z}[\mathcal{M}]^+$ is free as a left and right module separately, the balancing property of Tor groups yields

$$\begin{aligned} \mathrm{Tor}_*^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \mathcal{P}_{1+n} \otimes A) \;&\cong\; \mathrm{Tor}_*^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} \mathcal{P}_n \otimes A) \\ &\cong\; \mathrm{Tor}_*^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z} \otimes_{\mathcal{M}} \mathbb{Z}[\mathcal{M}]^+, \mathcal{P}_n \otimes A) \;\cong\; \mathrm{Tor}_*^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \mathcal{P}_n \otimes A) \end{aligned}$$

since $\mathbb{Z} \otimes_{\mathcal{M}} \mathbb{Z}[\mathcal{M}]^+$ is again the trivial right $\mathcal{M}$-module $\mathbb{Z}$. So induction on $n$ shows that the groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \mathcal{P}_n \otimes A)$ vanish in positive dimensions.

(iii) Since $\mathcal{P}_n$ is free as a right $\Sigma_n$-module, the functor $\mathcal{P}_n \otimes_{\Sigma_n} -$ is exact. The functor takes the free $\Sigma_n$-module of rank 1 to $\mathcal{P}_n$, so by part (ii) it takes projective $\Sigma_n$-modules to tame $\mathcal{M}$-modules which are acyclic for the functor $\mathbb{Z} \otimes_{\mathcal{M}} -$.

Thus if $P_\bullet \longrightarrow B$ is a projective resolution of $B$ by $\Sigma_n$-modules, then $\mathcal{P}_n \otimes_{\Sigma_n} P_\bullet$ is a resolution of $\mathcal{P}_n \otimes_{\Sigma_n} B$ which can be used to calculate the desired Tor groups. Thus we have isomorphisms

$$\mathrm{Tor}_*^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \mathcal{P}_n \otimes_{\Sigma_n} B) \;=\; H_*(\mathbb{Z} \otimes_{\mathcal{M}} \mathcal{P}_n \otimes_{\Sigma_n} P_\bullet) \;\cong\; H_*(\mathbb{Z} \otimes_{\Sigma_n} P_\bullet) \;=\; H_*(\Sigma_n; B) \ .$$

□

**4.4. Examples.** We discuss several classes of symmetric spectra with a view towards the $\mathcal{M}$-action on the stable homotopy groups.

EXAMPLE 4.27 (Eilenberg-Mac Lane spectra). Every tame $\mathcal{M}$-module $W$ can be realized as the homotopy group of a symmetric spectrum. For this purpose we modify the construction of the symmetric Eilenberg-Mac Lane spectrum of an abelian group. We define a symmetric spectrum $HW$ of simplicial sets by

$$(HW)_n \;=\; W^{(n)} \otimes \mathbb{Z}[S^n] \ ,$$

where $W^{(n)}$ is the filtration $n$ subgroup of $W$ and $\mathbb{Z}[S^n]$ refers to the simplicial abelian group freely generated by the simplicial set $S^n = S^1 \wedge \ldots \wedge S^1$, divided by the subgroup generated by the basepoint. The symmetric group $\Sigma_n$ takes $W^{(n)}$ to itself and we let it act diagonally on $(HW)_n$, i.e., on $S^n$ by permuting the smash factors. If $\mathcal{M}$ acts trivially on $W$, then this is just the ordinary Eilenberg-Mac Lane spectrum introduced in Example 2.7. Note that $HW$ is an $\Omega$-spectrum if and only if the $\mathcal{M}$-action on $W$ is trivial.

Since $(HW)_n$ is an Eilenberg-Mac Lane space of type $(W^{(n)}, n)$ the homotopy groups of the symmetric spectrum $HW$ are concentrated in dimension zero where we have $\pi_0 HW \cong \bigcup_{n \geq 0} W^{(n)} = W$ as $\mathcal{M}$-modules. As an aside we note that instead of the system $\mathbf{n} \mapsto W^{(n)}$ we could use any $I$-functor in the definition above; this shows that every $I$-functor arises as the $I$-functor $\underline{\pi}_0$ of a symmetric spectrum.

EXAMPLE 4.28 (Twisted smash products). We describe the homotopy groups of a twisted smash product $L \triangleright_m X$ (see Example 2.20) as a functor of the homotopy groups of $L \wedge X$, using all available structure on those. Since $L \triangleright_m X$ is isomorphic to $G_m L \wedge X$ this gives a description of the homotopy groups of smash products with semifree spectra. Since free and semifree symmetric spectra are special cases of twisted smash products, this will specialize to formulas for the homotopy groups of free and semifree symmetric spectra.

So we consider a symmetric spectrum $X$ and a non-degenerately based $\Sigma_m$-space (or pointed $\Sigma_m$-simplicial set) $L$ for some $m \geq 0$. We construct a natural isomorphism of $\mathcal{M}$-modules

$$\pi_k(L \triangleright_m X) \;\cong\; \mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m \times \mathcal{M}} (\mathrm{sgn}_m \otimes \pi_{k+m}(L \wedge X)) \;. \tag{4.29}$$

Here we use the left $\Sigma_m$-action on $\pi_{k+m}(L \wedge X)$ through the action on $L$, twisted by the sign representation $\mathrm{sgn}_m$ of $\Sigma_m$, and $\mathbb{Z}[\mathcal{M}]^{+m}$ is the monoid ring of $\mathcal{M}$ with its natural left multiplication action and right action of $\Sigma_m \times \mathcal{M}$ through the homomorphism $\times : \Sigma_m \times \mathcal{M} \longrightarrow \mathcal{M}$, compare (4.19).

We remark that as a right $\Sigma_m \times \mathcal{M}$-module, $\mathbb{Z}[\mathcal{M}]^{+m}$ is free of countably infinite rank. One possible basis is given by the '$(m, \infty)$-shuffles', i.e., by those bijections $f \in \mathcal{M}$ which satisfy $f(i) < f(i+1)$ for all $i \neq m$. In other words, all bijective $f$ which keep the sets $\mathbf{m} = \{1, \dots, m\}$ and $\{m+1, m+2, \dots\}$ in their natural order. So the isomorphism (4.29) in particular implies that the underlying abelian group of $\pi_k(L \triangleright_m X)$ is a countably infinite sum of copies of the underlying abelian group of $\pi_{k+m}(L \wedge X)$.

To establish the isomorphism (4.29) we calculate the $I$-functor $\underline{\pi}^{\mathrm{s}}_k(L \triangleright_m X)$ consisting of the *stable* homotopy groups of the spaces in the spectrum $L \triangleright_m X$ and exploit that for any symmetric spectrum $Y$ the $\mathcal{M}$-module $\pi_k Y$ can also be calculated as the colimit of $\underline{\pi}^{\mathrm{s}}_k Y$ instead of the $I$-functor $\underline{\pi}_k Y$ of *unstable* homotopy groups, see Remark 4.15. We denote by $\mathrm{sgn}_m$ the sign representation of the symmetric group $\Sigma_m$ and recall that for every spectrum $Y$ the action of $\Sigma_m = I(\mathbf{m}, \mathbf{m})$ on $(\underline{\pi}^s_k Y)(\mathbf{m}) = \pi^s_{k+m} Y_m$ is induced by the action on $Y_m$ twisted by $\mathrm{sgn}_m$.

Since stable homotopy groups takes wedges to sums we have an isomorphism of $\Sigma_{m+n}$-modules

$$\begin{aligned} \underline{\pi}^{\mathrm{s}}_k(L \triangleright_m X)(\mathbf{m}+\mathbf{n}) \;&=\; \mathrm{sgn}_{m+n} \otimes \pi^{\mathrm{s}}_{k+(m+n)} \left(\Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} (L \wedge X_n)\right) \\ &\cong\; \mathbb{Z}[\Sigma_{m+n}] \otimes_{\mathbb{Z}[\Sigma_m \times \Sigma_n]} \Big(\mathrm{sgn}_m \otimes \mathrm{sgn}_n \otimes \pi^{\mathrm{s}}_{(k+m)+n}(L \wedge X_n)\Big) \\ &=\; \mathbb{Z}[\Sigma_{m+n}] \otimes_{\mathbb{Z}[\Sigma_m \times \Sigma_n]} \left(\mathrm{sgn}_m \otimes \underline{\pi}^{\mathrm{s}}_{k+m}(L \wedge X)\right) \end{aligned} \tag{4.30}$$

The isomorphism (4.30) is also compatible with stabilization maps, so it is an isomorphism of $I$-functors

$$\underline{\pi}^{\mathrm{s}}_k(L \triangleright_m X) \;\cong\; \triangleright_m(\mathrm{sgn}_m \otimes \underline{\pi}^{\mathrm{s}}_{k+m}(L \wedge X))$$

where $\triangleright_m$ is the construction for $I$-functors with $\Sigma_m$-action defined in (4.18). Lemma 4.20 provides an isomorphism of $\mathcal{M}$-modules

$$(\triangleright_m(\mathrm{sgn}_m \otimes \underline{\pi}^{\mathrm{s}}_{k+m}(L \wedge X)))(\omega) \;\cong\; \mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m \times \mathbb{Z}[\mathcal{M}]} (\mathrm{sgn}_m \otimes \pi_{k+m}(L \wedge X)) \;.$$

Combining these two isomorphisms gives (4.29).

Example 4.31 (Free and semifree symmetric spectra). We saw in Example 2.12 that the zeroth stable homotopy group of the free symmetric spectrum $F_1 S^1$ is free abelian of countably infinite rank. We now refine this calculation to an isomorphism of $\mathcal{M}$-modules $\pi_0(F_m S^m) \cong \mathcal{P}_m$, see (4.33) below; here $\mathcal{P}_m$ is the $\mathcal{M}$-module which represents taking filtration $m$ elements, see Example 4.25. So while the groups $\pi_0(F_m S^m)$ are all additively isomorphic for different positive $m$, the $\mathcal{M}$-action distinguishes them. In particular, there cannot be a chain of $\pi_*$-isomorphisms between $F_m S^m$ and $F_n S^n$ for $n \neq m$.

The calculation of the $\mathcal{M}$-action on free and semifree symmetric spectra is a special case of the very general formula (4.29) for the homotopy groups of a twisted smash product. Let $L$ be a pointed space (or simplicial set) with a left action by the symmetric group $\Sigma_m$, for some $m \geq 0$. Recall that $G_m L$ denotes the semifree symmetric spectrum generated by $L$ in level $m$, defined in Example 2.13, which is also equal to the twisted smash product $L \triangleright_m \mathbb{S}$ of $L$ with the sphere spectrum. The functor $G_m$ is left adjoint to evaluating a symmetric spectrum at level $m$, viewed as a functor with values in pointed $\Sigma_m$-spaces. When we apply formula (4.29) to $G_m L = L \triangleright_m \mathbb{S}$ we obtain an isomorphism of $\mathcal{M}$-modules

$$\pi_k(G_m L) \;=\; \pi_k(L \triangleright_m \mathbb{S}) \;\cong\; \mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m \times \mathcal{M}} (\mathrm{sgn}_m \otimes \pi_{k+m}(L \wedge \mathbb{S})) \;.$$

The homotopy groups of the spectrum $L \wedge \mathbb{S} = \Sigma^\infty L$ are the stable homotopy groups of $L$. Since the $\mathcal{M}$-action on $\pi_{k+m}(L \wedge \mathbb{S})$ is trivial [justify] we get

$$\mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m \times \mathcal{M}} (\mathrm{sgn}_m \otimes \pi_{k+m}(L \wedge \mathbb{S})) \cong (\mathbb{Z}[\mathcal{M}]^{+m} \otimes_{1 \times \mathcal{M}} \mathbb{Z}) \otimes_{\Sigma_m} (\mathrm{sgn}_m \otimes \pi^s_{k+m} L) .$$

The tame $\mathcal{M}$-module $\mathcal{P}_m$ has a compatible right $\Sigma_m$-action which is given on the basis by permuting the components of an $m$-tuple, i.e., $(x_1, \dots, x_m)\gamma = (x_{\gamma(1)}, \dots, x_{\gamma(m)})$. The map

$$\mathbb{Z}[\mathcal{M}]^{+m} \otimes_{1 \times \mathcal{M}} \mathbb{Z} \longrightarrow \mathcal{P}_m , \quad f \otimes 1 \longmapsto (f(1), \dots, f(m))$$

is an isomorphism of $\mathcal{M}$-$\Sigma_m$-bimodules; so combining all these isomorphisms we finally get a natural isomorphism of $\mathcal{M}$-modules

$$\pi_k(G_m L) \cong \mathcal{P}_m \otimes_{\Sigma_m} (\pi^{\mathrm{s}}_{k+m} L)(\mathrm{sgn}) . \tag{4.32}$$

On the right of the tensor symbol, the group $\Sigma_m$ acts by what is induced on stable homotopy groups by the action on $L$, twisted by sign.

Free symmetric spectra are special cases of semifree symmetric spectra. For a pointed space $K$ (without any group action) we have $F_m K \cong G_m(\Sigma_m^+ \wedge K)$ and $\pi^{\mathrm{s}}_{k+m}(\Sigma_m^+ \wedge K) \cong \mathbb{Z}[\Sigma_m] \otimes \pi^{\mathrm{s}}_{k+m} K$ as $\Sigma_m$-modules. So (4.32) specializes to a natural isomorphism of $\mathcal{M}$-modules

$$\pi_k(F_m K) \cong \mathcal{P}_m \otimes \pi^{\mathrm{s}}_{k+m} K . \tag{4.33}$$

Here $\pi^{\mathrm{s}}_{k+m} K$ is the $(k+m)$th stable homotopy group of $K$; the monoid $\mathcal{M}$ acts only on $\mathcal{P}_m$.

Example 4.34 (Loop and suspension). The loop $\Omega X$ and suspension $S^1 \wedge X$ of a symmetric spectrum $X$ are defined by applying the functors $\Omega$ respectively $S^1 \wedge -$ levelwise, where the structure maps do not interact with the new loop or suspension coordinates, compare Examples 2.16 and 2.23. We already saw that loop and suspension simply shift the homotopy groups, and we shall now prove that the $\mathcal{M}$-action is unchanged in this process.

For every symmetric spectrum $X$ the map $S^1 \wedge - : \pi_{k+n} X_n \longrightarrow \pi_{1+k+n}(S^1 \wedge X_n)$ is $\Sigma_n$-equivariant and a natural transformations of $I$-functors as $\mathbf{n}$ varies. So the induced morphism $S^1 \wedge - : \pi_k X \longrightarrow \pi_{1+k}(S^1 \wedge X)$ on colimits is $\mathcal{M}$-linear, and hence, by Example 2.16, an isomorphism of $\mathcal{M}$-modules.

The isomorphism $\alpha : \pi_{k+n} \Omega(X_n) \cong \pi_{1+k+n} X_n$ which we used in Example 2.23 sends a representing continuous map $f : S^{k+n} \longrightarrow \Omega(X_n)$ to the class of the adjoint $\hat{f} : S^{1+k+n} \longrightarrow X_n$ given by $\hat{f}(s \wedge t) = f(t)(s)$, where $s \in S^1$, $t \in S^{k+n}$. As $n$ varies, these particular isomorphisms are compatible with the symmetric group actions and stabilization maps, so they form an isomorphism of $I$-functors $\alpha : \underline{\pi}_k(\Omega X) \cong \underline{\pi}_{1+k} X$. Hence the induced isomorphism on colimits $\alpha : \pi_k(\Omega X) \cong \pi_{1+k} X$ is $\mathcal{M}$-linear.

Example 4.35 (Shift). The shift is another construction for symmetric spectra which reindexes the homotopy groups, but unlike the suspension, this construction changes the $\mathcal{M}$-action in a systematic way. The shift of a symmetric spectrum $X$ was defined in Example 2.18 by $(\mathrm{sh}\, X)_n = X_{1+n}$ with action of $\Sigma_n$ via the monomorphism $(1 \times -) : \Sigma_n \longrightarrow \Sigma_{1+n}$. The structure maps of $\mathrm{sh}\, X$ are the reindexed structure maps for $X$.

If we view $\Sigma_n$ as the subgroup of $\mathcal{M}$ of maps which fix all numbers bigger than $n$, then the homomorphism $(1 \times -) : \Sigma_n \longrightarrow \Sigma_{1+n}$ has a natural extension to a monomorphism $(1 \times -) : \mathcal{M} \longrightarrow \mathcal{M}$ given by $(1 \times f)(1) = 1$ and $(1 \times f)(i) = f(i-1) + 1$ for $i \geq 2$. The image of the monomorphism $1 \times -$ is the submonoid of those $g \in \mathcal{M}$ with $g(1) = 1$. If $W$ is an $\mathcal{M}$-module, we denote by $W(1)$ the $\mathcal{M}$-module with the same underlying abelian group, but with $\mathcal{M}$-action through the endomorphism $1 \times -$. We call $W(1)$ the *shift* of $W$. Since $|1 \times f| = 1 + |f|$, shifting an $\mathcal{M}$-module shifts the filtration subgroups, i.e., we have $W(1)^{(n)} = W^{(1+n)}$ for all $n \geq 0$. Thus the $\mathcal{M}$-module $W(1)$ is tame if and only if $W$ is.

For any symmetric spectrum $X$, integer $k$ and large enough $n$ we have

$$\pi_{(k+1)+n}(\mathrm{sh}\, X)_n = \pi_{k+(1+n)} X_{1+n} ,$$

and the maps in the colimit system for $\pi_{k+1}(\mathrm{sh}\, X)$ are the same as the maps in the colimit system for $\pi_k X$. Thus we get $\pi_{k+1}(\mathrm{sh}\, X) = \pi_k X$ as abelian groups. However, the action of a permutation on $\pi_{k+1+n}(\mathrm{sh}\, X)_n$

is shifted by the homomorphism $1 \times -$, so we have

$$\pi_{*+1}(\mathrm{sh}\, X) \;=\; (\pi_* X)(1) \tag{4.36}$$

as $\mathcal{M}$-modules.

Example 4.37 (Shift adjoint). The shift functor has a left adjoint $S^0 \triangleright_1$ given by $(S^0 \triangleright_1 X)_0 = *$ and

$$(S^0 \triangleright_1 X)_{1+n} \;=\; \Sigma_{1+n}^+ \wedge_{\Sigma_n} X_n$$

for $n \geq 0$. Here $\Sigma_n$ acts from the right on $\Sigma_{1+n}$ via the monomorphism $(1 \times -) : \Sigma_n \longrightarrow \Sigma_{1+n}$. The structure map $(\Sigma_{1+n}^+ \wedge_{\Sigma_n} X_n) \wedge S^1 \longrightarrow \Sigma_{1+n+1}^+ \wedge_{\Sigma_{n+1}} X_{n+1}$ is induced by $(- \times 1) : \Sigma_{1+n} \longrightarrow \Sigma_{1+n+1}$ (the 'inclusion') and the structure map of $X$.

The effect on homotopy groups of the functor $S^0 \triangleright_0$ is given as a special case of the very general formula (4.29) for the homotopy groups of a twisted smash product. Indeed, that formula specializes to a natural isomorphism of $\mathcal{M}$-modules

$$\pi_k(S^0 \triangleright_1 X) \;\cong\; \mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} \pi_{k+1} X \;. \tag{4.38}$$

Here $\mathbb{Z}[\mathcal{M}]^+$ denotes the monoid ring of $\mathcal{M}$ with its usual left action, but with right action through the monomorphism $(1 \times -) : \mathcal{M} \longrightarrow \mathcal{M}$ given by $(1 \times f)(1) = 1$ and $(1 \times f)(i) = f(i-1) + 1$ for $i \geq 2$. As a right $\mathcal{M}$-module, $\mathbb{Z}[\mathcal{M}]^+$ is free of countably infinite rank (one possible basis is given by the transpositions $(1, n)$ for $n \geq 1$). So the isomorphism (4.38) in particular implies that the underlying abelian group of $\pi_k(S^0 \triangleright_1 X)$ is a countably infinite sum of copies of the underlying abelian group of $\pi_{k+1} X$.

The functor $\mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathcal{M}} -$ is left adjoint to $\mathrm{Hom}_{\mathcal{M}}(\mathbb{Z}[\mathcal{M}]^+, -)$, which is a fancy way of writing the algebraic shift functor $W \mapsto W(1)$. Under the isomorphism (4.38) and the identification (4.36), the adjunction between shift and $S^0 \triangleright_1$ as functors of symmetric spectra corresponds exactly to the adjunction between $W \mapsto W(1)$ and $\mathbb{Z}[\mathcal{M}]^+ \otimes_{\mathbb{Z}[\mathcal{M}]} -$ as functors of tame $\mathcal{M}$-modules.

Example 4.39 (Infinite products). Finite products of symmetric spectra are $\pi_*$-isomorphic to finite wedges, so stable homotopy groups commute with finite products. But homotopy groups do not in general commute with infinite products. This should not be surprising because stable homotopy groups involves a sequential colimit, and these generally do not preserve infinite products.

There are even two different ways in which commutation with products can fail. First we note that an infinite product of a family $\{W_i\}_{i \in I}$ of tame $\mathcal{M}$-modules is only tame if almost all the modules $W_i$ have trivial $\mathcal{M}$-action. Indeed, if there are infinitely many $W_i$ with non-trivial $\mathcal{M}$-action, then by Lemma 4.24 (ii) the product $\prod_{i \in I} W_i$ contains tuples of elements whose filtrations are not bounded. We define the *tame product* of the family $\{W_i\}_{i \in I}$ by

$$\prod_{i \in I}^{\mathrm{tame}} W_i \;=\; \bigcup_{n \geq 0} \left( \prod_{i \in I} W_i^{(n)} \right) \;,$$

which is the largest tame submodule of the product and thus the categorical product in the category of tame $\mathcal{M}$-modules.

Now we consider a family $\{X_i\}_{i \in I}$ of symmetric spectra. Since the monoid $\mathcal{M}$ acts tamely on the homotopy groups of any symmetric spectrum, the natural map from the homotopy groups of the product spectrum to the product of the homotopy groups always lands in the tame product. But in general, this natural map

$$\pi_k \left( \prod_{i \in I} X_i \right) \;\longrightarrow\; \prod_{i \in I}^{\mathrm{tame}} \pi_k X_i \tag{4.40}$$

need not be an isomorphism. As an example we consider the symmetric spectra $(F_1 S^1)^{\leq i}$ obtained by truncating the free symmetric spectrum $F_1 S^1$ above level $i$, i.e.,

$$((F_1 S^1)^{\leq i})_n \;=\; \begin{cases} (F_1 S^1)_n & \text{for } n \leq i, \\ * & \text{for } n \geq i+1 \end{cases}$$

with structure maps as a quotient spectrum of $F_1S^1$. Then $(F_1S^1)^{\leq i}$ has trivial homotopy groups for all $i$. The 0th homotopy group of the product $\prod_{i\geq 1}(F_1S^1)^{\leq i}$ is the colimit of the sequence of maps

$$\prod_{i\geq n}\mathcal{P}_1^{(n)} \longrightarrow \prod_{i\geq n+1}\mathcal{P}_1^{(n+1)}$$

which first projects away from the factor indexed by $i = n$ and then takes a product of inclusions $\mathcal{P}_1^{(n)} \longrightarrow \mathcal{P}_1^{(n+1)}$. The colimit is the quotient of the tame product $\prod_{i\geq 1}^{\mathrm{tame}}\mathcal{P}_1$ by the sum $\bigoplus_{i\geq 1}\mathcal{P}_1$; so $\pi_0$ of the product is non-zero and even has a non-trivial $\mathcal{M}$-action.

Remark 4.41. The injection monoid $\mathcal{M}$ gives essentially all natural operations on the homotopy groups of symmetric spectra. More precisely, we now identify the ring of natural operations $\pi_0X \longrightarrow \pi_0X$ with a completion of the monoid ring $\mathbb{Z}[\mathcal{M}]$. Moreover, tame $\mathcal{M}$-modules can equivalently be described as the discrete modules over the ring of operations. We will not need this information later, so we will be brief.

We define the ring $\mathbb{Z}[[\mathcal{M}]]$ as the endomorphism ring of the functor $\pi_0 : \mathcal{S}p^\Sigma \longrightarrow \mathcal{A}b$. So an element of $\mathbb{Z}[[\mathcal{M}]]$ is a natural self-transformation of the functor $\pi_0$, and composition of transformations gives the product. The following calculation of this ring depends on the fact that the homotopy group functor $\pi_0$ is pro-represented, in the level homotopy category of symmetric spectra, by the inverse system of free symmetric spectra $F_nS^n$, and that we know $\pi_0(F_nS^n)$ by Example 4.31.

In more detail: for every $n \geq 0$ we let $j_n \in \pi_n(F_nS^n)_n$ be the wedge summand inclusion $S^n \longrightarrow \Sigma_n^+ \wedge S^n = (F_nS^n)_n$ indexed by the unit element of $\Sigma_n$. Then evaluation at $j_n$ is a bijection

$$[F_nS^n, X] \longrightarrow \pi_nX_n\ , \quad [f] \mapsto f_*(j_n)$$

where the left hand side means homotopy classes of morphisms of symmetric spectra. We write $\lambda : F_{n+1}S^{n+1} \longrightarrow F_nS^n$ for the morphism adjoint the wedge summand inclusion $S^{n+1} \longrightarrow \Sigma_{n+1}^+ \wedge (S^n \wedge S^1) = (F_nS^n)_{n+1}$ indexed by the unit element of $\Sigma_{n+1}$. Then we have

$$\lambda_*(j_{n+1}) = \iota_*(j_n)$$

in the group $\pi_{n+1}(F_nS^n)_{n+1}$ which implies that the squares

$$\begin{tikzcd}
{[F_nS^n, X]} \arrow[r, "\cong"] \arrow[d, "{[\lambda,X]}"'] & \pi_nX_n \arrow[d, "\iota_*"] \\
{[F_{n+1}S^{n+1}, X]} \arrow[r, "\cong"'] & \pi_{n+1}X_{n+1}
\end{tikzcd}$$

commute. Passage to colimits give a natural isomorphism

$$\mathrm{colim}_n\ [F_nS^n, X] \longrightarrow \pi_0X\ .$$

From here the Yoneda lemma shows that we get an isomorphism of abelian groups

$$\beta\ :\ \mathbb{Z}[[\mathcal{M}]] \longrightarrow \lim_n\ \pi_0(F_nS^n)\ , \tag{4.42}$$

(where the limit is taken over the maps $\pi_0\lambda$) by sending a natural transformation $\tau : \pi_0 \longrightarrow \pi_0$ to the tuple $\{\tau_{F_nS^n}[j_n]\}_n$.

It remains to exhibit the ring $\mathbb{Z}[[\mathcal{M}]]$ as a completion of the monoid ring $\mathbb{Z}[\mathcal{M}]$. The natural action of $\mathcal{M}$ on the 0th homotopy group of a symmetric spectrum provides a ring homomorphism $\mathbb{Z}[\mathcal{M}] \longrightarrow \mathbb{Z}[[\mathcal{M}]]$. We define a left ideal $I_n$ of $\mathbb{Z}[\mathcal{M}]$ as the subgroup generated by all differences of the form $f - g$ for all $f, g \in \mathcal{M}$ such that $f$ and $g$ agree on $\mathbf{n}$. If $W$ is a tame $\mathcal{M}$-module and if $x \in W^{(n)}$ has filtration $n$, then $I_n \cdot x = 0$. So the action of the monoid ring $\mathbb{Z}[\mathcal{M}]$ on any tame module automatically extends to an additive map

$$(\lim_n\ \mathbb{Z}[\mathcal{M}]/I_n) \otimes W \longrightarrow W\ .$$

(Warning: $I_n$ is *not* a right ideal for $n \geq 1$, so the completion does not a priori have a ring structure). Since the homotopy groups of every symmetric spectrum form tame $\mathcal{M}$-modules, this gives a map of abelian groups

$$\alpha \ : \ \lim_n \ \mathbb{Z}[\mathcal{M}]/I_n \ \longrightarrow \ \mathbb{Z}[[\mathcal{M}]]$$

which extends the map from the monoid ring $\mathbb{Z}[\mathcal{M}]$.

To prove that $\alpha$ is a bijection we show that the composite $\beta\alpha : \lim_n \mathbb{Z}[\mathcal{M}]/I_n \longrightarrow \lim_n \pi_0(F_n S^n)$ with the isomorphism (4.42) is bijective. But this holds because the composite arise from compatible isomorphisms

$$\mathbb{Z}[\mathcal{M}]/I_n \ \longrightarrow \ \pi_0(F_n S^n) \ , \quad f + I_n \ \longmapsto \ f \cdot [j_n] \ ,$$

which in turn uses the isomorphisms $\mathcal{P}_n \cong \pi_0(F_n S^n)$ from Example 4.31.

We end this remark by claiming without proof that the extended action of $\mathbb{Z}[[\mathcal{M}]]$ on a tame $\mathcal{M}$-module $W$ makes it a *discrete module* in the sense that the action map

$$\mathbb{Z}[[\mathcal{M}]] \times W \ \longrightarrow \ W$$

is continuous with respect to the discrete topology on $W$ and the filtration topology on $\mathbb{Z}[[\mathcal{M}]]$. Conversely, if $W$ is discrete module over $\mathbb{Z}[[\mathcal{M}]]$, then its underlying $\mathcal{M}$-module is tame. This establishes an isomorphism between the category of tame $\mathcal{M}$-modules and the category of discrete $\mathbb{Z}[[\mathcal{M}]]$-modules.

**4.5. Semistable symmetric spectra.** The semistable spectra form an important class of symmetric spectra since for these, the naively defined homotopy groups of (1.5) coincide with the 'true' homotopy groups, i.e., morphisms in the stable homotopy category from the sphere spectra (a more precise definition of the true homotopy groups will be given in Section 4.2 of Chapter II). As a slogan, for semistable spectra the homotopy groups are 'correct', and they are pathological otherwise. The rigorous meaning of this statement will only become clear later when we have introduced stable equivalences and the stable homotopy category. Many symmetric spectra which arise naturally are semistable, compare Example 4.48.

In Theorem 4.44 below we define semistable symmetric spectra via several equivalent conditions. To state these, we need a few definitions. As we have seen in Examples 2.16 and 2.18, the suspension and shift construction both shift the homotopy groups. However, there is in general no morphism of symmetric spectra which realizes an isomorphism on homotopy groups. The two constructions are related by the natural morphism $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ which was defined in (2.19). We set $RX = \Omega(\operatorname{sh} X)$ and obtain a morphism $\lambda^*_X : X \longrightarrow RX$ as the adjoint of $\lambda_X$. The $n$-th level of $\lambda^*$ differs from the adjoint structure map $\tilde\sigma_n : X_n \longrightarrow \Omega X_{n+1}$ by the isomorphism $\Omega(\chi_{n,1}) : \Omega(X_{n+1}) \longrightarrow \Omega(X_{1+n})$. So $X$ (levelwise Kan when in the simplicial context) is an $\Omega$-spectrum if and only if the morphism $\lambda^*_X : X \longrightarrow RX$ is a level equivalence.

We iterate this construction and let $R^\infty X$ be the colimit of the sequence

$$X \xrightarrow{\lambda^*} RX \xrightarrow{R(\lambda^*)} R^2 X \xrightarrow{R^2(\lambda^*)} \cdots \ . \tag{4.43}$$

This construction comes with a canonical natural morphism $\lambda^\infty_X : X \longrightarrow R^\infty X$.

THEOREM 4.44. *For every symmetric spectrum $X$ the following conditions are equivalent.*

(i) *The action of the monoid $\mathcal{M}$ is trivial on all homotopy groups of $X$.*
(ii) *The morphism $\lambda_X : S^1 \wedge X \longrightarrow \operatorname{sh} X$ is a $\pi_*$-isomorphism.*
(iii) *The morphism $\lambda^*_X : X \longrightarrow \Omega(\operatorname{sh} X)$ is a $\pi_*$-isomorphism.*
(iv) *The morphism $\lambda^\infty_X : X \longrightarrow R^\infty X$ is a $\pi_*$-isomorphism.*
(v) *The symmetric spectrum $R^\infty X$ is an $\Omega$-spectrum.*
(vi) *There exists a $\pi_*$-isomorphism from $X$ to an $\Omega$-spectrum.*

*If conditions* (i)-(vi) *hold, then $X$ is called* semistable. [For symmetric spectra of simplicial sets, $X$ should be levelwise Kan in (iii), (iv) and (v).]

PROOF. For symmetric spectra of simplicial sets all relevant notions are defined by geometrically realizing to a topological spectrum. So it suffices to prove the theorem for symmetric spectra of topological spaces.

To see that condition (i) implies condition (iii) we compare the effect of $\lambda^*_X : X \longrightarrow \Omega(\mathrm{sh}\, X)$ on homotopy with the action of the special monoid element $d$ given by $d(i) = i + 1$. The level $n$ component $\lambda^*_n : X_n \longrightarrow \Omega(X_{1+n})$ is adjoint to the composite

$$S^1 \wedge X_n \xrightarrow[\text{twist}]{\cong} X_n \wedge S^1 \xrightarrow{\sigma_n} X_{n+1} \xrightarrow{(1,\dots,n,n+1)} X_{1+n}$$

So the square

$$\begin{tikzcd}
\pi_{k+n} X_n \arrow[r, "\iota_*"] \arrow[d, "\pi_{k+n}(\lambda^*_n)"'] & \pi_{k+n+1} X_{n+1} \arrow[d, "(-1)^n (1,\dots,n,n+1)_*"] \\
\pi_{k+n}\Omega(X_{1+n}) \arrow[r, "\cong", "(-1)^k\alpha"'] & \pi_{k+1+n} X_{1+n}
\end{tikzcd}$$

commutes, where $\iota_*$ is the stabilization map, and the isomorphism $\alpha$ is as in Example 2.23. The signs arise as the effect of moving a sphere coordinate past $k$ respectively $n$ other coordinates. As $n$ increases, the maps $(-1)^n(1, \dots, n, n + 1)_* \circ \iota_* : \pi_{k+n}X_n \longrightarrow \pi_{k+1+n}X_{1+n}$ stabilize to the left multiplication of $d \in \mathcal{M}$ on $\pi_k X$, see Example 4.14. So the square

(4.45)
$$\begin{tikzcd}
\pi_k X \arrow[r, "d\cdot"] \arrow[d, "\pi_k(\lambda^*_X)"'] & (\pi_k X)(1) \arrow[d, equal] \\
\pi_k \Omega(\mathrm{sh}\, X) \arrow[r, "\cong", "(-1)^k\alpha"'] & \pi_{k+1}(\mathrm{sh}\, X)
\end{tikzcd}$$

commutes, and if $\mathcal{M}$, hence in particular $d$, acts trivially, then $\lambda^*_X$ is a $\pi_*$-isomorphism.

Conditions (ii) and (iii) are equivalent since we have a commutative square

$$\begin{tikzcd}
\pi_k X \arrow[r, "\pi_k(\lambda^*_X)"] \arrow[d, "S^1\wedge -"', "\cong"] & \pi_k \Omega(\mathrm{sh}\, X) \arrow[d, "\cong"', "\alpha"] \\
\pi_{1+k}(S^1 \wedge X) \arrow[r, "\pi_{1+k}(\lambda_X)"'] & \pi_{1+k}(\mathrm{sh}\, X)
\end{tikzcd}$$

in which both vertical maps are isomorphisms.

(iii) $\Rightarrow$ (iv): The commutative square (4.45) implies a natural isomorphism of $\mathcal{M}$-modules

(4.46)
$$\pi_k(R^\infty X) \;\cong\; (\pi_k X)(\infty) \; .$$

Here for an $\mathcal{M}$-module $V$ we denote by $V(\infty)$ the colimit of the sequence

$$V \xrightarrow{d\cdot} V(1) \xrightarrow{d\cdot} V(2) \xrightarrow{d\cdot} \cdots$$

(note that $(1 \times f)d = df$ for all $f \in \mathcal{M}$, which means that $d\cdot : V \longrightarrow V(1)$ is indeed $\mathcal{M}$-linear, and so the colimit $V(\infty)$ is naturally an $\mathcal{M}$-module). If condition (iii) holds for $X$ then it also holds for $R^n X$ for all $X$ and the morphism $X \longrightarrow R^\infty X$ is a $\pi_*$-isomorphism.

(iv) $\Rightarrow$ (v): As we saw in the proof of the previous implication, the effect of $\lambda^\infty_X$ on homotopy groups coincides with the natural injection $\pi_k X \longrightarrow (\pi_k X)(\infty)$ obtained by iterated left multiplication by $d \in \mathcal{M}$. If that injection is a bijection the $d$ acts bijectively on the homotopy groups of $X$. Hence every monoid element acts identically on $\pi_k X$. The commutative square

$$\begin{tikzcd}
\pi_k X \arrow[r, "(1_n\times d)\cdot"] \arrow[d, "\cong"'] & \pi_k X \arrow[d, "\cong"] \\
\pi_{k+n}(R^\infty X)_n \arrow[r, "\iota_*"'] & \pi_{k+n+1}(R^\infty X)_{n+1}
\end{tikzcd}$$

[define the vertical iso; check] thus shows that $R^\infty X$ is an $\Omega$-spectrum.

To see that condition (vi) implies condition (i) we show that the $\mathcal{M}$-action on the homotopy groups of an $\Omega$-spectrum is always trivial. If $X$ is an $\Omega$-spectrum, then for every integer $k$ the canonical map $\pi_k X_0 \longrightarrow \pi_k X$ (for $k \geq 0$) or the map $\pi_0 X_{-k} \longrightarrow X_k$ (for $k \leq 0$) is bijective. So every element of $\pi_k X$ has filtration $\max(0, -k)$. But tame $\mathcal{M}$-modules with bounded filtration necessarily have trivial $\mathcal{M}$-action by Lemma 4.24 (ii). □

Proposition 4.47. *Let $X$ be a symmetric spectrum which satisfies one of the following conditions.*

(i) *For every $k \in \mathbb{Z}$ there is an $n \geq 0$ such that the canonical map $\pi_{k+n} X_n \longrightarrow \pi_k X$ is surjective.*
(ii) *Every even permutation $\gamma \in \Sigma_n$ acts as the identity on the homotopy groups of $X_n$.*
(iii) *$X$ is underlying an orthogonal spectrum.*
(iv) *The homotopy groups of $X$ are dimensionwise finitely generated as abelian groups.*

*Then the $\mathcal{M}$-action on all homotopy groups of $X$ is trivial and so $X$ is semistable.*

Proof. (i) Under the assumption every element of $\pi_k X$ has filtration $n$. But tame $\mathcal{M}$-modules with bounded filtration necessarily have trivial $\mathcal{M}$-action by Lemma 4.24 (ii).

(ii) We consider more generally any $I$-functor $F$ which takes all even permutations to identity maps. Given $f \in \mathcal{M}$ and an element $[x] \in F(\omega)$ in the colimit represented by $x \in F(\mathbf{n})$, then we can find $m \geq \max\{f(\mathbf{n})\}$ and an *even* permutation $\gamma \in \Sigma_m$ such that $\gamma$ agrees with $f$ on $\mathbf{n}$. Since $\gamma$ is even, we then have $f_*[x] = [(f|_{\mathbf{n}})_*(x)] = [(\gamma|_{\mathbf{n}})_*(x)] = [\gamma_*(\iota_*^{m-n}(x))] = [\iota_*^{m-n}(x)] = [x]$. Thus the monoid $\mathcal{M}$ acts trivially on the colimit $F(\omega)$.

(iii) The inclusion $\Sigma_n \longrightarrow O(n)$ as permutation matrices sends all even permutations to the path component of the unit in $O(n)$. So if the $\Sigma_n$-action on a pointed space $X_n$ extends to an $O(n)$-action, then all even permutations act as the identity on the homotopy groups of $X_n$. So part (ii) applies.

(iv) If $\pi_k X$ is finitely generated as an abelian group, then tameness forces the $\mathcal{M}$-action to be trivial on $\pi_k X$ (Lemma 4.24 (iv)). □

Example 4.48. An important special case where condition (i) in Proposition 4.47 above holds is when the homotopy groups of a symmetric spectrum $X$ stabilize, i.e., for each $k \in \mathbb{Z}$ there exists an $n \geq 0$ such that from the group $\pi_{k+n} X_n$ on, all maps in the sequence (1.5) defining $\pi_k X$ are isomorphisms.

Examples of symmetric spectra with stabilizing homotopy groups include all suspension spectra (Example 2.16), $\Omega$-spectra, or $\Omega$-spectra from some point $X_n$ on. So it includes Eilenberg-Mac Lane spectra $HA$ associated to an abelian group (see Example 2.7) as well as spectra of topological $K$-theory (Example 2.10) and algebraic $K$-theory (Example 2.11). The symmetric spectrum obtained from a $\Gamma$-space $A$ (see Example 2.39) by evaluation on spheres is another example since the structure map $A(S^n) \wedge S^1 \longrightarrow A(S^{n+1})$ is $(2n+1)$-connected [**34**, Prop. 5.21]. So all these kinds of symmetric spectra are semistable.

The various Thom spectra $MO$, $MSO$ and $MSpin$ of Example 2.8 or $MU$, $MSU$ and $MSp$ of Example 2.9 are underlying orthogonal spectra, so they are all semistable.

Example 4.49. We collect some examples of symmetric spectra which are *not* semistable. Example 4.31 identifies the homotopy groups of free and semifree symmetric spectra as

$$\pi_k(F_m K) \;\cong\; \mathcal{P}_m \otimes \pi^{\mathrm{s}}_{k+m} K \qquad \text{respectively} \qquad \pi_k(G_m L) \;\cong\; \mathcal{P}_m \otimes_{\Sigma_m} (\pi^{\mathrm{s}}_{k+m} L)(\mathrm{sgn}) \ .$$

Since $\mathcal{P}_m$ is free of countably infinite rank as a right $\Sigma_m$-module, the free or semifree symmetric spectra generated in positive level $m$ are never semistable unless $K$ respectively $L$ has trivial stable homotopy groups.

If $W$ is a tame $\mathcal{M}$-module with non-trivial $\mathcal{M}$-action, then $\pi_0 HW \cong W$ as $\mathcal{M}$-modules and so the generalized Eilenberg-Mac Lane spectrum $HW$ as defined in Example 4.27 is not semistable.

Example 4.39 shows that an infinite product of symmetric spectra with trivial homotopy groups can have homotopy groups with non-trivial $\mathcal{M}$-action. In particular, infinite products of semistable symmetric spectra need not be semistable.

If $X$ has at least one non-trivial homotopy group, then $S^0 \triangleright_1 X \cong F_1 S^0 \wedge X$ is not semistable by Example 4.37.

The 'trivial $\mathcal{M}$-action' criterion is often handy for showing that semistability is preserved by certain constructions. We give a few examples of this in the following proposition.

EXAMPLE 4.50. If $f : X \longrightarrow Y$ is any morphism of symmetric spectra, then the homotopy groups of the spectra $X$, $Y$ and the mapping cone $C(f) = [0,1]^+ \wedge X \cup_f Y$ are related by a long exact sequence of tame $\mathcal{M}$-modules (we use that the $\mathcal{M}$-action does not change under loop and suspension). Trivial tame $\mathcal{M}$-modules are closed under taking submodules, quotient modules and extensions (Lemma 4.22 (iv)); so if two out of three graded $\mathcal{M}$-modules $\pi_*X$, $\pi_*Y$ and $\pi_*C(f)$ have trivial $\mathcal{M}$-action, then so does the third. Thus the mapping cone of any morphism between semistable symmetric spectra is semistable.

If $f : X \longrightarrow Y$ is an h-cofibration [define] of symmetric spectra, or simply an injective morphism when in the simplicial context, then the mapping cone $C(f)$ is $\pi_*$-isomorphic to the quotient $Y/X$. Thus if two of the spectra $X$, $Y$ and $Y/X$ are semistable, then so is the third.

EXAMPLE 4.51. Semistability is preserved under suspension, loop, wedges, shift and sequential colimits along h-cofibrations (or injective morphisms when in the simplicial context) since these operations preserve the property of $\mathcal{M}$ acting trivially on homotopy groups.

We shall see later (see Theorem II.5.17) that the smash product of two semistable symmetric spectra is semistable if at least one of the factors is flat. Moreover, if $X$ is a semistable symmetric spectrum and $A$ is a $\Gamma$-space, then $A(X)$ is semistable (see Proposition II.5.18 (iii)) [also true for simplicial functors?]

EXAMPLE 4.52. For a symmetric spectrum $X$ and a pointed space $K$ we let $K \wedge X$ be the symmetric spectrum obtained by smashing $K$ levelwise with $X$ (compare Example 2.15). For example, when $K = S^1$ is the circle, this specializes to the suspension of $X$. We claim that if $X$ is semistable and $K$ is a CW-complex, then the symmetric spectrum $K \wedge X$ is again semistable.

We first prove the claim for finite dimensional CW-complexes by induction over the dimension. If $K$ is 0-dimensional, then $K \wedge X$ is a wedge of copies of $X$, thus semistable. If $K$ has positive dimension $n$ and $K_{(n-1)}$ is its $(n-1)$-skeleton, then $K/K_{(n-1)}$ is a wedge of $n$-spheres and so the quotient of $K \wedge X$ by the subspectrum $K_{(n-1)} \wedge X$ is a wedge of $n$-fold suspension of $X$. By induction the subspectrum $K_{(n-1)} \wedge X$ is semistable; since the inclusion is an h-cofibration and the quotient spectrum is also semistable, so is $K \wedge X$. For a general CW-complex $K$ the symmetric spectrum $K \wedge X$ is the sequential colimit, over h-cofibrations, of the smash product of $X$ with the skeleta of $K$. So $K \wedge X$ is semistable.

The geometric realization of any simplicial set is a CW-complex, so in the simplicial context we conclude that for any pointed simplicial set $K$ and any semistable symmetric spectrum $X$ the symmetric spectrum $K \wedge X$ is again semistable.

EXAMPLE 4.53. Let $F : J \longrightarrow \mathcal{S}p^\Sigma$ be a functor from a small category $J$ to the category of symmetric spectra. If $F(j)$ is semistable for each object $j$ of $J$, then the homotopy colimit of $F$ over $J$ is semistable.

Indeed, the homotopy colimit is the geometric realization of the *simplicial replacement* $\amalg_* F$ in the sense of Bousfield and Kan [**10**, Ch. XII, 5.1], a simplicial object of symmetric spectra. The spectrum of $n$-simplices of $\amalg_* F$ is a wedge, indexed over the $n$-simplices of the nerve of $J$, of spectra which occur as values of $F$. The geometric realization $|\amalg_* F|$ is the sequential colimit, over h-cofibrations, of the realizations of the skeleta $\mathrm{sk}_n \amalg_* F$ in the simplicial direction, so it suffices to show that each of these is semistable. The skeleton inclusion realizes to an h-cofibration $|\,\mathrm{sk}_{n-1} \amalg_* F| \longrightarrow |\,\mathrm{sk}_n \amalg_* F|$ whose quotient symmetric spectrum is a wedge, indexed over the *non-degenerate* $n$-simplices of the nerve of $J$, of $n$-fold suspensions of spectra which occur as values of $F$. So the quotient spectra are semistable, and so by induction the symmetric spectra $|\,\mathrm{sk}_n \amalg_* F|$ are semistable.

**4.6. Homotopy groups of ring spectra.** In Example 2.4 we constructed a product on the stable stems and an action of this graded ring on the homotopy groups of a symmetric spectrum. We now extend this type of multiplication to symmetric ring spectra; however, the situation is more subtle, and in general only those stable homotopy classes which are fixed by the $\mathcal{M}$-action support a multiplication. More precisely, we show:

THEOREM 4.54. *Let $R$ be a symmetric ring spectrum. There is a natural structure of a graded ring on the graded subgroup $(\pi_*R)^{(0)}$ of $\mathcal{M}$-fixed elements in the homotopy groups of $R$. If $R$ is commutative, then* *the product on $(\pi_*R)^{(0)}$ is graded-commutative. The homotopy groups of every right $R$-module naturally form a graded right module over the graded ring $(\pi_*R)^{(0)}$.*

*Thus if $R$ is semistable then the homotopy groups $\pi_*R$ form a graded ring which acts naturally on the homotopy groups of every right $R$-module.*

To prove Theorem 4.54 we consider a right $R$-module $M$ and try to define a biadditive pairing from $\pi_kM \times \pi_lR$ to $\pi_{k+l}M$ in the same way as for $R = \mathbb{S}$ in Example 2.4 (an $\mathbb{S}$-module is the same as a symmetric spectrum). As we shall see, the case of general ring spectra is more subtle, and the condition of being fixed by $\mathcal{M}$ comes up naturally when checking that an unstable product is well-defined on stable homotopy classes. Given two homotopy classes $f \in \pi_{k+n}M_n$ and $g \in \pi_{l+m}R_m$ we denote by $f \cdot g$ the homotopy class in $\pi_{k+l+n+m}M_{n+m}$ given by the composite

$$S^{k+l+n+m} \xrightarrow{\mathrm{Id}\wedge\chi_{l,n}\wedge\mathrm{Id}} S^{k+n+l+m} \xrightarrow{f\wedge g} M_n \wedge R_m \xrightarrow{\mu_{n,m}} M_{n+m} \ . \tag{4.55}$$

This dot operation is associative, i.e., if $h \in \pi_{j+q}R_q$ is another homotopy class, then we have $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ in $\pi_{k+l+j+n+m+q}R_{n+m+q}$. After spelling out the definitions, this associativity ultimately boils down to the equality

$$(k \times \chi_{l,n} \times m \times j \times q) \circ (k \times l \times \chi_{j,n+m} \times q) \ = \ (k \times n \times l \times \chi_{j,m} \times q) \circ (k \times \chi_{l+j,n} \times m \times q)$$

in the symmetric group $\Sigma_{k+l+j+q+n+m}$ and the associativity of the action of $R$ on $M$.

Using the dot product we can rewrite the stabilization map $\iota_* : \pi_{k+n}M_n \longrightarrow \pi_{k+n+1}M_{n+1}$ as $\iota_*(f) = f \cdot \iota_1$, where $\iota_1 \in \pi_1R_1$ is the class of the unit map of $R$. In the special case $M = R$ we can also multiply with the unit map $\iota_1$ from the left, and then $\iota_1 \cdot g$ is the composite of the top row in the diagram

$$\begin{tikzcd}
S^{l+1+m} \arrow[r, "\chi_{l,1}\times m"] \arrow[dr, "l\times\chi_{1,m}"'] & S^{1+l+m} \arrow[r, "S^1\wedge g"] \arrow[d, "\chi_{1,l+m}"] & S^1 \wedge R_m \arrow[r, "\iota_1\wedge R_m"] \arrow[d, "\mathrm{twist}"] & R_1 \wedge R_m \arrow[r, "\mu_{1,m}"] & R_{1+m} \\
 & S^{l+m+1} \arrow[r, "g\wedge S^1"'] & R_m \wedge S^1 \arrow[r, "R_m\wedge\iota_1"'] & R_m \wedge R_1 \arrow[r, "\mu_{m,1}"'] & R_{m+1} \arrow[u, "\chi_{m,1}"']
\end{tikzcd}$$

The diagram commutes, using centrality of $\iota_1$, so that we have the relation

$$\iota_1 \cdot g \ = \ (-1)^m(\chi_{m,1})_*(g \cdot \iota_1)$$

in $\pi_{l+1+m}R_{1+m}$. The image of the right hand side in the stable group $\pi_lR$ is precisely $d \cdot [g]$, where $d\cdot$ is the action of the special monoid element $d \in \mathcal{M}$, see Example 4.14. So we deduce the relation $[\iota_1 \cdot g] = d \cdot [g]$ in $\pi_lR$.

Now it is easy to investigate to what extent the dot product passes to an operation stable homotopy groups. If we replace the right factor $g \in \pi_{l+m}R_m$ by the next representative $\iota_*(g)$ then associativity gives

$$f \cdot \iota_*(g) \ = \ f \cdot (g \cdot \iota_1) \ = \ (f \cdot g) \cdot \iota_1 \ = \ \iota_*(f \cdot g) \ ,$$

so the class $[f \cdot g]$ in the stable group $\pi_{k+l}M$ only depends on the class $[g]$ in $\pi_lR$, and not on the unstable representative.

However, the dot product interacts differently with stabilization in the left factor, for we have

$$\iota_*(f) \cdot g \ = \ (f \cdot \iota_1) \cdot g \ = \ f \cdot (\iota_1 \cdot g) \tag{4.56}$$

which in general will not have the same image in $\pi_{k+l}M$ as $f \cdot g$. So in general we cannot expect that the dot product passes to a well-defined operation on $\pi_kM \times \pi_lR$.

Let us now make the additional assumption that the stable class $[g] \in \pi_lR$ is fixed by the special monoid element $d \in \mathcal{M}$. Because of the relation $[\iota_1 \cdot g] = d \cdot [g] = [g]$ in $\pi_lR$ and equation (4.56) we then get that $\iota_*(f) \cdot g$ and $f \cdot g$ do have the same image in $\pi_{k+l}M$. Note that if $d[g] = [g]$, then $[g]$ in fact has filtration zero (by Lemma 4.22 (iii)) which means that it is fixed by all monoid elements. So by restricting to classes in $\pi_lR$ which are $\mathcal{M}$-fixed we get a well-defined paring

$$\cdot \ : \ \pi_kM \ \times \ (\pi_lR)^{(0)} \ \longrightarrow \ \pi_{k+l}M \ .$$

In particular, if $R$ is semistable, then we end up with a well-defined map $\cdot : \pi_k M \times \pi_l R \longrightarrow \pi_{k+l} M$.

The proof that the dot operation is biadditive is the same as in the special case $R = \mathbb{S}$, compare Example 2.4. We claim that the product if $\mathcal{M}$-linear in the left variable, i.e., we have $(\alpha \cdot x) \cdot y = \alpha \cdot (x \cdot y)$ for $\alpha \in \mathcal{M}$, $x \in \pi_k M$ and $y \in (\pi_l R)^{(0)}$.[justify]

If we specialize to $M = R$ then the product provides a biadditive graded pairing $\cdot : \pi_k R \times (\pi_l R)^{(0)} \longrightarrow \pi_{k+l} R$ on the homotopy groups of the symmetric ring spectrum $R$. By $\mathcal{M}$-linearity in the left variable, this restricts to a biadditive pairing $\cdot : (\pi_k R)^{(0)} \times (\pi_l R)^{(0)} \longrightarrow (\pi_{k+l} R)^{(0)}$. Associativity for the unstable dot product immediately implies associativity for its stabilized version, i.e., for every right $R$-module $M$ the diagram

$$\begin{tikzcd}
\pi_k M \times (\pi_l R)^{(0)} \times (\pi_j R)^{(0)} \arrow[r, "\cdot \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \cdot"'] & \pi_{k+l} M \times (\pi_j R)^{(0)} \arrow[d, "\cdot"] \\
\pi_k M \times (\pi_{l+j} R)^{(0)} \arrow[r, "\cdot"'] & \pi_{k+l+j} M
\end{tikzcd}$$

commutes. This finishes the proof of Theorem 4.54, except for the verification that the product on $(\pi_* R)^{(0)}$ is graded-commutative if $R$ is commutative. This will come out of the study of the product on the opposite ring spectrum, which we discuss now.

EXAMPLE 4.57 (Opposite ring spectrum). For every symmetric ring spectrum $R$ we can define the *opposite* ring spectrum $R^{op}$ by keeping the same spaces (or simplicial sets), symmetric group actions and unit maps, but with new multiplication $\mu^{op}_{n,m}$ on $R^{op}$ given by the composite

$$R^{op}_n \wedge R^{op}_m = R_n \wedge R_m \xrightarrow{\text{twist}} R_m \wedge R_n \xrightarrow{\mu_{m,n}} R_{m+n} \xrightarrow{\chi_{m,n}} R_{n+m} = R^{op}_{n+m} .$$

As a consequence of centrality of $\iota_1$, the higher unit maps for $R^{op}$ agree with the higher unit maps for $R$. By definition, a symmetric ring spectrum $R$ is commutative if and only if $R^{op} = R$.

For example, we have $(HA)^{op} = H(A^{op})$ for the Eilenberg-Mac Lane ring spectra (Example 2.7) of an ordinary ring $A$ and its opposite, we have $(T \wedge R)^{op} = T^{op} \wedge R^{op}$ for the smash product of an $I$-space with multiplication and a ring spectrum (Example 2.31), and $R[M]^{op} = (R^{op})[M^{op}]$ for the monoid ring spectra (Example 2.32) of a simplicial or topological monoid $M$ and its opposite.

We claim that for every ring spectrum $R$ and integer $k$ the homotopy groups $\pi_k R$ and $\pi_k(R^{op})$ are equal (not just isomorphic) as $\mathcal{M}$-modules and

$$(\pi_*(R^{op}))^{(0)} \;=\; ((\pi_* R)^{(0)})^{op} \tag{4.58}$$

(again equality) as graded rings, where the right hand side is the graded-opposite ring, i.e., the graded abelian group $\pi_* R$ with new product $x \cdot_{op} y = (-1)^{kl} y \cdot x$ for $x \in \pi_k R$ and $y \in \pi_l R$. In particular, if $R$ is semistable, then we have $\pi_*(R^{op}) = (\pi_* R)^{op}$ as graded rings.

Let us first check that $\pi_k R$ and $\pi_k(R^{op})$ are the same $\mathcal{M}$-modules. This follows from the stronger claim that the $I$-functors $\underline{\pi}_k R$ and $\underline{\pi}_k(R^{op})$ are equal. It is immediate from the definition of $R^{op}$ that these two $I$-functors agree on objects and bijective morphisms in the category $I$. There is something to check though to see that $\underline{\pi}_k R$ and $\underline{\pi}_k(R^{op})$ agree on the inclusions $\mathbf{n} \longrightarrow \mathbf{n+1}$ (and thus on all injections), i.e., that they have the same stabilization maps. The stabilization map $\iota^{op}_*$ for the opposite ring spectrum takes the homotopy class of a map $f : S^{k+n} \longrightarrow R_n$ to the class of the composite

$$\begin{tikzcd}
S^{k+n+1} \arrow[r, "f \wedge \mathrm{Id}"] & R_n \wedge S^1 \arrow[r, "\mathrm{Id} \wedge \iota_1"] & R_n \wedge R_1 \arrow[r, "\text{twist}"] \arrow[rrr, bend right, "\mu^{op}_{n,1}"'] & R_1 \wedge R_n \arrow[r, "\mu_{1,n}"] & R_{1+n} \arrow[r, "\chi_{1,n}"] & R_{n+1}
\end{tikzcd} \tag{4.59}$$

Because the unit map $\iota_1$ is central, the composite of the last four maps in (4.59) is actually equal to $\mu_{n,1}(\mathrm{Id} \wedge \iota_1)$. Thus the total composite in (4.59) equals the composite

$$S^{k+n+1} \xrightarrow{f \wedge \mathrm{Id}} R_n \wedge S^1 \xrightarrow{\mathrm{Id} \wedge \iota_1} R_n \wedge R_1 \xrightarrow{\mu_{n,1}} R_{n+1} ,$$

which is precisely the effect of the stabilization map for $R$. So the $I$-functors $\underline{\pi}_k R$ and $\underline{\pi}_k(R^{op})$ are equal and hence their colimits $\pi_k R$ and $\pi_k(R^{op})$ are equal (or at least as equal as a colimit is canonical) as $\mathcal{M}$-modules.

Now consider two homotopy classes $f \in \pi_{k+n} R_n$ and $g \in \pi_{l+m} R_m$ which represents $\mathcal{M}$-fixed classes in $\pi_k R$ respectively $\pi_l R$. Spelling out the definitions leads to the equation

$$f \cdot_{op} g \ = \ (\chi_{m,n})_* \circ (g \cdot_R f) \circ (\chi_{k,l} \times \chi_{n,m}) \ = \ (-1)^{kl+nm} (\chi_{m,n})_*(g \cdot f)$$

as elements of $\pi_{k+l+n+m} R_{n+m}$. We already know that the element represented by this class in $\pi_{k+l} R$ is $\mathcal{M}$-fixed, and hence

$$[(-1)^{nm}(\chi_{m,n})_*(g \cdot f)] \ = \ d \cdot [g \cdot f] \ = \ [g \cdot f]$$

in $\pi_{k+l} R$. Thus $[f \cdot_{op} g] = (-1)^{kl}[g \cdot f]$, which proves the equation (4.58).

As a special case of (4.58) we obtain the claim about commutative ring spectrum made in Theorem 4.54. If $R$ is commutative, then $R = R^{op}$ and so then $(\pi_* R)^{(0)}$ agrees with its graded opposite ring, which means that the multiplication on $(\pi_* R)^{(0)}$ is commutative in the graded sense.

The product on the homotopy groups of a symmetric ring spectrum can also be explained from a different angle using the smash product. For symmetric spectra $X$ and $Y$ smashing of representatives and composing with the universal bimorphism $i_{n,m} : X_n \wedge Y_m \longrightarrow (X \wedge Y)_{n+m}$ passes to an $\mathcal{M}$-linear biadditive pairing

$$\cdot \ : \ \pi_k X \times (\pi_l Y)^{(0)} \ \longrightarrow \ \pi_{k+l}(X \wedge Y) \ .$$

The verification that this well-defined and the reason for restricting to $\mathcal{M}$-fixed classes in the second factor are the same as above. It is straightforward to check that for a symmetric ring spectrum $R$ and a right $R$-module $M$ the multiplication as defined in Theorem 4.54 agrees with the composite

$$\pi_k M \times (\pi_l R)^{(0)} \ \xrightarrow{\ \cdot\ } \ \pi_{k+l}(M \wedge R) \ \xrightarrow{\ \alpha\ } \ \pi_{k+l} R \ .$$

where $\alpha : M \wedge R \longrightarrow M$ is the 'external' form of the action of $R$ on $M$.

Again we have associativity, i.e., for every triple of symmetric spectra $X, Y$ and $Z$ the diagram

$$\begin{tikzcd}
\pi_k X \times (\pi_l Y)^{(0)} \times (\pi_j Z)^{(0)} \arrow[r, "\cdot \times \mathrm{Id}"] \arrow[d, "\mathrm{Id} \times \cdot"'] & \pi_{k+l}(X \wedge Y) \times (\pi_j Z)^{(0)} \arrow[d, "\cdot"] \\
\pi_k X \times (\pi_{l+j}(Y \wedge Z))^{(0)} \arrow[r, "\cdot"'] & \pi_{k+l+j}(X \wedge Y \wedge Z)
\end{tikzcd}$$

commutes. The graded commutativity of the multiplication in the homotopy of a commutative ring spectrum also has a precursor: for all symmetric spectra $X$ and $Y$ the square

$$\begin{tikzcd}
(\pi_k X)^{(0)} \times (\pi_l Y)^{(0)} \arrow[r, "\cdot"] \arrow[d, "\mathrm{twist}"'] & \pi_{k+l}(X \wedge Y)^{(0)} \arrow[d, "\pi_{k+l}(\tau_{X,Y})"] \\
(\pi_l Y)^{(0)} \times (\pi_k X)^{(0)} \arrow[r, "\cdot"'] & \pi_{l+k}(Y \wedge X)^{(0)}
\end{tikzcd}$$

commutes up to the sign $(-1)^{kl}$.

Example 4.60 (Killing a homotopy class). We describe a construction that can be used to 'kill' the action of a homotopy class in a ring spectrum on a given module. We consider a symmetric ring spectrum $R$ and a map $x : S^{l+m} \longrightarrow R_m$. For every right $R$-module $M$ we define a morphism of symmetric spectra

$$\rho_x \ : \ S^{l+m} \wedge M \ \longrightarrow \ \mathrm{sh}^m M$$

in level $n$ as the composite

$$S^{l+m} \wedge M_n \xrightarrow{\mathrm{twist}} M_n \wedge S^{l+m} \xrightarrow{\mathrm{Id} \wedge x} M_n \wedge R_m \xrightarrow{\alpha_{n,m}} M_{n+m} \xrightarrow{\chi_{n,m}} M_{m+n} = (\mathrm{sh}^m M)_n \ .$$

The name $\rho_x$ stands for 'right multiplication by $x$'. For example, if $x = \iota_1 : S^1 \longrightarrow R_1$ is the unit map then $\rho_{\iota_1} : S^1 \wedge M \longrightarrow \mathrm{sh}\, M$ equals the map $\lambda_M$ defined in (2.19). [relate the composite of $\rho_y$ and $\rho_x$ to $\rho_{x \cdot y}$]

While the construction of the morphism $\rho_x : S^{l+m} \wedge M \longrightarrow \mathrm{sh}^m M$ makes sense for any map $x$ and $R$-module $M$, we can only analyze it homotopically under additional assumptions.

PROPOSITION 4.61. *Let $R$ be a symmetric ring spectrum, $x : S^{l+m} \longrightarrow R_m$ a based map and $M$ a right $R$-module.*

(i) *Suppose that the map $x$ is central, i.e., the square*

$$\begin{tikzcd} R_n \wedge S^{l+m} \arrow[r, "\mathrm{Id} \wedge x"] \arrow[d, "\tau"'] & R_n \wedge R_m \arrow[r, "\mu_{n,m}"] & R_{n+m} \arrow[d, "\chi_{n,m}"] \\ S^{l+m} \wedge R_n \arrow[r, "x \wedge \mathrm{Id}"'] & R_m \wedge R_n \arrow[r, "\mu_{m,n}"'] & R_{m+n} \end{tikzcd}$$

*commutes for all $n \geq 0$. Then the morphism of symmetric spectra $\rho_x : S^{l+m} \wedge M \longrightarrow \mathrm{sh}^m M$ is a homomorphism of $R$-modules.*

(ii) *If the class $[x] \in \pi_l R$ in the $l$-th stable homotopy group is $\mathcal{M}$-fixed and $M$ is semistable, then the morphism $\rho_x : S^{l+m} \wedge M \longrightarrow \mathrm{sh}^m M$ realizes right multiplication by $[x]$ in homotopy. More precisely, the diagram*

$$\begin{tikzcd} \pi_k M \arrow[r, "\cdot [x]"] \arrow[d, "S^{l+m} \wedge -"', "\cong"] & \pi_{k+l} M \arrow[d, equal] \\ \pi_{l+m+k}(S^{l+m} \wedge M) \arrow[r, "\pi_{l+m+k}(\rho_x)"'] & \pi_{l+m+k}(\mathrm{sh}^m M) \end{tikzcd}$$

*commutes up to the sign $(-1)^{k(l+m)}$.*

As usual, the sign in part (ii) above can be predicted by remembering that the group $\pi_{k+l} M$ is 'naturally' equal to $\pi_{k+l+m}(\mathrm{sh}^m M)$ (compare Example 2.18), whereas the 'natural' target of the lower vertical map is $\pi_{l+m+k}(\mathrm{sh}^m M)$; so secretly, $k$ sphere coordinates move past $l+m$ other coordinates, hence the sign.

PROOF OF PROPOSITION 4.61. Part (i) is straightforward from the definitions. For (ii) we consider an element $f \in \pi_{k+n} M_n$ which represents a class in $\pi_k M$ and chase it around both sides of the square. We then have

$$(\rho_x)_*(S^{l+m} \wedge f) \; = \; (\chi_{n,m})_*(f \cdot x) \circ \gamma_* \tag{4.62}$$

in the group $\pi_{(l+m)+(k+n)} M_{m+n}$, where $\gamma_* : S^{(l+m)+(k+n)} \longrightarrow S^{(k+l)+(n+m)}$ is the coordinate permutation induced by the element $\gamma \in \Sigma_{l+m+k+n}$ given by

$$\gamma(i) \; = \; \begin{cases} i+k & \text{for } 1 \leq i \leq l, \\ i+k+n & \text{for } l+1 \leq i \leq l+m, \\ i-l-m & \text{for } l+m+1 \leq i \leq l+m+k, \\ i-m & \text{for } l+m+k+1 \leq i. \end{cases}$$

The permutation $\gamma$ has sign $(-1)^{k(l+m)+nm}$ and since $M$ is semistable the effect of the shuffle $\chi_{n,m}$ in the stable homotopy group is the sign $(-1)^{nm}$. So when we pass from the unstable homotopy group $\pi_{(l+m)+(k+n)} M_{m+n}$ to the stable group $\pi_{l+m+k}(\mathrm{sh}^m M)$, the equation (4.62) becomes $\pi_{l+m+k}(\rho_x)(S^{l+m} \wedge [f]) = (-1)^{k(l+m)}[f] \cdot [x]$, as claimed. $\square$

Now suppose that the hypothesis of both (i) and (ii) hold in Proposition 4.61, i.e., the map $x$ is central, its class $[x]$ in $\pi_l R$ is $\mathcal{M}$-fixed and $M$ is semistable. We let $M/x$ denote the mapping cone of the morphism $\tilde{\rho}_x : S^l \wedge M \longrightarrow \Omega^m(\mathrm{sh}^m M)$ which is adjoint to $\rho_x$. Then by Proposition 4.61 the morphism $\tilde{\rho}_x$ also realizes multiplication by $[x]$ on homotopy [levelwise Kan] in the sense that the square

$$\begin{tikzcd} \pi_k M \arrow[rr, "\cdot[x]"] \arrow[d, "S^l\wedge -"', "\cong"] & & \pi_{k+l} M \arrow[d, "\cong"', "\lambda^*"] \\ \pi_{l+k}(S^l \wedge M) \arrow[rr, "\pi_{l+k}(\tilde\rho_x)"'] & & \pi_{l+k}(\Omega^m(\mathrm{sh}^m M)) \end{tikzcd}$$

commutes up to the sign $(-1)^{k(l+m)}$. The long exact homotopy sequence of a mapping cone (Proposition 4.7 (i)) breaks up into a short exact sequence of $(\pi_* R)^{(0)}$-modules [the connecting morphism is $(\pi_* R)^{(0)}$-linear]

$$0 \longrightarrow \pi_* M/(\pi_{*-l} M)\cdot[x] \longrightarrow \pi_*(M/x) \longrightarrow \{\pi_{*+l-1} M\}_{[x]} \longrightarrow 0$$

where the first map is induced by the composite of $\lambda^* : M \longrightarrow \Omega^m(\mathrm{sh}^m M)$ and the mapping cone inclusion $\Omega^m(\mathrm{sh}^m M) \longrightarrow M/x$ and $\{-\}_{[x]}$ denotes the submodule of homotopy classes annihilated by $[x]$. So we conclude that if $x$ acts injectively on the homotopy groups of $M$, then the morphism $M \longrightarrow M/x$ realizes the quotient map $\pi_* M \longrightarrow \pi_* M/(\pi_{*-l} M)\cdot[x]$ on homotopy groups.

In contrast, if the class $[x]$ annihilates nonzero classes in the homotopy of $M$ then we may not be able realize the module $\pi_* M/(\pi_{*-l} M)\cdot[x]$ as the homotopy of an $R$-module. In fact, Toda brackets give the first obstructions to such a realization. If there are homogeneous elements $m \in \pi_* M$ and $x, y \in (\pi_* R)^{(0)}$ such that $mx = 0 = xy$ and the Toda bracket $\langle m, x, y\rangle$ does *not* contain 0, then the projection $\pi_* M \longrightarrow \pi_* M/(\pi_{*-l} M)\cdot[x]$ is *not* realizable as the effect on homotopy of any $R$-module homomorphism out of $M$. [explain]

Example 4.63 (Killing a regular sequence). We can iteratively kill homotopy classes as in the previous example and thereby kill the action of certain ideals in the homotopy groups of a symmetric ring spectrum. We just saw that we can only control the homotopy groups of $M/x$ if the homotopy class $[x]$ which is killed is not a zero divisor on $\pi_* M$. So iterating the construction naturally leads us to consider regular sequences.

Recall that a sequence, finite or countably infinite, of homogeneous elements $y_i$ in a graded commutative ring $R_*$ is a *regular sequence* for a graded $R_*$-module $M_*$ if $y_1$ acts injectively on $M_*$ and for all $i \geq 2$ the element $y_i$ acts injectively on $M_*/M_*\cdot(y_1, \dots, y_{i-1})$. A homogeneous ideal $I$ of $R_*$ is a *regular ideal* for $M_*$ if it can be generated by a regular sequence, finite or countably infinite, for $M_*$.

To simplify the exposition we now assume that the ring spectrum $R$ we work over is semistable and commutative. As in Example 4.60 this can be relaxed to the assumption that all relevant homotopy classes are $\mathcal{M}$-fixed and can be represented by central maps.

Proposition 4.64. *Let $R$ be a commutative semistable symmetric ring spectrum, $M$ a semistable right $R$-module and $I$ a homogeneous ideal of $\pi_* R$. If $I$ is a regular ideal for the module $\pi_* M$ then there exists a semistable $R$-module $M/I$ and a homomorphism $q : M \longrightarrow M/I$ of $R$-modules such that the induced homomorphism of homotopy group*

$$\pi_*(q) \ : \ \pi_* M \longrightarrow \pi_*(M/I)$$

*is surjective and has kernel equal to $(\pi_* M)I$.*

Proof. We choose a sequence $y_1, y_2, \dots$ of homogeneous elements of $\pi_* R$ which generate the ideal $I$ and form a regular sequence for $\pi_* M$. We construct inductively a sequence of semistable $R$-modules $M^i$ and homomorphisms

$$M = M^0 \xrightarrow{q_1} M^1 \xrightarrow{q_2} M^2 \xrightarrow{q_3} \cdots$$

such that the composite morphism $M \longrightarrow M^i$ is surjective on homotopy groups and has kernel equal to $(\pi_* M)\cdot(y_1, \dots, y_i)$.

The induction starts with $i = 0$, where there is nothing to show. In the $i$th step we let $l$ be the dimension of the homotopy class $y_i$ and choose a based map $x : S^{l+m} \longrightarrow R_m$ which represents $y_i \in \pi_l R$. By induction the homotopy groups of $M^{i-1}$ realize the $\pi_* R$-module $\pi_* M/(\pi_* M)\cdot(y_1, \dots, y_{i-1})$. Since we have a regular sequence for $\pi_*M$ the class $y_i = [x]$ acts injectively on the homotopy of $M^{i-1}$, so the morphism $q_i : M^{i-1} \longrightarrow M^{i-1}/x$ constructed in Example 4.60 realizes the projection $\pi_*M^{i-1} \longrightarrow \pi_*M^{i-1}/(\pi_*M^{i-1}) \cdot y_i$. We can thus take $M^i = M^{i-1}/x$; then the composite morphism $M \longrightarrow M^i$ is again surjective on homotopy groups and its kernel is

$$(\pi_*M) \cdot (y_1, \dots, y_{i-1}) \ + \ (\pi_*M) \cdot y_i \ = \ (\pi_*M) \cdot (y_1, \dots, y_i) \ .$$

This finishes the argument if $I$ is generated by a *finite* regular sequence.

If the generating sequence is countably infinite we define $M/I$ as the diagonal (see Example 2.27) of the above sequence of $R$-modules $M^i$. Then the natural map

$$\mathrm{colim}_i \ \pi_*M_i \ \longrightarrow \ \pi_*(M/I)$$

is an isomorphism (see (2.30)), and the left hand side is isomorphic to

$$\mathrm{colim}_i \ (\pi_*M/(\pi_*M) \cdot (x_1, \dots, x_i)) \ \cong \ \pi_*M/(\pi_*M) \cdot (x_1, x_2, \dots) \ = \ \pi_*M/(\pi_*M) \cdot I \ .$$

□

EXAMPLE 4.65 (Inverting a homotopy class). In Example 2.36 we defined a new symmetric ring spectrum $R[1/x]$ from a given symmetric ring spectrum $R$ and a central map $x : S^1 \longrightarrow R_1$. We now generalize this construction to central maps $x : S^{l+m} \longrightarrow R_m$ and also analyze it homotopically.

First we extend the localization construction to right $R$-modules $M$. We define a right $R[1/x]$-module $M[1/x]$ by

$$M[1/x]_p \ = \ \mathrm{map}(S^{(l+m)p}, M_{(1+m)p}) \ .$$

The symmetric group $\Sigma_p$ acts on $S^{(l+m)p}$ and $M_{(1+m)p}$ by permuting the $p$ blocks of $l+m$ respectively $1+m$ coordinates, i.e., by restriction along the diagonal embeddings

$$\Delta : \Sigma_p \longrightarrow \Sigma_{(l+m)p} \quad \text{respectively} \quad \Delta : \Sigma_p \longrightarrow \Sigma_{(1+m)p} \ .$$

More precisely, the diagonal embedding $\Delta : \Sigma_p \longrightarrow \Sigma_{np}$ is defined by $\Delta(\gamma) = \mathrm{Id}_{\mathbf{n}} \cdot \gamma$ (using the notation of Remark 1.12) which unravels to

$$\Delta(\gamma)(j + (i-1)n) \ = \ j + (\gamma(i) - 1)$$

for $i \leq j \leq n$ and $1 \leq i \leq p$. The action of $\Sigma_p$ on the whole mapping space $M[1/x]_p$ is then by conjugation.

A special case of this is $M = R$ and we now define $\Sigma_p \times \Sigma_q$-equivariant action maps

$$\alpha_{p,q} \ : \ M[1/x]_p \wedge R[1/x]_q \ \longrightarrow \ M[1/x]_{p+q}$$

as the composite

$$\begin{array}{ccc} \mathrm{map}(S^{(l+m)p}, M_{(1+m)p}) \wedge \mathrm{map}(S^{(l+m)q}, R_{(1+m)q}) & \longrightarrow & \mathrm{map}(S^{(l+m)(p+q)}, M_{(1+m)(p+q)}) \\ f \wedge g & \longmapsto & \mu_{(1+m)p,(1+m)q} \circ (f \wedge g) \ . \end{array}$$

The action maps are associative because smashing and the original action maps are. The verification the maps $\alpha_{p,q}$ are $\Sigma_p \times \Sigma_q$-equivariant ultimately boils down to the equivariant for the original action maps and the fact that the diagonal embeddings $\Delta_p : \Sigma_p \longrightarrow \Sigma_{np}$, $\Delta_q : \Sigma_q \longrightarrow \Sigma_{nq}$ and $\Delta_{p+q} : \Sigma_{p+q} \longrightarrow \Sigma_{n(p+q)}$ satisfy

$$\Delta_p(\gamma) \times \Delta_q(\tau) \ = \ \Delta_{p+q}(\gamma \times \tau) \ .$$

The next piece of data we define are maps $j_p : M_p \longrightarrow M[1/x]_p$ which we will later recognize as a homomorphism of $R$-modules. We denote by $x^p : S^{(l+m)p} \longrightarrow R_{mp}$ the composite

$$S^{(l+m)p} \xrightarrow{x^{(p)}} R_m^{(p)} \xrightarrow{\mu_{m,\dots,m}} R_{mp} \ ;$$

because $x$ is central the map $x^p$ is $\Sigma_p$-equivariant if we let $\Sigma_p$ act on source and target through the diagonal embeddings. We define

$$j_p \ : \ M_p \ \longrightarrow \ \mathrm{map}(S^{(l+m)p}, M_{(1+m)p}) \ = \ M[1/x]_p$$

as the adjoint to

$$M_p \wedge S^{(l+m)p} \xrightarrow{\mathrm{Id} \wedge x^p} M_p \wedge R_{mp} \xrightarrow{\alpha_{p,mp}} M_{p+mp} \xrightarrow{\xi_*} M_{(1+m)p}$$

where $\xi \in \Sigma_{(1+m)p}$ is given by

$$\xi(i) \ = \ \begin{cases} 1+(i-1)(1+m) & \text{if } 1 \le i \le p \\ (j+1)+(k-1)(1+m) & \text{if } i = p+mk+j \text{ for } 1 \le k \le p,\ 1 \le j \le m. \end{cases}$$

The map $j_p$ is $\Sigma_p$-equivariant. In terms of the adjoint of $j_p$ this means that for every permutation $\gamma \in \Sigma_p$ the outer composites in the diagram

$$\begin{tikzcd} M_p \wedge S^{(l+m)p} \arrow[r, "\mathrm{Id} \wedge x^p"] \arrow[d, "\gamma\wedge\Delta(\gamma)"'] & M_p \wedge R_{mp} \arrow[r, "\alpha_{p,mp}"] \arrow[d, "\gamma\wedge\Delta(\gamma)"'] & M_{p+mp} \arrow[r, "\xi"] \arrow[d, "\gamma\times\Delta(\gamma)"] & M_{(1+m)p} \arrow[d, "\Delta(\gamma)"] \\ M_p \wedge S^{(l+m)p} \arrow[r, "\mathrm{Id} \wedge x^p"'] & M_p \wedge R_{mp} \arrow[r, "\alpha_{p,mp}"'] & M_{p+mp} \arrow[r, "\xi"'] & M_{(1+m)p} \end{tikzcd}$$

agree. The left square commutes by equivariance of $x^p$, the middle square by the equivariance of the actions maps of $M$. The right square commutes because the relation

$$\xi \circ (\gamma \times \Delta(\gamma)) \ = \ \Delta(\gamma) \circ \xi$$

holds in the symmetric group $\Sigma_{(1+m)p}$.

The collection of maps $j_p$ is multiplicative in the sense that the squares

$$\begin{tikzcd} M_p \wedge R_q \arrow[r, "\alpha_{p,q}"] \arrow[d, "j_p\wedge j_q"'] & M_{p+q} \arrow[d, "j_{p+q}"] \\ M[1/x]_p \wedge R[1/x]_q \arrow[r, "\alpha_{p,q}"'] & M[1/x]_{p+q} \end{tikzcd}$$

commute [elaborate].

For $M = R$ we define the unit maps $\iota_n : S^n \longrightarrow R[1/x]_n$ as the composite of the unit map $\iota_n : S^n \longrightarrow R_n$ for $R$ with $j_n : R_n \longrightarrow R[1/x]_n$. [central] Then $j : R \longrightarrow R[1/x]$ is a homomorphism of symmetric ring spectra and $j : M \longrightarrow M[1/x]$ is a morphism of $R$-modules if we view $M[1/x]$ as an $R$-module by restriction of scalars along $j : R \longrightarrow R[1/x]$. In other words, we have constructed a functor

$$\text{mod-}R \ \longrightarrow \ \text{mod-}R[1/x] \ , \quad M \ \longmapsto \ M[1/x]$$

and a natural transformation of $R$-modules $j : M \longrightarrow j^*(M[1/x])$.

Remark 4.66. The permutation $\xi$ can be predicted as follows by the general rules which we introduced in Remark 1.12. The natural 'coordinate free' target of the map $\alpha_{p,m,\dots,m}$ is indexed by the set

$$\mathbf{p} + \underbrace{\mathbf{m} + \cdots + \mathbf{m}}_{p} \ = \ \mathbf{1}\cdot\mathbf{p} + \mathbf{m}\cdot\mathbf{p} \ .$$

This set is equal to the set $(\mathbf{1}+\mathbf{m})\cdot\mathbf{p}$, but the way the parenthesis arise naturally reminds us that we should use the right distributivity isomorphism

$$\mathbf{p} + \mathbf{m}\cdot\mathbf{p} = \mathbf{1}\cdot\mathbf{p} + \mathbf{m}\cdot\mathbf{p} \ \xrightarrow{\cong} \ (\mathbf{1}+\mathbf{m})\cdot\mathbf{p}$$

to identify the two sets. The right distributivity isomorphism is defined using the multiplicative shuffles as $\chi^{\times}_{p,1+m} \circ (\chi^{\times}_{1,p} \times \chi^{\times}_{m,p})$, which is precisely the permutation $\xi$. Note that in contrast, the definition of the actions map $\alpha_{p,q}$ does not need any permutations. Indeed, the natural coordinate free target of $\alpha_{p,q}$ is indexed by the set $(\mathbf{1}+\mathbf{m})\cdot\mathbf{p} + (\mathbf{1}+\mathbf{m})\cdot\mathbf{q}$, which is equal to the set $(\mathbf{1}+\mathbf{m})\cdot(\mathbf{p}+\mathbf{q})$. Here, however, the parenthesis suggest using the left distributivity isomorphism, which is the identity permutation.

The relation $\xi \circ (\gamma \times \Delta(\gamma)) = \Delta(\gamma) \circ \xi$ which came up in the proof of the equivariance on the map $j_p$ just expresses the fact that the multiplicative shuffle $\chi^{\times}_{n,m} : \mathbf{n} \cdot \mathbf{m} \longrightarrow \mathbf{m} \cdot \mathbf{n}$ is a natural isomorphism

$$\begin{tikzcd}
\mathbf{1} \cdot \mathbf{p} + \mathbf{m} \cdot \mathbf{p} \arrow[r, "\chi^{\times}_{1,p} + \chi^{\times}_{m,p}"'] \arrow[rr, bend left, "\xi"] \arrow[d, "\gamma + \Delta(\gamma)"', "\mathrm{Id}_{\mathbf{1}} \cdot \gamma + \mathrm{Id}_{\mathbf{m}} \cdot \gamma"] & \mathbf{p} \cdot \mathbf{1} + \mathbf{p} \cdot \mathbf{m} = \mathbf{p} \cdot (\mathbf{1} + \mathbf{m}) \arrow[r, "\chi^{\times}_{p,1+m}"] \arrow[d, "\gamma \cdot \mathrm{Id}_{\mathbf{1}} + \gamma \cdot \mathrm{Id}_{\mathbf{m}}"', "\gamma \cdot \mathrm{Id}_{\mathbf{1+m}}"] & (\mathbf{1} + \mathbf{m}) \cdot \mathbf{p} \arrow[d, "\gamma \cdot \mathrm{Id}_{\mathbf{1}}"', "\Delta(\gamma)"] \\
\mathbf{1} \cdot \mathbf{p} + \mathbf{m} \cdot \mathbf{p} \arrow[r, "\chi^{\times}_{1,p} + \chi^{\times}_{m,p}"'] \arrow[rr, bend right, "\xi"'] & \mathbf{p} \cdot \mathbf{1} + \mathbf{p} \cdot \mathbf{m} = \mathbf{p} \cdot (\mathbf{1} + \mathbf{m}) \arrow[r, "\chi^{\times}_{p,1+m}"'] & (\mathbf{1} + \mathbf{m}) \cdot \mathbf{p}
\end{tikzcd}$$

Now we analyze the construction homotopically, under the additional assumption that the class of $x$ in $\pi_l R$ is $\mathcal{M}$-fixed. The centrality condition on the map $x$ implies that then the stable homotopy class $[x]$ is central in the graded ring $(\pi_* R)^{(0)}$. [when is $R[1/x]$ equivalent to just changing the unit map ?]

Proposition 4.67. *Let $R$ be a symmetric ring spectrum and $x : S^{l+m} \longrightarrow R_m$ a central map whose class $[x]$ in $\pi_l R$ is $\mathcal{M}$-fixed. Then for every semistable right $R$-module $M$* [levelwise Kan] *the $R[1/x]$-module $M[1/x]$ is again semistable* [and levelwise Kan] *and the morphism $j : M \longrightarrow M[1/x]$ of $R$-modules induces a natural isomorphism*

$$(\pi_* M)[1/[x]] \ \cong \ \pi_*\big(M[1/x]\big) \tag{4.68}$$

*of graded modules over $(\pi_* R)^{(0)}$. In the special case $M = R$ the morphism of symmetric ring spectra $j : R \longrightarrow R[1/x]$ induces an isomorphism of graded rings*

$$(\pi_* R)^{(0)}[1/[x]] \ \cong \ \pi_*\big(R[1/x]\big)^{(0)} \ .$$

Proof. As an $R$-module $M[1/x]$ is equal to the diagonal [No: not correct $\Sigma_n$- action] (see Example 2.27) of the sequence of $R$-module homomorphisms

$$M \xrightarrow{\ \tilde\rho_x\ } \Omega^{l+m}(\mathrm{sh}^m M) \xrightarrow{\Omega^{l+m}\,\mathrm{sh}^m(\tilde\rho_x)} \Omega^{(l+m)2}(\mathrm{sh}^{m2} M) \ \cdots$$
$$\cdots \ \ \Omega^{(l+m)p}(\mathrm{sh}^{mp} M) \xrightarrow{\Omega^{(l+m)p}\,\mathrm{sh}^{mp}(\tilde\rho_x)} \ \cdots$$

where $\tilde\rho_x$ is adjoint to the 'right multiplication' homomorphism defined in Example 4.60. The map $j : M \longrightarrow M[1/x]$ is the canonical morphism from the initial term of a sequence to the diagonal spectrum.

By Proposition 4.61 (ii) the effect of $\tilde\rho_x$ on homotopy groups is right multiplication by the class $[x] \in \pi_l R$. So since the homotopy groups of the diagonal are isomorphic to the colimit of the homotopy groups (see (2.30)), we deduce that $j$ induces the isomorphism

$$\pi_*(M[1/x]) \ = \ \pi_*\, \mathrm{diag}_p \left(\Omega^{(l+m)p}(\mathrm{sh}^{mp} M)\right) \ \cong \ \mathrm{colim}_p\, \pi_* \left(\Omega^{(l+m)p}(\mathrm{sh}^{mp} M)\right) \ \cong \ \mathrm{colim}_p\, \pi_{*+lp} M$$

with the last colimit being taken over iterated multiplication by $[x]$. Since the right hand side is the result of inverting $[x]$ in $\pi_* M$, this proves the first claim.

In the special case $M = R$ we now know that the morphism of symmetric ring spectra $j : R \longrightarrow R[1/x]$ induces an isomorphism of graded $(\pi_* R)^{(0)}$-modules

$$(\pi_* R)^{(0)}[1/[x]] \ \cong \ \pi_*\big(R[1/x]\big)^{(0)} \ .$$

This is necessarily a multiplicative isomorphism. □

An important special case of the above is when the symmetric ring spectrum $R$ is commutative (which makes centrality of the map $x$ is automatic) and semistable (so that all of $\pi_* R$ is $\mathcal{M}$-fixed). For easier reference we spell out Proposition 4.67 in this special case.

Corollary 4.69. *Let $R$ be a commutative semistable symmetric ring spectrum [levelwise Kan] and $x : S^{l+m} \longrightarrow R_m$ a based map. Then $R[1/x]$ is a commutative symmetric ring spectrum, the homomorphism of symmetric ring spectra $j : R \longrightarrow R[1/x]$ sends the class $[x] \in \pi_l R$ to a unit in the $l$-th homotopy group of $R[1/x]$ and the induced morphism of graded commutative rings*

$$(\pi_* R)[1/[x]] \;\cong\; \pi_*\big(R[1/x]\big)$$

*is an isomorphism.*

Example 4.70 (Brown-Peterson, Johnson-Wilson spectra and Morava $K$-theory). If we apply the method of 'killing a regular sequence' to the Thom spectrum $MU$ we can construct a whole collection of important spectra. In Example 2.9 we constructed $MU$ as a commutative symmetric ring spectrum, and $MU$ is semistable because it underlies an orthogonal spectrum (compare Proposition 4.47). As input for the following construction we need the knowledge of the homotopy ring of $MU$. The standard way to perform this calculation is in the following sequence of steps:

- calculate, for each prime $p$, the mod-$p$ cohomology of the spaces $BU(n)$ and $BU$,
- use the Thom isomorphism to calculate the mod-$p$ cohomology of the Thom spectrum $MU$ as a module over the mod-$p$ Steenrod algebra,
- use the Adams spectral sequence, which for $MU$ collapses at the $E_2$-term, to calculate the $p$-completion of the homotopy groups of $MU$,
- and finally assemble the $p$-local calculations into the integral answer.

When the dust settles, the result is that $\pi_* MU$ is a polynomial algebra generated by infinitely many homogeneous elements $x_i$ of dimension $2i$ for $i \geq 1$. The details of this calculation can be found in [**61**] and [**50**] [check this; other sources?]. A very different geometric approach to this calculation was described by Quillen [**47**], who determines the ring of cobordism classes of stably almost complex manifolds, which by Thom's theorem is isomorphic to $\pi_* MU$. (Quillen's argument, however, needs as an input the a priori knowledge that the homotopy groups of $MU$ are finitely generated in each dimension.)

Now fix a prime number $p$. Using the close connection between the ring spectrum $MU$ and the theory of formal groups laws one can make a particular choices for the $(p^n - 1)$-th generator $x_{p^n-1}$, the so-called *Hazewinkel generator*, which is then denoted $v_n$. Killing all polynomial generators *except* those of the form $x_{p^n-1}$ produces a semistable $MU$-module $\overline{BP}$ with homotopy groups $\pi_*(\overline{BP}) = \mathbb{Z}[v_1, v_2, v_3, \dots]$ where the degree of $v_n$ is $2p^n - 2$. Localizing at $p$ produces a semistable $MU$-module $BP$, called the *Brown-Peterson spectrum*, with homotopy groups

$$\pi_* BP \;\cong\; \mathbb{Z}_{(p)}[v_1, v_2, v_3, \dots]\ .$$

The original construction of this spectrum by Brown and Peterson was quite different, and we say more about the history of $BP$ in the 'History and credits' section at the end of this chapter.

Now we can keep going and kill more of the polynomial generators $v_i$ in the homotopy of $BP$, and possibly also invert another generator. In this way we can produce various $MU$-modules $BP/I$ and $(BP/I)[v_n^{-1}]$ together with $MU$-homomorphisms from $BP$ whose underlying stable homotopy types play important roles in stable homotopy theory. Some examples of spectra which we can obtained in this way are given in the following table, along with their homotopy groups:

| | | | |
|---|---|---|---|
| $BP\langle n\rangle$ | $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n]$ | $E(n)$ | $\mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n, v_n^{-1}]$ |
| $P(n)$ | $\mathbb{F}_p[v_n, v_{n+1}, \dots]$ | $B(n)$ | $\mathbb{F}_p[v_n^{-1}, v_n, v_{n+1}, \dots]$ |
| $k(n)$ | $\mathbb{F}_p[v_n]$ | $K(n)$ | $\mathbb{F}_p[v_n, v_n^{-1}]$ |

[discuss uniqueness] The spectrum $E(n)$ is referred to as the *Johnson-Wilson spectrum* and $k(n)$ respectively $K(n)$ are the connective and periodic *Morava $K$-theory spectra*.

We have so far only constructed the spectra above as $MU$-modules. The way we have presented the homotopy groups of the various spectra above does not only give graded modules over the homotopy ring of $MU$, but in fact graded commutative *algebras*. This already hints that the spectra have more structure. In fact, all the spectra above can be constructed as $MU$-algebra spectra, so in particular as symmetric ring spectra. We may or may not get back to this later.

CONSTRUCTION 4.71 (Toda brackets). The homotopy groups of a symmetric ring spectrum have even more structure than that of a graded ring, they also have 'secondary' (and higher…) forms of multiplications, called *Toda brackets*. We will restrict ourselves to the simplest kind of such brackets, namely triple brackets (as opposed to four-fold, five-fold,…) with single entries (as opposed to 'matric' Toda brackets). In order to simplify the exposition we will only consider semistable ring spectra; the discussion works more generally if the operations and relations are suitably restricted to $\mathcal{M}$-fixed classes.

So we let $R$ be a semistable symmetric ring spectrum and $M$ a right $R$-module. The Toda bracket $\langle x, y, z\rangle$ is defined for every triple of homogeneous elements $x \in \pi_* M$ and $y, z \in \pi_* R$ which satisfies the relations $xy = 0 = yz$. If the dimensions of $x, y$ and $z$ are $k, l$ and $j$ respectively, then the bracket is not a single homotopy class, but an entire coset in $\pi_{k+l+j+1}M$ for the subgroup $x \cdot \pi_{l+j+1}R + \pi_{k+l+1}M \cdot z$. This subgroup is called the *indeterminacy* of the bracket. [construct the bracket]

Here are some examples of non-trivial Toda bracket. In the stable stems, i.e., the homotopy groups of the sphere spectrum (compare the table in Example 2.4) we have

$$\begin{array}{rll} \eta^2 & \in \langle 2, \eta, 2\rangle & \text{mod } (0) \\ \nu^2 & \in \langle \eta, \nu, \eta\rangle & \text{mod } (0) \\ \eta\sigma + \epsilon & \in \langle \nu, \eta, \nu\rangle & \text{mod } (0) \end{array} \qquad \begin{array}{rll} 6\nu & \in \langle \eta, 2, \eta\rangle & \text{mod } (12\nu) \\ 40\sigma & \in \langle \nu, 24, \nu\rangle & \text{mod } (0) \\ \epsilon & \in \langle \eta, \nu, 2\nu\rangle & \text{mod } (\eta\sigma) \end{array}$$

Toda brackets satisfy a number of relations. The first are a kind of 'higher form of associativity' and are often referred to as 'juggling formulas'. For the juggling formula we need another homogeneous class $w \in \pi_* R$ satisfying $zw = 0$. Then we have

$$x \cdot \langle y, z, w\rangle \;=\; \pm\langle x, y, z\rangle \cdot w$$

module the common indeterminacy $x \cdot \pi^s_{|yz|+1} \cdot w$. [fix the sign] An example of the juggling formula is

$$\eta \cdot \langle 2, \eta, 2\rangle \;=\; \langle \eta, 2, \eta\rangle \cdot 2$$

which holds in $\pi_3^s$ without indeterminacy. By the table above, the first bracket contains $\eta^3$ while the second bracket contains $12\nu$. So we get the multiplicative relation $\eta^3 = 12\nu$ as a consequence of the Toda brackets involving 2 and $\eta$.

The first brackets $\eta^2 \in \langle 2, \eta, 2\rangle$ in the table above is a special case of Toda's relation

$$\eta x \;\in\; \langle 2, x, 2\rangle$$

which holds for all 2-torsion classes $x$ in the homotopy of every commutative [?] symmetric ring spectrum. This in turn is a special case of relations which hold between Toda brackets and power operations in the homotopy ring of commutative symmetric ring spectra. We plan to get back to this later. Another example of a non-trivial Toda bracket is $u \in \langle 1, \eta, 2\rangle$ (modulo $2u$) in $\pi_2 KU$. Here the complex topological $K$-theory spectrum $KU$ is viewed just as a symmetric spectrum (and not as a ring spectrum). So the Bott class $u$ and 1 are homotopy classes of $KU$, while $\eta$ and 2 have to be viewed as elements in the stable stems.

It is straight forward to check that homomorphisms of symmetric ring spectra preserve Toda brackets. More precisely, for every homomorphism $f : S \longrightarrow R$ we have

$$f_*\left(\langle x, y, z\rangle\right) \;\subseteq\; \langle f_*(x), f_*(y), f_*(z)\rangle$$

whenever the bracket on the left is defined. The indeterminacy of the right hand side may be larger than the image of the indeterminacy of the bracket $\langle x, y, z\rangle$, which is why in general we only have containment, not necessarily equality, as subsets on $\pi_{k+l+j+1}R$. For example, the relation $\eta^2 \in \langle 2, \eta, 2\rangle$ holds in the homotopy of every ring spectrum (with possibly bigger indeterminacy, and possibly with $\eta^2 = 0$); since $\eta^2$ is non-zero, in the homotopy of real topological $K$-theory $KO$ (compare the table in Example 2.10), we get a non-trivial bracket $\eta^2 \in \langle 2, \eta, 2\rangle$ (modulo 0) in $\pi_2 KO$. In $\pi_4 KO$ we also have $\xi \in \langle 2, \eta, \eta^2\rangle$ (modulo $2\xi$).

## Exercises

EXERCISE 5.1. The definition of a symmetric spectrum contains some redundancy. Show that the equivariance condition for the iterated structure map is already satisfied if for every $n \geq 0$ the following two conditions hold:

(i) the structure map $\sigma_n : X_n \wedge S^1 \longrightarrow X_{n+1}$ is $\Sigma_n$-equivariant where $\Sigma_n$ acts on the target by restriction from $\Sigma_{n+1}$ to the subgroup $\Sigma_n$.
(ii) the composite

$$X_n \wedge S^2 \xrightarrow{\sigma_n \wedge \mathrm{Id}} X_{n+1} \wedge S^1 \xrightarrow{\sigma_{n+1}} X_{n+2}$$

is $\Sigma_2$-equivariant.

EXERCISE 5.2. Let $X$ be a symmetric spectrum such that for infinitely many $n$ the action of $\Sigma_n$ on $X_n$ is trivial. Show that all stable homotopy groups of $X$ are trivial. (Hint: identify the quotient space of the $\Sigma_2$-action on $S^2$.) What can be said if infinitely many of the alternating groups act trivially?

EXERCISE 5.3. Let $\mathbb{S}^{[k]}$ denote the symmetric subspectrum of the sphere spectrum obtained by truncating below level $k$, i.e.,

$$(\mathbb{S}^{[k]})_n = \begin{cases} * & \text{for } n < k \\ S^n & \text{for } n \geq k. \end{cases}$$

Show that for every symmetric spectrum $X$ and all $k \geq 0$ the inclusion $\mathbb{S}^{[k]} \longrightarrow \mathbb{S}$ induces a $\pi_*$-isomorphism $\mathbb{S}^{[k]} \wedge X \longrightarrow \mathbb{S} \wedge X \cong X$.

EXERCISE 5.4. Let $X$ be a symmetric spectrum. For each finite set $A$, choose a bijection $\kappa_A : A \longrightarrow \mathbf{n}$ where $n = |A|$ is the cardinality of $A$, insisting that $\kappa_{\mathbf{n}}$ is the identity. Define $X_A = X_n$. For each injective map $\alpha : A \longrightarrow B$ of finite sets define a structure map $\alpha_* : X_A \wedge S^{B-\alpha(A)} \longrightarrow X_B$ as the composite

$$X_n \wedge S^{B-\alpha(A)} \xrightarrow{\mathrm{Id} \wedge (\gamma\kappa_B)_*} X_n \wedge S^{m-n} \xrightarrow{\sigma^{m-n}} X_m \xrightarrow{\gamma_*^{-1}} X_m$$

where $n = |A|$ and $m = |B|$ and $\gamma \in \Sigma_m$ is any permutation such that $\gamma\kappa_B\alpha = \iota^{m-n}\kappa_A$. The first isomorphism comes from the bijection $B - \alpha(A) \cong \mathbf{m} - \mathbf{n} = \{n+1, \dots, m\}$ given by the restriction of $\gamma\kappa_B : B \longrightarrow \mathbf{m}$.

Show that the above data defines a coordinate free symmetric spectrum and that together with restriction it defines an equivalence between the category of symmetric spectra and the category of coordinate free symmetric spectra.

EXERCISE 5.5. Here is yet another perspective on what a symmetric spectrum is. We define a based topological category $\Sigma$ as follows. The objects of $\Sigma$ are the natural numbers $0, 1, 2, \dots$ and the based space of morphisms from $n$ to $m$ is given by $\Sigma(n,m) = \Sigma_m^+ \wedge_{1 \times \Sigma_{m-n}} S^{m-n}$, which is to be interpreted as a one-point space if $m < n$. The identity in $\Sigma(n,n) = \Sigma_n^+ \wedge S^0$ is the identity of $\Sigma_n$ (smashed with the non-basepoint of $S^0$) and composition $\circ : \Sigma(m,k) \wedge \Sigma(n,m) \longrightarrow \Sigma(n,k)$ is defined by

$$[\tau \wedge z] \circ [\gamma \wedge y] = [\tau(\gamma \times 1) \wedge (y \wedge z)]$$

where $\tau \in \Sigma_k$, $\gamma \in \Sigma_m$, $z \in S^{k-m}$ and $y \in S^{m-n}$.

(i) Given a symmetric spectrum $X$ we define a map $\bar{X} : \Sigma(n,m) \wedge X_n \longrightarrow X_m$ by

$$[\tau \wedge z] \wedge x \longmapsto \gamma_*(\sigma^{m-n}(x \wedge z)) \ .$$

Verify that this assignment defines a based continuous functor $\bar{X} : \Sigma \longrightarrow \mathcal{T}$ from the category $\Sigma$ to the category of pointed spaces. Show that $X \mapsto \bar{X}$ is an isomorphism between the category of symmetric spectra and the category of based continuous functors $\Sigma \longrightarrow \mathcal{T}$. How can one modify this to accommodate symmetric spectra of simplicial sets?

(ii) We define a functor $+ : \Sigma \times \Sigma \longrightarrow \Sigma$ on objects by addition of natural numbers and on morphisms by

$$[\tau \wedge z] + [\gamma \wedge y] = [(\tau \times \gamma) \wedge (1 \times \chi_{n,\bar{m}} \times 1)_*(z \wedge y)] \in \Sigma(n+\bar{n}, m+\bar{m})$$

for $[\tau \wedge z] \in \Sigma(n,m)$ and $[\gamma \wedge y] \in \Sigma(\bar{n},\bar{m})$. Show that '+' is strictly associative and unital, i.e., a strict monoidal product on the category $\Sigma$. Define a symmetry isomorphism to make this into a *symmetric* monoidal product.

(iii) Show that the construction in (i) can be extended to an isomorphism between the categories of symmetric ring spectra and strong monoidal functors from $\Sigma$ to $\mathcal{T}$ such that it takes commutative symmetric ring spectra isomorphically onto the full subcategory of symmetric monoidal functors.

Exercise 5.6. Let $R$ be a symmetric ring spectrum. Define mapping spaces (simplicial sets) and function symmetric spectra of homomorphisms between two given $R$-modules. Check that for all $k \geq 0$ the endomorphism ring spectrum $\mathrm{Hom}_R(k^+ \wedge R, k^+ \wedge R)$ of the $R$-module $k^+ \wedge R$ is isomorphic, as a symmetric ring spectrum, to the matrix ring spectrum $M_k(R)$ (see Example 2.34).

Exercise 5.7. This exercise is supposed to motivate the term 'bimorphism' which we used in the first way to introduce the smash product of symmetric spectra. Let $X, Y$ and $Z$ be symmetric spectra.

(i) Let $b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$ be a collection of $\Sigma_p \times \Sigma_q$-equivariant maps. Show that the commutativity of the left part of (3.1) is equivalent to the condition that for every $p \geq 0$ the maps $b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}$ form a morphism $b_{p,\bullet} : X_p \wedge Y \longrightarrow \mathrm{sh}^p Z$ of symmetric spectra as $q$ varies. Show that the commutativity of the right part of (3.1) is equivalent to the condition that for every $q \geq 0$ the composite maps

$$Y_q \wedge X_p \xrightarrow{\text{twist}} X_p \wedge Y_q \xrightarrow{b_{p,q}} Z_{p+q} \xrightarrow{\chi_{p,q}} Z_{q+p}$$

form a morphism $Y_q \wedge X \longrightarrow \mathrm{sh}^q Z$ of symmetric spectra as $p$ varies.

(ii) Let $b = \{b_{p,q} : X_p \wedge Y_q \longrightarrow Z_{p+q}\}$ be a bimorphism. Define $\bar{b}_p : X_p \longrightarrow \mathrm{map}(Y, \mathrm{sh}^p Z)$ as the adjoint of the morphism of symmetric spectra $b_{p,\bullet} : X_p \wedge Y \longrightarrow \mathrm{sh}^p Z$ (compare part (i)). Show that as $p$ varies, the maps $\bar{b}_p$ form a morphism of symmetric spectra $\bar{b} : X \longrightarrow \mathrm{Hom}(Y, Z)$. Show then that the assignment

$$\mathrm{Bimor}((X,Y),Z) \longrightarrow \mathcal{S}p^{\Sigma}(X, \mathrm{Hom}(Y,Z)) , \quad b \mapsto \bar{b}$$

is bijective and natural in all three variables.

Exercise 5.8. The way Hovey, Shipley and Smith introduce the smash product in their original paper [**25**] is quite different from our exposition, and this exercise makes the link. Thus the paper [**25**] has the solutions to this exercise. A *symmetric sequence* consists of pointed spaces (or simplicial set) $X_n$, for $n \geq 0$, with based, continuous (respectively simplicial) $\Sigma_n$-action on $X_n$. Morphisms $f : X \longrightarrow Y$ are sequences of equivariant based maps $f_n : X_n \longrightarrow Y_n$. The *tensor product* $X \otimes Y$ of two symmetric sequences $X$ and $Y$ is the symmetric sequence with $n$th term

$$(X \otimes Y)_n = \bigvee_{p+q=n} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_q} X_p \wedge Y_q .$$

(i) Make the tensor product into a closed symmetric monoidal product on the category of symmetric sequences.

(ii) Show that the sequence of spheres $\mathbb{S} = \{S^n\}_{n \geq 0}$ forms a commutative monoid in the category of symmetric sequences. Show that the category of symmetric spectra is isomorphic to the category of right $\mathbb{S}$-modules in the monoidal category of symmetric sequences.

(iii) Given a commutative monoid $R$ in the monoidal category of symmetric sequences and two right $R$-modules $M$ and $N$, show that the coequalizer $M \wedge_R N$ of the two morphisms

$$\alpha_M \otimes \mathrm{Id}, \ \mathrm{Id} \otimes (\alpha_N \circ \tau_{R,N}) \ : \ M \otimes R \otimes N \ \to \ M \otimes N$$

is naturally a right $R$-module. Show that the smash product over $R$ is a closed symmetric monoidal product on the category of right $R$-modules.

(iv) Show that the smash product over $\mathbb{S}$ corresponds to the smash product of symmetric spectra under the isomorphism of categories of part (ii).

Exercise 5.9. Define a notion of 'commuting homomorphisms' between symmetric ring spectra such that homomorphism of symmetric ring spectra $R \wedge S \longrightarrow T$ are in natural bijection with pairs of commuting homomorphisms $(R \longrightarrow T, S \longrightarrow T)$. Deduce that the smash product is the categorical coproduct for *commutative* symmetric ring spectra.

Exercise 5.10. In Remark 4.41 we introduced the ring $\mathbb{Z}[[\mathcal{M}]]$ of natural operations on the 0-th homotopy group of symmetric spectra and identified it with a certain completion of the monoid ring $\mathbb{Z}[\mathcal{M}]$.

(i) An *$I$-set* is a functor from the category $I$ of finite sets and injections to the category of sets. Show that the endomorphism monoid of the 'colimit over inclusions' functor

$$(I\text{-sets}) \longrightarrow (\text{sets}) , \quad F \mapsto F(\omega)$$

is isomorphic to $\mathcal{M}$.

(ii) Show that the endomorphism ring of the 'colimit over inclusions' functor

$$(I\text{-functors}) \longrightarrow (\text{ab. groups}) , \quad F \mapsto F(\omega)$$

is isomorphic to the ring $\mathbb{Z}[[\mathcal{M}]]$.

## History and credits

I now summarize the history of symmetric spectra and symmetric ring spectra, and the genesis of the examples which were discussed above, to the best of my knowledge. My point with respect to the examples is not when certain spectra first appeared as homotopy types or ring spectra 'up to homotopy', but rather when a 'highly structured' multiplication was first noticed in one form or another. Additions, corrections and further references are welcome.

Symmetric spectra and symmetric ring spectra were first introduced under this name in the article [**25**] by Hovey, Shipley and Smith. However, these mathematical concepts had been used before, in particular in several papers related to topological Hochschild homology and algebraic $K$-theory. For example, symmetric ring spectra appeared as *strictly associative ring spectra* in [**22**, Def. 6.1] and as *FSPs defined on spheres* in [**23**, 2.7].

There is a key observation, however, which is due to Jeff Smith and which was essential for the development of symmetric spectra and related spectra categories. Smith noticed that symmetric ring spectra are the monoids in a category of symmetric spectra which has a smash product and a compatible stable model structure. Smith gave the first talks on this subject in 1993. In the fall of 1995, Hovey, Shipley and Smith started a collaboration in which many remaining issues and in particular the model structures were worked out. The results first appeared in a joint preprint on the Hopf algebraic topology server (at `hopf.math.purdue.edu`), the $K$-theory preprint server (at `www.math.uiuc.edu/K-theory/`) and the ArXiv (under `math.AT/9801077`) in January 1998. This preprint version has a section about symmetric spectra based on topological spaces which did not make it into the published version [**25**] because the referee requested that the paper be shortened.

Several of the examples which we gave in Section 2.1 had been around with enough symmetries before symmetric spectra were formally introduced. For example, Bökstedt and Waldhausen introduced *functors with smash product*, or FSPs for short, in [**6**], from which symmetric ring spectra are obtained by restricting to spheres. Eilenberg-Mac Lane spectra (Example 2.7) and monoid ring spectra (Example 2.32) arise in this way from FSPs and seem to have first appeared in [**6**] (or already in Gunnarson's preprint [**22**] ?). Matrix ring spectra (Example 2.34) were also treated as FSP in [**6**] [first published reference ?].

Cobordism spectra first appeared as highly structured ring spectra in the form of as '$\mathscr{I}_*$-prefunctors' in [**42**]. $\mathscr{I}_*$-prefunctors are the same as [commutative ?] orthogonal ring spectra, and the underlying symmetric ring spectra are what we present in Example 2.8. The construction in Example 2.41 which turns unitary spectra into orthogonal spectra by looping with the imaginary spheres appears to be new.

The model for the complex topological $K$-theory spectrum in Example 2.10 is a specialization of a more general construction for $C^*$-algebras by Joachim and Stolz [**29**]. Earlier, Joachim had given a different model for real topological $K$-theory as a commutative symmetric ring spectrum in [**28**].

Waldhausen notes on p. 330 of [**66**] that the iterated $S.$-construction defines a (sequential) spectrum which is an $\Omega$-spectrum from level 1 upwards. Waldhausen's construction predates symmetric spectra, and it was later noticed by Hesselholt [**20**, Appendix] that iterating the $S.$-construction in fact has all the symmetries needed to form a symmetric spectrum. Moreover, bi-exact pairings of input data yields multiplications of associated $K$-theory spectra. Our treatment of the algebraic $K$-theory spectrum on Example 2.11 follows very closely the Appendix of [**20**].

Free and semifree symmetric spectra, suspensions, loop and shifts of symmetric spectra were first discussed in the original paper [**25**] of Hovey, Shipley and Smith.

The particular method for inverting a homotopy elements in a symmetric ring spectrum described in Examples 2.35, 2.36 and 4.65 seems to be new. The construction of Example 2.37 for adjoining roots of unity to a symmetric ring spectrum is due to Schwänzl, Vogt and Waldhausen [**51**]. They originally wrote up the construction in the context of $S$-modules, but their argument only needs that one can form monoid rings and invert homotopy elements within the given framework of commutative ring spectra. So as soon as these constructions are available, their argument carries over to symmetric ring spectra.

I learned the model of the periodic complex cobordism spectrum $MUP$ given in Example 2.42 from Morten Brun, who adapted a construction of Strickland [**60**, Appendix] from 'complex $S$-modules' to unitary spectra.

The category of $\Gamma$-spaces was introduced by Segal [**55**], who showed that it has a homotopy category equivalent to the usual homotopy category of connective spectra. Bousfield and Friedlander [**11**] considered a bigger category of $\Gamma$-spaces in which the ones introduced by Segal appeared as the *special* $\Gamma$-spaces. Their category admits a closed simplicial model category structure with a notion of stable weak equivalences giving rise again to the homotopy category of connective spectra. Then Lydakis [**34**] showed that $\Gamma$-spaces admit internal function objects and a symmetric monoidal smash product with good homotopical properties.

After the discovery of smash products and compatible model structures for $\Gamma$-spaces and symmetric spectra it became obvious that variations of this theme are possible. Simplicial functors were first used for the purposes of describing stable homotopy types by Bökstedt and Waldhausen when they introduced 'FSPs' in [**6**]. Various model structures and the smash product of simplicial functors were systematically studied by Lydakis in [**35**]. The paper [**39**] contains a systematic study of 'diagram spectra', their model structures and smash products, which includes symmetric spectra, $\Gamma$-spaces and simplicial functors. Here orthogonal spectra and continuous functors (defined on finite CW-complexes) make their first explicit appearance. The category of $S$-modules is very different in flavor from the categories diagram spectra, and it is defined and studied in the monograph [**19**].

The smash product of symmetric spectra was defined by Hovey, Shipley and Smith in their original paper [**25**]. However, their exposition of the smash product differs substantially from ours. Hovey, Shipley and Smith use the category of *symmetric sequences* (sequences of pointed spaces $X_n$, for $n \geq 0$, with $\Sigma_n$-action on $X_n$) as an intermediate step towards symmetric spectra and in the construction of the smash product, compare Exercise 5.8. I chose to present the smash product of symmetric spectra in a different way because I want to highlight its property as the universal target for bimorphisms.

The $\mathcal{M}$-action on the homotopy groups of symmetric spectra was first studied systematically by the author in [**53**]. However, several results related to the $\mathcal{M}$-action on homotopy groups are already contained, mostly implicitly, in the papers [**25**] and [**56**]. The definition of semistable symmetric spectra and the characterizations (ii)-(v) of Theorem 4.44 appear in Section of [**25**]; the criterion of trivial $\mathcal{M}$-action on homotopy groups (Theorem 4.44 (i)) first appears in [**53**]. I owe the proof of Lemma 4.21 to Neil Strickland.

Multiplications on the 'classical' homotopy groups of a symmetric ring spectrum have not previously been discussed in the literature. One reason may be that the naive approach to defining a product on the homotopy groups of a symmetric ring spectrum does not always work because it is not in general compatible with stabilization. The preferred way to bypass this issue has so far been to consider 'derived homotopy groups', i.e., to redefine homotopy groups as the classical homotopy groups of a stably fibrant replacement (which we will discuss in the next chapter). In this approach, a symmetric spectrum is replaced by an $\Omega$-spectrum, which is in particular semistable, and thus has a well-defined multiplication.

The idea to construct various spectra from the Thom spectrum $MU$ by killing a regular sequence and possibly inverting an element (see Example 4.70) is taken from Chapter V of [**19**] where this process is carried out in the world of $S$-modules. This strategy had previously been adapted to symmetric spectra in Weiner's *Diplomarbeit* [**67**].

The original construction of the Brown-Peterson spectrum in the paper [**12**] by Brown and Peterson was quite different. They constructed a spectrum whose mod-$p$ homology realizes a certain polynomial subalgebra of the dual Steenrod algebra. Later Quillen gave a construction of the spectrum $BP$ using the theory of formal groups, and Quillen's approach is still at the heart of most current applications of $BP$. Quillen used $p$-typical formal groups to produce an idempotent endomorphism $e : MU_{(p)} \longrightarrow MU_{(p)}$ of the $p$-localization of $MU$ in the stable homotopy category (see Section II) which is even a homomorphism of homotopy ring spectra (see Section II.5 below). The 'image' of this idempotent is isomorphic, in the stable homotopy category, to the spectrum $BP$, and Quillen's construction produces it as a homotopy ring spectrum. Part II of Adams' notes [**2**] are a good exposition of Quillen's results in this area. [original paper?]

CHAPTER II

# The stable homotopy category

## 1. Injective Ω-spectra

Our definition of the stable homotopy category uses symmetric spectra of simplicial sets. We define the stable homotopy category as the homotopy category of injective Ω-spectra. In Chapter III we establish various stable model category structures for symmetric spectra and can then interpret the stable homotopy category as introduced here as the homotopy category, in the sense of model category theory, with respect to the stable model structures.

In order to make sense of the 'homotopy category of injective Ω-spectra' we still have to define the homotopy relation and 'injective' symmetric spectra. Recall from Example 2.15 of Chapter I that we can smash a symmetric spectrum levelwise with a pointed simplicial set $K$, in such a way that the symmetric groups and the structure maps do not act on $K$.

Definition 1.1. Two morphisms of symmetric spectra $f_0, f_1 : A \to X$ are called *homotopic* if there is a morphism

$$H \ : \ \Delta[1]^+ \wedge A \ \longrightarrow \ X \ ,$$

called a *homotopy*, such that $f_0 = H \circ d_0$, and $f_1 = H \circ d_1$, where $d_j : A \longrightarrow \Delta[1]^+ \wedge A$ for $j = 0, 1$ is induced by the face morphisms $d_j : \Delta[0] \longrightarrow \Delta[1]$ and $A$ is identified with $\Delta[0]^+ \wedge A$. We denote by $[A, X]$ the set of homotopy classes of morphisms of symmetric spectra, i.e., the classes under the equivalence relation generated by homotopy.

The homotopy relation is not in general an equivalence relation, but it is when the target is injective, see Proposition 1.6 (ii) below. The notion of homotopy makes just as much sense for symmetric spectra of topological spaces. The only change is that the simplicial 1-simplex $\Delta[1]$ should be replaced by the unit interval $[0, 1]$ in the definition of a homotopy, and the morphisms $d_j : \Delta[0] \longrightarrow \Delta[1]$ are replaced by the endpoint inclusions of the interval. In the context of symmetric spectra of spaces 'homotopy' is always an equivalence relation, with no restriction on the target spectra.

A homotopy between spectrum morphisms is really the same as levelwise pointed homotopies between $(f_0)_n$ and $(f_1)_n : A_n \to X_n$ compatible with the $\Sigma_n$-actions and structure maps. Hence for a morphism of spectra we have implications

$$\text{homotopy equivalence} \implies \text{level equivalence} \implies \pi_*\text{-isomorphism},$$

but the converse implications are not true in general. Yet another way to look at homotopies is via the mapping space. Since the spectrum $\Delta[0]^+ \wedge A$ is canonically isomorphic to $A$, we can identify the vertices of the morphism space $\mathrm{map}(A, X)$ with the spectrum morphisms from $A$ to $X$. Then a morphism $f_0$ is homotopic to a morphism $f_1$ if and only if there exists a 1-simplex $H \in \mathrm{map}(A, X)_1$ satisfying $d_0(H) = f_0$ and $d_1(H) = f_1$.

Example 1.2. Homotopy groups are represented by homotopy classes of morphisms from the sphere spectrum. In more detail, if we denote by $1 \in \pi_0\, \mathbb{S}$ the class of the identity map, then for every integer $n$ and every Ω-spectrum $X$ which is levelwise Kan, the evaluation map

$$\big[\mathbb{S}, X\big] \ \longrightarrow \ \pi_0 X \ , \quad [f] \ \longmapsto \ (\pi_0 f)(1) \ ,$$

is bijective. Then the evaluation map factors as a composite

$$\big[\mathbb{S}, X\big] \ \longrightarrow \ \pi_0 |X_0| \ \longrightarrow \ \pi_0\, X$$

where the first map takes the homotopy class of a morphism $\mathbb{S} \longrightarrow X$ to the component of the image of the non-basepoint in $\mathbb{S}_0 = S^0$. The second map is the canonical map from $\pi_0|X_0|$ to the stable homotopy group in dimension 0. Since the sphere spectrum $\mathbb{S}$ is a suspension spectrum, a morphism from $\mathbb{S}$ to any other spectrum is uniquely determined by its restriction to the zeroth level, and similarly for homotopies. So the first map in the composition is a bijection for any spectrum $X$, no matter whether it is an $\Omega$-spectrum or not. However, if $X$ is an $\Omega$-spectrum, then the second map from $\pi_0|X_0|$ to $\pi_0 X$ is also bijective.

DEFINITION 1.3. A symmetric spectrum $X$ of simplicial sets is *injective* if for every monomorphism which is also a level equivalence $i : A \longrightarrow B$ and every morphism $f : A \longrightarrow X$ there exists an extension $g : B \longrightarrow X$ with $f = gi$.

Injective spectra do not arise 'in nature' very often, but we prove in Proposition 4.21 (i) below that injectivity can always be arranged up to level equivalence.

EXAMPLE 1.4 (Co-free symmetric spectrum). Let $P_m : \Sigma_m\text{-}\mathbf{sset}_* \longrightarrow \mathcal{S}p^\Sigma$ be right adjoint to evaluation at level $m$, considered as a $\Sigma_m$-simplicial set. We call $P_m L$ the *co-semifree* symmetric spectrum generated by the $\Sigma_m$-simplicial set $L$ in level $m$. The spectrum $P_m L$ can explicitely be described as follows: it is just a point above level $m$ and for $n \leq m$ we have

$$(P_m L)_n \ = \ \mathrm{map}^{1\times\Sigma_{m-n}}(S^{m-n}, L) \ ,$$

the subspace of $1 \times \Sigma_{m-n}$-equivariant maps in $\mathrm{map}(S^{m-n}, L)$, with restricted $\Sigma_n$-action from $L$. The structure map $\sigma_n : (P_m L)_n \wedge S^1 \longrightarrow (P_m L)_{n+1}$ is adjoint to the map

$$\mathrm{map}^{1\times\Sigma_{m-n}}(S^{m-n}, L) \xrightarrow{\text{incl.}} \mathrm{map}^{1\times\Sigma_{m-n-1}}(S^{m-n}, L) \ \cong \ \Omega\left(\mathrm{map}^{1\times\Sigma_{m-n-1}}(S^{m-n-1}, L)\right) \ .$$

The forgetful functor $\Sigma_m\text{-}\mathbf{sset}_* \longrightarrow \mathbf{sset}_*$ also has a right adjoint given by $K \mapsto \mathrm{map}(\Sigma_m^+, K)$, the function space from the set $\Sigma_m$ into $K$ (i.e., a product of $m!$ copies of $K$). So the composite forgetful functor $\mathcal{S}p^\Sigma \longrightarrow \mathbf{sset}_*$ which takes $X$ to the pointed simplicial set $X_m$ has a right adjoint $R_m : \mathbf{sset}_* \longrightarrow \mathcal{S}p^\Sigma$ given by $R_m K = P_m(\mathrm{map}(\Sigma_m^+, K))$.

Every Kan simplicial set has the right lifting property with respect to all injective weak equivalences of simplicial sets. So by adjointness, the co-free symmetric spectrum $R_m K$ is injective for every Kan simplicial set $K$. More generally, let $L$ be a pointed $\Sigma_m$-simplicial set with the property that for all subgroups $H \leq \Sigma_m$ the $H$-fixed simplicial set $L^H$ is a Kan complex. Then $H$ has the right lifting property with respect to all injective based morphisms of $\Sigma_m$-simplicial sets which are weak equivalences after forgetting the $\Sigma_m$-action. So again by adjointness, the co-semifree symmetric spectrum $P_n L$ is injective.

Here is the main definition of this chapter.

DEFINITION 1.5. The *stable homotopy category* $\mathcal{SHC}$ is the homotopy category of injective $\Omega$-spectra. In other words, the objects of $\mathcal{SHC}$ are all injective $\Omega$-spectra of simplicial sets and for such spectra, the morphisms from $X$ to $Y$ in $\mathcal{SHC}$ are given by $[X, Y]$, the set of homotopy classes of spectrum morphisms.

As we already mentioned, injective spectra rarely occur in nature. In fact, at this point, trivial spectra (with all levels a point) are the only injective $\Omega$-spectra we can write down explicitly. However, we explain in Section 4 below how every symmetric spectrum can be replaced, up to a notion called 'stable equivalence', by an injective $\Omega$-spectrum. In that way every symmetric spectrum 'represents a stable homotopy type', i.e., gives rise to an object in the stable homotopy category.

In Chapter III we will show that the stable equivalences can be complemented by various useful choices of cofibrations and fibrations, thus arriving at different stable model category structures for symmetric spectra. For one particular choice (the *injective stable model structure*), every symmetric spectrum is cofibrant and the fibrant objects are precisely the injective $\Omega$-spectra. Moreover, the 'concrete' homotopy relation using homotopies defined on $\Delta[1]^+ \wedge A$ coincides with the model category theoretic homotopy relation using abstract cylinder objects. Thus the stable homotopy category as introduced above turns out to be the homotopy category, in the sense of model category theory, with respect to the injective stable model structure.

Our next aim in Sections 2 and 3 of this chapter will be to show that the stable homotopy category just defined has the structure of a triangulated category. We first develop some tools needed in these later sections.

PROPOSITION 1.6. *Let $X$ be an injective spectrum.*

(i) *For every injective morphism $i : A \longrightarrow B$ of symmetric spectra the map* $\mathrm{map}(i, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ *is a Kan fibration of simplicial sets. If in addition $i$ is a level equivalence, then* $\mathrm{map}(i, X)$ *is a weak equivalence.*
(ii) *For every symmetric spectrum $B$ the function space* $\mathrm{map}(B, X)$ *is a Kan complex and the homotopy relation for morphisms from $B$ to $X$ is an equivalence relation.*
(iii) *For every $n \geq 0$ the simplicial set $X_n$ is a Kan complex.*

PROOF. (i) We have to check that $\mathrm{map}(i, X)$ has the right lifting property with respect to every injective weak equivalence $j : K \longrightarrow L$ of pointed simplicial sets. By the adjunction between the smash pairing and mapping spaces, a lifting problem in the form of a commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d, "j"'] & \mathrm{map}(B, X) \arrow[d, "\mathrm{map}(i,X)"] \\ L \arrow[r] & \mathrm{map}(A, X) \end{tikzcd}$$

corresponds to a morphism $K \wedge B \cup_{K \wedge A} L \wedge A \longrightarrow X$, and a lifting corresponds to a morphism $L \wedge B \longrightarrow X$ which restricts to the previous morphism along the 'pushout product' map $j \wedge i : K \wedge B \cup_{K \wedge A} L \wedge A \longrightarrow L \wedge B$. Since $j$ is an injective weak equivalence and $i$ is injective, the pushout product morphism $j \wedge i$ is an injective level equivalence of symmetric spectra. So the lifting exists since we assumed that $X$ is injective.

The second part is very similar. If $i$ is injective and a level equivalence, then for every injective morphism $j : K \longrightarrow L$ (not necessarily a weak equivalence) of pointed simplicial sets, the pushout product map $j \wedge i$ is an injective level equivalence of symmetric spectra. So $\mathrm{map}(i, X)$ has the right lifting property with respect to all injective morphisms of pointed simplicial sets.

Part (ii) is the special case of (i) where $A$ is the trivial spectrum so that $\mathrm{map}(A, X)$ is a one-point simplicial set. Vertices of the simplicial set $\mathrm{map}(B, X)$ correspond bijectively to morphisms $B \longrightarrow X$ in such a way that 1-simplices correspond to homotopies. So the second claim of (ii) follows since in every Kan complex, the relation $x \sim y$ on vertices defined by existence of a 1-simplex $z$ with $d_0 z = x$ and $d_1 z = y$ is an equivalence relation.

The simplicial set $X_n$ is naturally isomorphic to the mapping space $\mathrm{map}(F_n S^0, X)$ with source the free symmetric spectrum generated by $S^0$ in level $n$. So (iii) is a special case of (ii). □

We now get a criterion for level equivalence by testing against injective spectra.

PROPOSITION 1.7. *A morphism $f : A \longrightarrow B$ of symmetric spectra of simplicial sets is a level equivalence if and only if for every injective spectrum $X$ the induced map $[f, X] : [B, X] \longrightarrow [A, X]$ on homotopy classes of morphisms is bijective.*

PROOF. Suppose first that $f$ is a level equivalence. We replace $f$ by the inclusion of $A$ into the mapping cylinder of $f$, which is homotopy equivalent to $B$. This way we can assume without loss of generality that $f$ is injective. By part (i) of Proposition 1.6 the map $\mathrm{map}(f, X) : \mathrm{map}(B, X) \longrightarrow \mathrm{map}(A, X)$ is then a weak equivalence of simplicial sets, so in particular a bijection of components. Since $\pi_0 \, \mathrm{map}(B, X) \cong [B, X]$, and similarly for $A$, this proves the claim.

Now suppose conversely that $[f, X] : [B, X] \longrightarrow [A, X]$ is bijective for every injective spectrum $X$. If $K$ is a pointed Kan complex and $m \geq 0$, then the co-free symmetric spectrum $R_m K$ of Example 1.4 is injective. The adjunction for morphisms and homotopies provides a natural bijection $[A, R_m K] \cong [A_m, K]_{\mathbf{sset}_*}$ to the based homotopy classes of morphisms of simplicial sets. So for every Kan complex $K$, the induced map $[f_m, X] : [B_m, K] \longrightarrow [A_m, K]$ is bijective, which is equivalent to $f_n$ being a weak equivalence of simplicial sets. Since this holds for all $m$, the morphism $f$ is a level equivalence. □

The next lemma can be used to recognize certain morphisms as homotopy equivalences, and thus as isomorphism in the stable homotopy category.

Proposition 1.8. (i) *Every level equivalence between injective spectra is a homotopy equivalence.*
(ii) *Every $\pi_*$-isomorphism between $\Omega$-spectra is a level equivalence.*

Proof. (i) Let $f : X \longrightarrow Y$ be a level equivalence between injective spectra. Using the mapping cylinder construction, $f$ can be factored as a monomorphism followed by a homotopy equivalence. So we can replace $Y$ by the mapping cylinder and assume without loss of generality that $f$ is also a monomorphism. By Proposition 1.6 (i) the induced map $\mathrm{map}(f, X) : \mathrm{map}(Y, X) \longrightarrow \mathrm{map}(X, X)$ is a weak equivalence and Kan fibration, thus surjective on vertices. So there is a morphism $g : Y \longrightarrow X$ satisfying $gf = \mathrm{Id}_X$.

Also by Proposition 1.6 (i) the induced map $\mathrm{map}(f, Y) : \mathrm{map}(Y, Y) \longrightarrow \mathrm{map}(X, Y)$ is a weak equivalence and Kan fibration. Since moreover $\mathrm{map}(f, Y)$ takes the vertices $fg$ and $\mathrm{Id}_Y$ of $\mathrm{map}(Y, Y)$ to the same vertex (namely $f \in \mathrm{map}(X, Y)$), they can be joined by a 1-simplex in $\mathrm{map}(Y, Y)$, i.e., a homotopy of spectrum morphisms.

(ii) For every $\Omega$-spectrum $X$ and all $k, n \geq 0$, the canonical map $\pi_k X_n \longrightarrow \pi_{k-n} X$ is a bijection. So if $f : X \longrightarrow Y$ is a $\pi_*$-isomorphism between $\Omega$-spectra, then for every $n \geq 0$, the morphism $f_n : X_n \longrightarrow Y_n$ induces a bijection of path components and isomorphisms of homotopy groups in positive dimensions, based at the distinguished basepoint of $X_n$. In particular, $f_n$ restricts to a weak equivalence between the components containing the distinguished basepoints. Since $X_n$ and $Y_n$ are loop spaces, their various path components are all weakly equivalent, and so $f_n$ restricts to a weak equivalence on every path component of $X$, i.e., $f_n$ is a weak equivalence of simplicial sets for every $n \geq 0$. □

Theorem 1.9. *For every $\pi_*$-isomorphism $f : A \longrightarrow B$ between symmetric spectra and every injective $\Omega$-spectrum $X$ the induced map on homotopy classes $[f, X] : [B, X] \longrightarrow [A, X]$ is a bijection.*

Proof. We use the functor $R^\infty$ introduced in (4.43) of Chapter I. Since $X$ is an $\Omega$-spectrum, the morphism $\lambda^* : X \longrightarrow RX = \Omega(\mathrm{sh}\, X)$ is a level equivalence, and so are all other morphisms in the sequence (4.43) whose colimit is $R^\infty X$. Thus also the morphism $\lambda^\infty_X : X \longrightarrow R^\infty X$ is a level equivalence, and $R^\infty X$ is again an $\Omega$-spectrum. Since $X$ is an injective spectrum the map $[\lambda^\infty_X, X] : [R^\infty X, X] \longrightarrow [X, X]$ is bijective by Proposition 1.7. So there exists a morphism $r : R^\infty X \longrightarrow X$ such that the composite $r\lambda^\infty$ is homotopic to the identity of $X$ (the other composite need not be homotopic to the identity of $R^\infty X$).

The functor $R^\infty$ preserves the homotopy relation, so we can define a natural transformation

$$[A, X] \longrightarrow [R^\infty A, X] , \quad [\varphi] \mapsto [r \circ R^\infty \varphi] .$$

There also is a natural transformation $[R^\infty A, X] \longrightarrow [A, X]$ in the other direction given by precomposition with $\lambda^\infty_A : A \longrightarrow R^\infty A$. Since $r$ is a retraction (up to homotopy) to $\lambda^\infty_X$, the composite of the two natural maps is the identity on $[A, X]$. In other words, for fixed injective $\Omega$-spectrum $X$, the functor $[-, X]$ is a retract of the functor $[R^\infty(-), X]$.

Now suppose that $f : A \longrightarrow B$ is a $\pi_*$-isomorphism. We assume first that both $A$ and $B$ are levelwise Kan complexes. In [prove] we established a natural isomorphism $\pi_k(R^\infty A)_m \cong \pi_{k-m} A$ (but beware that $R^\infty A$ is not an $\Omega$-spectrum unless $A$ is semistable). So $R^\infty f : R^\infty A \longrightarrow R^\infty B$ is a level equivalence [homotopy groups at other basepoints ?]. So by Proposition 1.7 the map $[R^\infty f, X] : [R^\infty B, X] \longrightarrow [R^\infty A, X]$ is bijective. Since this has $[f, X] : [B, X] \longrightarrow [A, X]$ as a retract, the latter is bijective.

In general we apply the functors 'geometric realization' and 'singular complex' to the morphism $f : A \longrightarrow B$ to replace it by a level equivalent morphism whose source and target are levelwise Kan. Since $[-, X]$ takes level equivalences to bijections, the general case follows from the special case. □

If $X$ is an $\Omega$-spectrum, then so is the shifted spectrum $\mathrm{sh}\, X$. The left adjoint $S^0 \triangleright_0$ to shifting (see Example I.4.37) preserves level equivalences and level injections, so shifting also preserves the property of being injective. Moreover, shifting preserves homotopies since $\mathrm{sh}(\Delta[1]^+ \wedge X) = \Delta[1]^+ \wedge \mathrm{sh}\, X$. If $X$ is an $\Omega$-spectrum and levelwise Kan, then so is the loop spectrum $\Omega X$, and the functor $\Omega$ preserves injective spectra and the homotopy relation. Moreover, for every $\Omega$-spectrum $X$ the natural map

$$\lambda^* \ : \ X \ \longrightarrow \ \Omega(\mathrm{sh}\, X)$$

is a level equivalence, thus a homotopy equivalence by part (i) of Proposition 1.8. And so on the level of the stable homotopy category, $\Omega$ and shift are inverse to each other. So we have shown

PROPOSITION 1.10. *The shift functor and the functor $\Omega$ are quasi-inverse self-equivalences of the stable homotopy category.*

## 2. Additive structure

Now we prove that the stable homotopy category is additive. We have to define an addition on morphisms sets in $\mathcal{SHC}$ for which composition is bilinear.

THEOREM 2.1. *For all symmetric spectra of simplicial sets $A$ and $B$ the natural morphism $A \vee B \to A \times B$ is a $\pi_*$-isomorphism.*

PROOF. This is a direct consequence of the fact that homotopy groups take both finite wedges and finite products of symmetric spectra to direct sums. More precisely, for every integer $k$ the composite map

$$\pi_k A \oplus \pi_k B \longrightarrow \pi_k(A \vee B) \longrightarrow \pi_k(A \times B) \longrightarrow \pi_k A \times \pi_k B$$

is the identity, where the first and last maps are the canonical ones. These canonical maps are isomorphisms by Corollary I.4.9, hence so is the middle map. $\square$

The stable homotopy category has sums (coproducts) and products of arbitrary size. For products this is easy to see: let $\{X^i\}_{i \in I}$ be a family of injective $\Omega$-spectra. Then the product $\prod_{i \in I} X^i$ is again an injective $\Omega$-spectrum, so it represents an object of the stable homotopy category. Moreover, a morphism to $\prod_{i \in I} X^i$ is just a family of morphisms, one to each $X^i$, and similarly for homotopies. So the map

$$[A, \prod_{i \in I} X^i] \longrightarrow \prod_{i \in I} [A, X^i]$$

induced by the projections from $\prod_{i \in I} X^i$ to each $X^i$ is a bijection for every symmetric spectrum $A$. In particular this holds when $A$ is an injective $\Omega$-spectrum, i.e., the pointset level product of symmetric spectra descends to a product in the stable homotopy category.

The case of sums is slightly more involved. On the level of symmetric spectra, the coproduct of a family of symmetric spectra $\{X^i\}_{i \in I}$ is given by the levelwise wedge. However, even if all $X^i$ are injective $\Omega$-spectra, the wedge is not an injective $\Omega$-spectrum. Still, the stable homotopy category has sums (coproducts): the product $X \times Y$ is also a *co*-product of $X$ and $Y$ in the stable homotopy category, as we will now show. Infinite coproducts in the stable homotopy are constructed differently, see Proposition 4.14 (iii) below.

Given symmetric spectra $A$ and $B$ we denote by $i_1 = (\mathrm{Id}, *) : A \longrightarrow A \times B$ respectively $i_2 = (*, \mathrm{Id}) : B \longrightarrow A \times B$ the 'inclusions' of the factors into the product.

PROPOSITION 2.2. *For every injective $\Omega$-spectrum $X$ and all symmetric spectra $A$ and $B$ the map*

$$\begin{array}{ccc} [A \times B, X] & \longrightarrow & [A, X] \times [B, X] \\ [f] & \longmapsto & ([fi_1], [fi_2]) \end{array} \tag{2.3}$$

*is bijective. Thus if $A$ and $B$ are also injective $\Omega$-spectra, then the morphisms $i_1$ and $i_2$ make $A \times B$ into a co(!)-product of $A$ and $B$ in the stable homotopy category.*

PROOF. The map (2.3) factors as a composite

$$[A \times B, X] \longrightarrow [A \vee B, X] \longrightarrow [A, X] \times [B, X] \ . \tag{2.4}$$

The first map in (2.4) is precomposition with the canonical injection $A \vee B \to A \times B$; this is a $\pi_*$-isomorphism by Theorem 2.1, so it induces a bijection of homotopy classes of maps into $X$ by Theorem 1.9. The second map in (2.4) is restriction to the respective wedge summands. A morphism from $X \vee Y$ is the same as two morphisms, one from $X$ and one from $Y$, and similarly for homotopies. So the second map in (2.4) is also bijection, which finishes the proof.

If $A$ and $B$ are injective $\Omega$-spectra, then the fact that the map (2.3) is bijective is precisely the universal property of a categorical coproduct in the special case of the stable homotopy category. $\square$

Construction 2.5. We define an operation '+' on the set $[A, X]$ of homotopy classes of morphisms from an arbitrary symmetric spectrum $A$ to an injective $\Omega$-spectrum $X$. For two homotopy classes $a, b \in [A, X]$ we denote by $a \vee b$ the unique class in $[A \times A, X]$ which restricts to the pair $(a, b)$ along the bijection of Proposition 2.2. Then we define $a + b = (a \vee b)\Delta$ where $\Delta : A \longrightarrow A \times A$ is the diagonal morphism.

Proposition 2.6. *For every symmetric spectrum $A$ and every injective $\Omega$-spectrum $X$, the binary operation $+$ makes the set $[A, X]$ of homotopy classes of morphisms into an abelian group. The neutral element is the class of the trivial morphism. Moreover, the group structure is natural for all morphisms in the source variable $A$ and all morphisms between injective $\Omega$-spectra in the target variable $X$. In particular, the stable homotopy category is an additive category.*

The additivity of the stable homotopy category is a fundamental result which deserves two different proofs.

First proof. The proof is a lengthy, but essentially formal consequence of Proposition 2.2 which says that to the eyes of $[-, X]$ coproducts are the same as products. For the associativity of '+' we consider three morphisms $a, b, c : A \longrightarrow X$. Then $a + (b + c)$ respectively $(a + b) + c$ are the two outer composites around the diagram

$$\begin{tikzcd}
 & A \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
A \times A \arrow[d, "\mathrm{Id} \times \Delta"'] & & A \times A \arrow[d, "\Delta \times \mathrm{Id}"] \\
A \times (A \times A) \arrow[dr, "a \vee (b \vee c)"'] & & (A \times A) \times A \arrow[dl, "(a \vee b) \vee c"] \\
 & X &
\end{tikzcd}$$

If we fill in the canonical associativity isomorphism $A \times (A \times A) \cong (A \times A) \times A$ then the upper part of the diagram commutes because the diagonal morphism is coassociative. The lower triangle then commutes up to homotopy since the two morphisms $a \vee (b \vee c), (a \vee b) \vee c : A \times (A \times A) \longrightarrow X$ have the same 'restrictions' to $X$, namely $a$, $b$ respectively $c$.

The commutativity is a consequence of two elementary facts: first, $b \vee a = (a \vee b)\tau$ as homotopy classes, where $\tau : A \times A \longrightarrow A \times A$ is the morphism which interchanges the two factors; this follows from $\tau i_1 = i_2$ and $\tau i_2 = i_1$. Second, the diagonal map is cocommutative, i.e., $\tau\Delta = \Delta : A \longrightarrow A \times A$. Altogether we get

$$a + b \;=\; (a \vee b)\Delta \;=\; (a \vee b)\tau\Delta \;=\; (b \vee a)\Delta \;=\; b + a\ .$$

We denote by $0 \in [A, X]$ the class of the trivial morphism. Then we have $a \vee 0 = ap_1$ in $[A \times A, X]$ where $p_1 : A \times A \longrightarrow A$ is the projection onto the first factor. Hence $a + 0 = (a \vee 0)\Delta = ap_1\Delta = a$; by commutativity we also have $0 + a = a$.

Now we know that the addition makes the set $[A, X]$ into an abelian monoid, and it remains to show that additive inverses exist. An arbitrary abelian monoid $M$ has additive inverses if and only if the map

$$M^2 \;\longrightarrow\; M^2\ , \quad (a, b) \longmapsto (a, a + b)$$

is bijective. Indeed, the inverse of $a \in A$ is the second component of the preimage of $(a, 0)$. For the abelian monoid $[A, X]$ we have a commutative square

$$\begin{tikzcd}
{[A \times A, X]} \arrow[r, "{[i_1 \vee \Delta, X]}"] \arrow[d, "\cong"'] & {[A \vee A, X]} \arrow[d, "\cong"] \\
{[A, X]^2} \arrow[r, "{(a,b) \mapsto (a, a+b)}"'] & {[A, X]^2}
\end{tikzcd}$$

The effect of the morphism $i_1 \vee \Delta : A \vee A \longrightarrow A \times A$ on homotopy groups

$$\pi_*A \oplus \pi_*A \;\cong\; \pi_*(A \vee A) \;\xrightarrow{\pi_*(i_1 \vee \Delta)}\; \pi_*(A \times A) \;\cong\; \pi_*A \oplus \pi_*A$$

is given by $(x,y)\mapsto(x+y,y)$, so $i_1\vee\Delta$ is a $\pi_*$-isomorphism since homotopy groups have additive inverses. Thus the map $[i_1\vee\Delta,X]$ is bijective by Theorem 1.9, and so the abelian monoid $[A,X]$ is a group.

Now we verify naturality of the addition on $[A,X]$ in $A$ and $X$. To check $(a+b)c=ac+bc$ for $a,b:A\longrightarrow X$ and $c:A'\longrightarrow A$ we consider the commutative diagram

$$\begin{tikzcd}
A' \arrow[r, "c"] \arrow[d, "\Delta"'] & A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A\times A \arrow[d, "a\vee b"] \\
A'\times A' \arrow[r, "c\times c"] \arrow[rr, bend right, "ac\vee bc"'] & A\times A \arrow[r, "a\vee b"] & X
\end{tikzcd}$$

in which the composite through the upper right corner is $(a+b)c$. The lower vertical composite $(a\vee b)(c\times c)$ equals $ac\vee bc$ since both have the same 'restrictions' $ac$ respectively $bc$ to the two factors of $A'\times A'$. Since the composite through the lower left corner is $ac+bc$, we have shown $(a+b)c=ac+bc$. Naturality in $X$ is even easier. For a morphism $d:X\longrightarrow Y$ between injective $\Omega$-spectra we have $d(a\vee b)=da\vee db:A\times A\longrightarrow Y$ since both sides have the same 'restrictions' $da$ respectively $db$ to the two factors of $A\times A$. Thus $d(a+b)=da+db$ by the definition of '+'. □

Second proof. If $X$ is an injective $\Omega$-spectrum then $\lambda^*:X\longrightarrow\Omega(\mathrm{sh}\,X)$ is a natural level equivalence, homotopy equivalence (by Proposition 1.8 (i)) between injective $\Omega$-spectra. So $\lambda^*$ induces a homotopy equivalence

$$\mathrm{map}(A,\lambda^*)\ :\ \mathrm{map}(A,X)\ \longrightarrow\ \mathrm{map}(A,\Omega(\mathrm{sh}\,X))\cong\Omega\,\mathrm{map}(A,\mathrm{sh}\,X)$$

on mapping spaces. Since the target is the simplicial loop space, the loop addition defines a group structure on the set of components $\pi_0\,\mathrm{map}(A,\Omega(\mathrm{sh}\,X))$ which we pull back along the bijection induced by $\mathrm{map}(A,\lambda^*)$ to a natural group structure on $\pi_0\,\mathrm{map}(A,X)$. Now we show that the natural bijection

$$[A,X]\ \cong\ \pi_0\,\mathrm{map}(A,X)$$

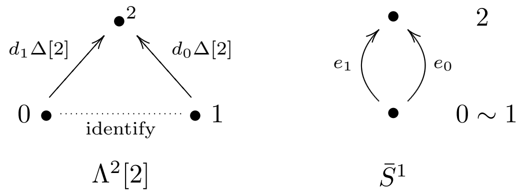
takes the operation '+' to the loop product in the components of the mapping space $\mathrm{map}(A,X)$ and we show simultaneously that the product on the right hand side is abelian. For this we consider the commutative diagram

$$\begin{tikzcd}
{[A,X]^2} \arrow[r, "\cong"] \arrow[dd, bend right=60, "+"'] & (\pi_0\,\mathrm{map}(A,X))^2 \arrow[dd, bend left=60, "\text{loop product}"] \\
{[A\times A,X]} \arrow[r, "\cong"] \arrow[u, "\cong", "(i_0^*,i_1^*)"'] \arrow[d, "{[\Delta,X]}"] & \pi_0\,\mathrm{map}(A\times A,X) \arrow[u, "(i_0^*,i_1^*)", "\cong"'] \arrow[d, "\pi_0\,\mathrm{map}(\Delta,X)"'] \\
{[A,X]} \arrow[r, "\cong"'] & \pi_0\,\mathrm{map}(A,X)
\end{tikzcd}$$

of sets in which all horizontal and the left upper vertical map are bijections. The left vertical composite defines '+'. The right vertical composite coincides with the loop product in $\pi_0\,\mathrm{map}(A,X)$ since it is a homomorphism which sends $(f,*)$ and $(*,f)$ to $f$. Since the group multiplication $(\pi_0\,\mathrm{map}(A,X))^2\longrightarrow\pi_0\,\mathrm{map}(A,X)$ is a homomorphism of groups, the group $\pi_0\,\mathrm{map}(A,X)$, and thus $[A,X]$, is abelian. □

For every injective $\Omega$-spectrum $X$, the identity morphism of the loop spectrum $\Omega X$ has an additive inverse in the group $[\Omega X,\Omega X]$. However, there is no natural endomorphism of the symmetric spectrum $\Omega X$ which represents the negative of the identity in the homotopy category. This is really a consequence of the rigidity of simplicial sets: if we work instead with symmetric spectra based on topological spaces, then we can 'invert the direction' of a loop, i.e., precompose a loop with the any selfmap of $S^1$ of degree $-1$ and thus realize $-\mathrm{Id}_{\Omega X}$ naturally on the pointset level. It will be convenient later to have something analogous for symmetric spectra based on simplicial sets, and we can arrange for this by using a simplicial model of the circle which is slightly larger than $S^1=\Delta[1]/\partial\Delta[1]$.

We define a simplicial set $\bar{S}^1$ as a quotient of the horn $\Lambda^2[2]$ (the simplicial subset of $\Delta[2]$ generated by $d_0\Delta[2]$ and $d_1\Delta[2]$) by identifying the two 'outer' vertices, compare the picture.




We use the common image of the two outer vertices in $\bar{S}^1$ as the basepoint and denote the two non-degenerate 1-simplices of $\bar{S}^1$ by $e_0$ and $e_1$. There is an involution $\tau : \bar{S}^1 \longrightarrow \bar{S}^1$ which interchanges $e_0$ and $e_1$. The geometric realization of $\bar{S}^1$ is homeomorphic to a circle and $\tau$ realizes a map of degree $-1$.

There are two based morphism $p_0, p_1 : \bar{S}^1 \longrightarrow S^1 = \Delta[1]/\partial\Delta[1]$ where $p_i$ sends $e_i$ to the generating 1-simplex of $S^1$ and it sends $e_{1-i}$ to the basepoint. We have $p_1 = p_0\tau$. Both $p_0$ and $p_1$ are weak equivalences, so for every symmetric spectrum $X$ which is levelwise Kan the induced maps $p_0^*, p_1^* : \Omega X = X^{S^1} \longrightarrow X^{\bar{S}^1}$ are level equivalences. If $X$ is an injective $\Omega$-spectrum, then so is $X^{\bar{S}^1}$ (by the same reasoning as for $\Omega X = X^{S^1}$).

Lemma 2.7. *For every injective $\Omega$-spectrum $X$ the relation $[p_1^*] = -[p_0^*]$ holds in the group $[\Omega X, X^{\bar{S}^1}]$. and the morphism $\tau^* : X^{\bar{S}^1} \longrightarrow X^{\bar{S}^1}$ represents the negative of the identity in the group $[X^{\bar{S}^1}, X^{\bar{S}^1}]$.*

Proof. The diagram of morphisms of symmetric spectra

$$\begin{tikzcd}
X^{S^1} \vee X^{S^1} \arrow[r] \arrow[drr, "p_0^* \vee p_1^*"'] & X^{S^1} \times X^{S^1} & X^{S^1 \vee S^1} \arrow[l, "\cong"'] \arrow[d, "\mathrm{pinch}^*"'] & X^{S^1} \arrow[l, "\mathrm{fold}^*"'] \arrow[dl, "(\mathrm{fold} \circ \mathrm{pinch})^*"] \\
 & & X^{\bar{S}^1} &
\end{tikzcd}$$

commutes. Here we use the fold map $S^1 \vee S^1 \longrightarrow S^1$ and the 'pinch' map $\bar{S}^1 \longrightarrow S^1 \vee S^1$ which sends $e_0$ and $e_1$ to the generating 1-simplex of $S^1$. By the very definition of the addition in the group $[\Omega X, X^{\bar{S}^1}]$ this proves that the sum of the homotopy classes of $p_0^*$ and $p_1^*$ is represented by the morphism $(\mathrm{fold} \circ \mathrm{pinch})^* : X^{S^1} \longrightarrow X^{\bar{S}^1}$.

The composite $\mathrm{fold} \circ \mathrm{pinch} : \bar{S}^1 \longrightarrow S^1$ factors as the composite

$$\bar{S}^1 = \Lambda^2[2]/(0 \sim 1) \xrightarrow{\mathrm{incl}} \Delta[2]/d_2\Delta[2] \xrightarrow{s_0} \Delta[1]/\partial\Delta[1] = S^1$$

where the second map is induced by the morphism $s_0 : \Delta[2] \longrightarrow \Delta[1]$ on quotients. Since the simplicial set $\Delta[2]/d_2\Delta[2]$ is contractible, the morphism $(\mathrm{fold} \circ \mathrm{pinch})^* : X^{S^1} \longrightarrow X^{\bar{S}^1}$ is null-homotopic. This proves $[p_1^*] + [p_0^*] = 0$ in $[\Omega X, X^{\bar{S}^1}]$. Since we have $[\tau^* p_0^*] = [p_1^*] = -[p_0^*]$ and $p_0^*$ is a homotopy equivalence, we conclude that $[\tau^*] = -\,\mathrm{Id}$. $\square$

Loops of symmetric spectra are defined levelwise, so we have $\Omega(\mathrm{sh}\, X) = \mathrm{sh}(\Omega X)$. We thus have the two morphisms $\lambda_{\Omega Y}^*$ and $\Omega(\lambda_Y^*)$ from $\Omega Y$ to $\Omega^2(\mathrm{sh}\, Y)$, and they differ by the involution of $\Omega^2(\mathrm{sh}\, Y) = Y^{S^2}$ which flips the two coordinates in $S^2$. So we have shown

Lemma 2.8. *The relations*

$$\lambda_{\Omega Y}^* = -\Omega(\lambda_Y^*) \quad \textit{and} \quad \lambda_{\mathrm{sh}\, Y}^* = -\,\mathrm{sh}(\lambda_Y^*)$$

*hold in the groups $[\Omega Y, \Omega^2(\mathrm{sh}\, Y)]$ respectively $[\mathrm{sh}\, Y, \Omega(\mathrm{sh}^2\, Y)]$.*

## 3. Triangulated structure

Besides the shift functor, for which we use the notation $X[1]$, a triangulated category has another piece of extra structure, namely a specified class of *distinguished triangles*. In the stable homotopy category, these arise from homotopy fibre sequences as follows.

Recall from (4.3) of Chapter I that the homotopy fibre $F(\varphi)$ of a morphism $\varphi : X \longrightarrow Y$ between symmetric spectra is the pullback in the cartesian square

$$\begin{tikzcd} F(\varphi) \arrow[r, "p"] \arrow[d] & X \arrow[d, "(*,\varphi)"] \\ Y^{\Delta[1]} \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & Y \times Y \end{tikzcd}$$

We write 'elements' of $F(\varphi)$ as pairs $(\omega, x)$ where $\omega$ is a path in $Y$ starting at the basepoint and $x$ is a point in $X$ such that $\varphi(x)$ is the endpoint of $\omega$. There are morphisms

$$\Omega Y \xrightarrow{i} F(\varphi) \xrightarrow{p} X \xrightarrow{\varphi} Y$$

the first two being given by

$$i(\omega) = (\omega, *) \quad \text{respectively} \quad p(\omega, x) = x\ .$$

The composite $pi$ is the trivial map and the composite $\varphi p$ comes with a preferred null-homotopy [specify it].

We call a morphism of symmetric spectra of simplicial sets an *injective fibration* if it has the right lifting property with respect to all monomorphisms which are also level equivalences. So $X$ is injective if and only if the unique morphism from $X$ to a trivial spectrum is an injective fibration. Note that here the adjective 'injective' refers to a lifting property and does *not* mean that such maps are monomorphisms.

Lemma 3.1. *Let $\varphi : X \longrightarrow Y$ be a morphism of symmetric spectra.*

(i) *If $X$ and $Y$ are $\Omega$-spectra and $\varphi$ is levelwise a Kan fibration, then the fibre of $\varphi$ over the basepoint is an $\Omega$-spectrum.*
(ii) *If $X$ and $Y$ are $\Omega$-spectra which are levelwise Kan complexes, then so is the homotopy fibre $F(\varphi)$.*
(iii) *Let $X$ be an injective symmetric spectrum and $K \longrightarrow L$ a monomorphism of pointed simplicial sets. Then the induced morphism $X^L \longrightarrow X^K$ is an injective fibration.*
(iv) *If $X$ and $Y$ are injective, then the projection $p : F(\varphi) \longrightarrow X$ is an injective fibration and the homotopy fibre $F(\varphi)$ is injective.*

Proof. (i) Standard. Uses that the geometric realization of a Kan fibration is a Serre fibration, the long exact sequence of homotopy groups of a Kan fibration and the five lemma.

(ii) The homotopy fibre $F(\varphi)$ is the fibre over the basepoint of the morphism $\mathrm{ev}_0 : Y^{\Delta[1]} \times_Y X \longrightarrow Y$. This morphism is levelwise a Kan fibration The spectrum $Y^{\Delta[1]} \times_Y X$ is homotopy equivalent to $X$ by 'contracting a path to its endpoint', so source and target of $\mathrm{ev}_0$ are $\Omega$-spectra. The morphism is also levelwise a Kan fibration: for every morphism $\psi : V \longrightarrow W$ between pointed Kan complexes, the lower horizontal map in the pullback square

$$\begin{tikzcd} W^{\Delta[1]} \times_W V \arrow[r, "(\mathrm{ev}_0,p)"] \arrow[d] & W \times V \arrow[d, "(\mathrm{Id},\psi)"] \\ W^{\Delta[1]} \arrow[r, "(\mathrm{ev}_0,\mathrm{ev}_1)"'] & W \times W \end{tikzcd}$$

is a Kan fibration (since $W$ is Kan), and hence so is its base change. Since $V$ is Kan, the projection $W \times V \longrightarrow W$ is a Kan fibration, hence also the composite $\mathrm{ev}_0 : W^{\Delta[1]} \times_W V \longrightarrow W$. So the morphism $\mathrm{ev}_0 : Y^{\Delta[1]} \times_Y X \longrightarrow Y$ satisfies the hypothesis of part (i), and so its (strict) fibre $F(\varphi)$ is an $\Omega$-spectrum.

(iv) Since $Y$ is injective and the boundary inclusion $\partial\Delta[1] \longrightarrow \Delta[1]$ is a monomorphism of simplicial set, the evaluation morphism $(\mathrm{ev}_0, \mathrm{ev}_1) : Y^{\Delta[1]} \longrightarrow Y^{\partial\Delta[1]} \cong Y \times Y$ is an injective fibration, by part (iii). So $p : F(\varphi) \longrightarrow X$ is an injective fibration since these are stable under basechange. If furthermore $X$ is injective, then so is $F(\varphi)$ by the composition property. $\square$

Now consider a morphism of symmetric spectra $\varphi : X \longrightarrow Y$. If we let $F$ denotes the (strict) fibre of $\varphi$ at the basepoint, then we have a morphism $j : F \longrightarrow F(\varphi)$ given by $x \mapsto (*, x)$ where we write $*$ for the constant path in $Y$ at the basepoint. If $X$ and $Y$ are levelwise Kan and $\varphi : X \longrightarrow Y$ is levelwise a Kan fibration, then the morphism $j : F \longrightarrow F(\varphi)$ is a level equivalence. This holds in particular whenever $X$ and $Y$ are injective $\Omega$-spectra and $\varphi : X \longrightarrow Y$ is an injective fibration, and then the level equivalence $j : F \longrightarrow F(\varphi)$ has injective $\Omega$-spectra as source and target, and so it is a homotopy equivalence, thus an isomorphism in the stable homotopy category.

An *elementary distinguished triangle* in the stable homotopy category is a diagram of the form

$$F \xrightarrow{\text{incl.}} X \xrightarrow{\varphi} Y \xrightarrow{\delta_\varphi} F[1]$$

for an injective fibration $\varphi : X \longrightarrow Y$ between injective $\Omega$-spectra. The 'connecting' morphism $\delta : Y \longrightarrow F[1]$ is the unique morphism in the stable homotopy category which makes the square

$$\begin{tikzcd}
Y \arrow[r, "\delta"] \arrow[d, "\lambda_Y^*"', "\cong"] & F[1] \arrow[d, "\cong"', "j[1]"] \\
\Omega Y[1] \arrow[r, "i[1]"'] & F(\varphi)[1]
\end{tikzcd}$$

commute. Both vertical maps are isomorphism in the stable homotopy category; the left one is the level equivalence $\lambda_Y^* : Y \longrightarrow \Omega(\mathrm{sh}\, Y) = \Omega Y[1]$. The lower morphism is the shift of the morphism $i : \Omega Y \longrightarrow F(\varphi)$. [remark naturality in the fibration and $\delta_{\mathrm{sh}\,\varphi} = -\,\mathrm{sh}(\delta_\varphi)$]

A *distinguished triangle* is any diagram

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

in the stable homotopy category which is isomorphic to an elementary distinguished triangle, i.e., such that there is an injective fibration $\varphi : X \longrightarrow Y$ between injective $\Omega$-spectra and isomorphisms $\alpha : A \longrightarrow F$, $\beta : B \longrightarrow X$ and $\gamma : C \longrightarrow Y$ in $\mathcal{SHC}$ such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\
F \arrow[r, "\text{incl.}"'] & X \arrow[r, "\varphi"'] & Y \arrow[r, "\delta_\varphi"'] & F[1]
\end{tikzcd}$$

commutes. Our aim is to show that the shift functor and the class of distinguished triangles make the stable homotopy category into a triangulated category. We first collect some ways to produce new distinguished triangles from old ones.

Proposition 3.2. (i) *If a triangle $(f, g, h)$ is distinguished, then so is the triangle*

$$\Omega C \xrightarrow{-\bar{h}} A \xrightarrow{f} B \xrightarrow{\lambda_C^* \circ g} \Omega C[1]$$

*where $\bar{h} : \Omega C \longrightarrow A$ is the unique morphism in the stable homotopy category such that $\bar{h}[1] \circ \lambda_C^* = h$.*
(ii) *If $(f, g, h)$ is a triangle such that the rotated triangle $(g, h, -f[1])$ is distinguished, then so is the original triangle $(f, g, h)$.*
(iii) *If the triangle $(f, g, h)$ is distinguished, then so is the triangle*

$$A[1] \xrightarrow{-f[1]} B[1] \xrightarrow{-g[1]} C[1] \xrightarrow{-h[1]} A[2] \ .$$

PROOF. (i) We can assume without loss of generality that we are dealing with the elementary distinguished triangle $(\mathrm{incl}, \varphi, \delta_\varphi)$ of an injective fibration $\varphi : X \longrightarrow Y$ between injective $\Omega$-spectra. By Lemma 3.1 (iv) the projection $p : F(\varphi) \longrightarrow X$ is again an injective fibration between injective $\Omega$-spectra. We claim that the diagram

$$\begin{tikzcd}
\Omega Y \arrow[r, "-\bar{\delta}_\varphi"] \arrow[d, "-\mathrm{Id}"'] & F \arrow[r, "\mathrm{incl.}"] \arrow[d, "j"'] & X \arrow[r, "\lambda_Y^* \circ \varphi"] \arrow[d, equal] & \Omega Y[1] \arrow[d, "-\mathrm{Id}"] \\
\Omega Y \arrow[r, "i"'] & F(\varphi) \arrow[r, "p"'] & X \arrow[r, "\delta_p"'] & \Omega Y[1]
\end{tikzcd}$$

commutes in the stable homotopy category where $\bar{\delta}_\varphi$ is the unique morphism such that $(\bar{\delta}_\varphi)[1] \circ \lambda_Y^* = \delta$. Indeed, by the definition of the connective morphism $\delta$ we have

$$i[1] \circ \lambda_Y^* \;=\; j[1] \circ \delta \;=\; j[1] \circ (\bar{\delta}_\varphi)[1] \circ \lambda_Y^* \;.$$

Since $\lambda_Y^*$ is an isomorphism and shifting is faithful this implies $i = j \circ \bar{\delta}_\varphi$, i.e., the left square commutes.

With respect to the right square we first claim that the relation $i_p = -j \circ \Omega(\varphi)$ holds in the group $[\Omega X, F(p)]$, where $j : \Omega Y \longrightarrow F(p)$ is given by $j(\omega) = (\mathrm{const}_*, \omega)$. Granted this for a moment, we have

$$\delta_p \;=\; (j^{-1} i_p)[1] \circ \lambda_X^* \;=\; -\Omega(\varphi)[1] \circ \lambda_X^* \;=\; -\lambda_Y^* \circ \varphi$$

(using naturality of $\lambda^*$), i.e., the right square commutes.

We are now reduced to proving the relation $i_p = -j \circ \Omega(\varphi)$ in $[\Omega X, F(p)]$. If we were using symmetric spectra based on topological spaces we could realize the negative of $\Omega(\varphi)$ by 'inversion of the loop' and write down an explicit homotopy between $i_p$ and the 'loop inverted' composite $j \circ \Omega(\varphi)$. Since we work simplicially and $S^1 = \Delta[1]/\partial\Delta[1]$ has no selfmap of degree $-1$, we have to say a bit more to make this idea rigorous. We use the level equivalent model $Y^{\bar{S}^1}$ for the loop spectrum $\Omega Y$, where $\bar{S}^1 = \Lambda^2[2]/(0 \sim 1)$ is the 'large' simplicial circle (compare Lemma 2.7). A morphism

$$g \;:\; F(p) = * \times_X X^{\Delta[1]} \times_X F(\varphi) \;\longrightarrow\; Y^{\bar{S}^1}$$

is defined levelwise by sending $(\lambda, \nu)$ to the map $\bar{S}^1 \longrightarrow Y$ which is $\varphi \circ \lambda$ on the 1-simplex $e_0$ of $\bar{S}^1$, and which is $\nu$ on 1-simplex $e_1$. The morphism $g$ is a level equivalence, and thus an isomorphism in $\mathcal{SHC}$, since the composite with the level equivalence $j : \Omega Y \longrightarrow F(p)$ is the level equivalence $p_1^* : \Omega X \longrightarrow Y^{\bar{S}^1}$ induced by the weak equivalence $p_1 : \bar{S}^1 \longrightarrow S^1$.

The two composites

$$gi_p,\; gj\Omega(\varphi) \;:\; \Omega X \;\longrightarrow\; Y^{\bar{S}^1}$$

differ precisely, on the pointset level, by the involution $\tau^* : Y^{\bar{S}^1} \longrightarrow Y^{\bar{S}^1}$ induced by the 'flip' $\tau$ of the 'large' circle $\bar{S}^1$. By Lemma 2.7 this involution realizes the negative of the identity, so we conclude that $[gi_p] = -[gj\Omega(\varphi)]$ in the group $[\Omega X, X^{\bar{S}^1}]$. Since $g$ becomes an isomorphism in the stable homotopy category [justify], this proves the claim.

The lower row of the commutative diagram is the elementary distinguished triangle associated to $p : F(\varphi) \longrightarrow X$ and all vertical morphisms are isomorphism. So the upper triangle is distinguished.

(ii) If $(g, h, -f[1])$ is a distinguished triangle, then so is the lower triangle in the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "-\lambda_A^*"'] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, equal] & A[1] \arrow[d, "-\lambda_A^*[1]"] \\
\Omega A[1] \arrow[r, "\overline{f[1]}"'] & B \arrow[r, "g"'] & C \arrow[r, "\lambda_{A[1]}^* \circ h"'] & \Omega A[2]
\end{tikzcd}$$

by part (i). The right square commutes by Lemma 2.7 which says that $\lambda_{A[1]}^* = -\lambda_A^*[1]$. The left square commutes since we have

$$f[1] \;=\; \overline{f[1]}[1] \circ \lambda_{A[1]} \;=\; -(\overline{f[1]} \circ \lambda_A)[1]$$

where the first equation is the defining property of $\overline{f[1]}$. So $(f,g,h)$ is isomorphic to a distinguished triangle, and thus itself distinguished.

(iii) If suffices to prove the claim for the elementary distinguished triangle associated to an injective fibration $\varphi : X \longrightarrow Y$ between injective $\Omega$-spectra. Shifting preserves fibres and injective fibrations. However, the two morphisms $\mathrm{sh}(\lambda_Y^*)$ and $\lambda^*_{\mathrm{sh}\,Y}$ are inverse to each other in the additive group $[\mathrm{sh}\,Y, \mathrm{sh}(\Omega(\mathrm{sh}\,Y))]$ by Lemma 2.8. This implies $\delta_{\mathrm{sh}\,\varphi} = -\,\mathrm{sh}(\delta_\varphi)$ and so in the stable homotopy category we have a commutative diagram

$$\begin{tikzcd}
\mathrm{sh}\,F \arrow[r, "-\,\mathrm{sh}(\mathrm{incl})"] \arrow[d, equal] & \mathrm{sh}\,X \arrow[r, "-\,\mathrm{sh}\,\varphi"] \arrow[d, "-\,\mathrm{Id}"'] & \mathrm{sh}\,Y \arrow[r, "-\,\mathrm{sh}(\delta_\varphi)"] \arrow[d, equal] & \mathrm{sh}^2\,F \arrow[d, equal] \\
F_{\mathrm{sh}\,\varphi} \arrow[r, "\mathrm{incl}"'] & \mathrm{sh}\,X \arrow[r, "\mathrm{sh}\,\varphi"'] & \mathrm{sh}\,Y \arrow[r, "\delta_{\mathrm{sh}\,\varphi}"'] & \mathrm{sh}\,F(\mathrm{sh}\,\varphi)
\end{tikzcd}$$

The lower row is the elementary distinguished triangle associated to the injective fibration $\mathrm{sh}\,\varphi : \mathrm{sh}\,X \longrightarrow \mathrm{sh}\,Y$. The upper row is thus distinguished. □

Now we can state and prove the main result of this section.

THEOREM 3.3. *The shift functor and the class of distinguished triangles make the stable homotopy category into a triangulated category.*

PROOF. We verify the axiom (T1), (T2) and (T3) as stated in Section 2 of Appendix A. These axioms seem weaker than Verdier's original axiom, but we recall in the appendix that they are in fact equivalent.

**(T1)** This axiom has three parts:
(a) For every injective $\Omega$-spectrum $X$ the unique morphism $X \longrightarrow *$ is an injective fibration and the triangle is $X \xrightarrow{\mathrm{Id}} X \longrightarrow 0 \longrightarrow X[1]$ the associated elementary distinguished triangle.
(b) Let $[\varphi] \in [X,Y]$ be a morphism in the stable homotopy category and let $\varphi : X \longrightarrow Y$ be a representing morphism between injective $\Omega$-spectra. By Lemma 3.1 (iv) the projection $p : F(\varphi) \longrightarrow X$ from the homotopy fibre of $\varphi$ to $X$ is an injective fibration between injective $\Omega$-spectrum, which thus has an associated elementary distinguished triangle $(i, p, \delta_p)$. By two applications of (T2) the lower row in the diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, equal] & Y \arrow[r, "-i[1]\circ\lambda_Y^*"] \arrow[d, "\lambda_Y^*"] & F(\varphi)[1] \arrow[r, "-p[1]"] \arrow[d, equal] & X[1] \arrow[d, equal] \\
X \arrow[r, "\delta_p"'] & \Omega Y[1] \arrow[r, "-i[1]"'] & F(\varphi)[1] \arrow[r, "-p[1]"'] & X[1]
\end{tikzcd}$$

is distinguished. Since the left square commutes in the stable homotopy category [...] the upper row is distinguished.

It is worth noting that the triangle which we get this way depends on the choice of representing morphism $\varphi$, and is thus not natural in the homotopy class $[\varphi]$. In fact, if $\varphi' : X \longrightarrow Y$ is homotopic to $\varphi$, then the homotopy fibre $F(\varphi)$ is homotopy equivalent to $F(\varphi')$, but any construction of such a homotopy equivalence involves a choice of homotopy between $\varphi$ and $\varphi'$. Different choices of homotopies will in general lead to different homotopy classes of homotopy equivalences.
(c) By definition, every triangle isomorphic to a distinguished triangle is itself distinguished.

**(T2)** If $(f,g,h)$ is a distinguished triangle, then the triangle $(-f[1], -g[1], -h[1])$ is distinguished by Proposition 3.2 (iii). Thus the triangle $(g, h, -f[1])$ is distinguished by two applications of part (ii) of Proposition 3.2.

**(T3)** The axiom is usually referred to as the *octahedral axiom.* Consider distinguished triangles $(f_1, g_1, h_1)$, $(f_2, g_2, h_2)$ and $(f_3, g_3, h_3)$ such that $f_1$ and $f_2$ are composable and $f_3 = f_2 f_1$. Then there exist morphisms $x$ and $y$ such that $(x, y, g_1[1] \circ h_2)$ is a distinguished triangle and the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"'] & C \arrow[r, "h_1"] \arrow[d, dotted, "x"] & A[1] \arrow[d, equal] \\
A \arrow[r, "f_3"'] & D \arrow[r, "g_3"'] \arrow[d, "g_2"'] & E \arrow[r, "h_3"'] \arrow[d, dotted, "y"] & A[1] \arrow[d, "f_1[1]"] \\
& F \arrow[r, equal] \arrow[d, "h_2"'] & F \arrow[r, "h_2"'] \arrow[d, "g_1[1]\circ h_2"] & B[1] \\
& B[1] \arrow[r, "g_1[1]"'] & C[1] &
\end{tikzcd}$$

We will use that for every distinguished triangle $(f, g, h)$ there exists an injective fibration $\varphi : X \longrightarrow B$ and isomorphisms $\alpha : A \longrightarrow X$ and $\gamma : C \longrightarrow F_\varphi$ in the stable homotopy category such that the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, equal] & C \arrow[r, "h"] \arrow[d, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\
X \arrow[r, "\varphi"'] & B \arrow[r, "\delta_\varphi"'] & F_\varphi[1] \arrow[r, "-i_\varphi[1]"'] & X[1]
\end{tikzcd}$$

commutes, where the bottom row is the rotation of the elementary distinguished triangle associated to $\varphi$. Indeed, the triangle $(-\bar{h}, f, \lambda_C^* \circ g)$ is distinguished by Proposition 3.2; so there is an isomorphism from $(-\bar{h}, f, \lambda_C^* \circ g)$ to an elementary distinguished triangle $(i, \varphi, \delta_\varphi)$, which we can rotate.

By the above we can assume without loss of generality that the triangle $(f_2, g_2, h_2)$ is the rotation of the elementary distinguished triangle $(\mathrm{incl}_\psi, \psi, \delta_\psi)$ associated to an injective fibration $\psi : Y \longrightarrow Z$. We can then assume that the triangle $(f_1, g_1, h_1)$ equals the rotation of $(\mathrm{incl}_\varphi, \varphi, \delta_\varphi)$ for another injective fibration $\varphi : X \longrightarrow Y$, where $X, Y$ and $Z$ are injective $\Omega$-spectra. The composite $\psi\varphi : X \longrightarrow Z$ is then again an injective fibration and we can finally assume that the triangle $(f_3, g_3, h_3)$ is the rotated elementary distinguished triangle of $\psi\varphi$. We are now in the situation of the following commutative diagram, where the two dotted morphisms have to be constructed:

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[d, equal] & Y \arrow[r, "\delta_\varphi"] \arrow[d, "\psi"'] & F_\varphi[1] \arrow[r, "-i_\varphi[1]"] \arrow[d, dotted, "\mathrm{incl}[1]"] & X[1] \arrow[d, equal] \\
X \arrow[r, "\psi\varphi"'] & Z \arrow[r, "\delta_{\psi\varphi}"'] \arrow[d, "\delta_\psi"'] & F_{\psi\varphi}[1] \arrow[r, "-i_{\psi\varphi}"'] \arrow[d, dotted, "\bar{\varphi}[1]"] & X[1] \arrow[d, "\varphi[1]"] \\
& F_\psi[1] \arrow[r, equal] \arrow[d, "-i_\psi[1]"'] & F_\psi[1] \arrow[r, "-i_\psi[1]"'] \arrow[d, "-\delta_{\bar{\varphi}}[1]"] & Y[1] \\
& Y[1] \arrow[r, "\delta_\varphi[1]"'] & F_\varphi[2] &
\end{tikzcd}$$

We can fill in the required morphisms in the third row taking the shift of the inclusion $F_\varphi \longrightarrow F_{\psi\varphi}$ respectively the restriction of $\varphi$ to the strict fibre of $\psi\varphi : X \longrightarrow Z$, which we denote by $\bar{\varphi} : F_{\psi\varphi} \longrightarrow F_\psi$.

With these fillers the four squares involving the spectrum $F_{\psi\varphi}$ commute because the construction of the triangle from an injective fibration is natural. In order to verify the octahedral axiom (T3) it remains to show that the third row is a distinguished triangle. The restriction $\bar{\varphi} : F_{\psi\varphi} \longrightarrow F_\psi$ of the injective fibration $\varphi$ is again an injective fibration, and the strict fibres of $\varphi : X \longrightarrow Y$ and its restriction $\bar{\varphi}$ are equal. So the triangle

$$F_\varphi \xrightarrow{\text{incl}} F_{\psi\varphi} \xrightarrow{\bar{\varphi}} F_\psi \xrightarrow{\delta_\varphi} F_\varphi[1]$$

is elementary distinguished; by part (iii) of Proposition 3.2, and closure under isomorphisms (to get rid of two signs) the triangle

$$F_\varphi[1] \xrightarrow{\text{incl}[1]} F_{\psi\varphi}[1] \xrightarrow{\bar{\varphi}[1]} F_\psi[1] \xrightarrow{-\delta_\varphi[1]} F_\varphi[2]$$

is distinguished. The relation $\delta_{\bar{\varphi}} = \delta_\varphi i_\psi$ holds in the group $[F_\psi, F_\varphi[1]]$ by another instance of the naturality of elementary distinguished triangles in the injective fibration defining it. □

## 4. Stable equivalences

By our definition, only injective $\Omega$-spectra are objects of the stable homotopy category. However, many constructions which one can perform with symmetric spectra do not preserve the property of being an injective $\Omega$-spectra, so it would be convenient if we can regard arbitrary symmetric spectra as objects of the stable homotopy category. In this section we introduce the notion of *stable equivalence* and show that up to stable equivalence, every symmetric spectrum can be replaced by an injective $\Omega$-spectrum. The ultimate consequence will be that the stable homotopy category arises as the localization of the category of symmetric spectra obtained by 'inverting stable equivalences', compare Theorem 4.12.

Definition 4.1. A morphism $f : A \longrightarrow B$ of symmetric spectra of simplicial sets is a *stable equivalence* if for every injective $\Omega$-spectrum $X$ the induced map

$$[f, X] \; : \; [B, X] \; \longrightarrow \; [A, X]$$

on homotopy classes of spectrum morphisms is a bijection.

In Theorem 4.3 below we give a list of several equivalent characterizations of stable equivalences. In Proposition 4.5 we prove that stable equivalences are closed under various operations. In our new language, Theorem 1.9 says that every $\pi_*$-isomorphisms of symmetric spectra is a stable equivalence; the converse does not in general hold, as Example 4.2 below shows. For morphisms of symmetric spectra we thus have the implications

$$\text{homotopy equivalence} \implies \text{level equivalence} \implies \pi_*\text{-isomorphism} \implies \text{stable equivalence.}$$

Example 4.2. While every $\pi_*$-isomorphism of symmetric spectra is a stable equivalence, the converse is not true. The standard example of this phenomenon is the following: consider the symmetric spectrum $F_1S^1$ freely generated by the circle $S^1$ in level 1, compare Example I. 2.12. We consider the morphism $\lambda : F_1S^1 \longrightarrow F_0S^0 = \mathbb{S}$ which is adjoint to the identity in level 1.

Since $F_1S^1$ is freely generated by the circle $S^1$ in level 1, it ought to be a desuspension of the suspension spectrum of the circle. And indeed, the morphism $\lambda : F_1S^1 \longrightarrow F_0S^0 = \mathbb{S}$ is a stable equivalence. To see this we consider an injective $\Omega$-spectrum $X$ and consider the commutative square

$$\begin{tikzcd}
{[\mathbb{S}, X]} \arrow[r, "{[\lambda, X]}"] \arrow[d, "\mathrm{eval}_0"', "\cong"] & {[F_1S^1, X]} \arrow[d, "\cong"', "\mathrm{eval}_1"] \\
{[S^0, X_0]_{\mathbf{sset}_*} = \pi_0 X_0} \arrow[r, "\iota_*"'] & {\pi_1 X_1 = [S^1, X_1]_{\mathbf{sset}_*}}
\end{tikzcd}$$

The vertical maps given by evaluation at levels 0 respectively 1 are adjunction bijections and the lower horizontal map is the stabilization map. Since $X$ is an $\Omega$-spectrum, the lower vertical map is a bijection, hence the upper vertical map is, which proves that $\lambda$ is a stable equivalence.

However, we calculated the 0th homotopy group of the free symmetric spectrum $F_1S^1$ in Example I.2.12 and, more systematically, in Example I.4.31. The group $\pi_0 F_1S^1$ is isomorphic to the $\mathcal{M}$-module $\mathcal{P}_1$ and in particular free abelian of countably infinite rank. But $\pi_0\mathbb{S}$ is free abelian of rank one, so $\lambda$ is not a $\pi_*$-isomorphism.

THEOREM 4.3. *The following are equivalent for a morphism $f : A \longrightarrow B$ of symmetric spectra of simplicial sets:*

(i) *$f$ is a stable equivalence;*
(ii) *for every injective $\Omega$-spectrum $X$ the induced map* $\mathrm{map}(f,X) : \mathrm{map}(B,X) \longrightarrow \mathrm{map}(A,X)$ *is a weak equivalence of simplicial sets;*
(iii) *for every injective $\Omega$-spectrum $X$ the induced map* $\mathrm{Hom}(f,X) : \mathrm{Hom}(B,X) \longrightarrow \mathrm{Hom}(A,X)$ *is a level equivalence of symmetric spectra;*
(iv) *the mapping cone $C(f)$ of $f$ is stably equivalent to the trivial spectrum.*
(v) *the suspension $S^1 \wedge f : S^1 \wedge A \longrightarrow S^1 \wedge B$ is a stable equivalence;*

*If $A$ and $B$ are levelwise Kan complexes, conditions* (i)-(v) *are furthermore equivalent to*

(vi) *the homotopy fibre $F(f)$ of $f$ is stably equivalent to the trivial spectrum;*
(vii) *the loop $\Omega f : \Omega A \longrightarrow \Omega B$ is a stable equivalence.*

PROOF. (i)$\Rightarrow$(ii) For every simplicial set $K$ and every injective $\Omega$-spectrum $X$ the function spectrum $X^K$ is again injective by Lemma 3.1 (iii) and an $\Omega$-spectrum by Example I.2.22. We have an adjunction bijection $[K, \mathrm{map}(A,X)] \cong [A, X^K]$ where the left hand side means homotopy classes of morphisms of simplicial sets. So if $f$ is a stable equivalence, then $[f, X^K]$ is bijective, hence $[K, \mathrm{map}(f,X)] : [K, \mathrm{map}(B,X)] \longrightarrow [K, \mathrm{map}(B,X)]$ is bijective. Since this holds for all simplicial sets $K$, $\mathrm{map}(f,X)$ is a weak equivalence.

(ii)$\Rightarrow$(iii) For every injective $\Omega$-spectrum $X$ and $n \geq 0$ the shifted spectrum $\mathrm{sh}^n X$ is again an injective $\Omega$-spectrum. So if $f : A \longrightarrow B$ satisfies (ii), it also satisfies (iii) since the $n$th level of the spectrum $\mathrm{Hom}(A,X)$ is defined as $\mathrm{map}(A, \mathrm{sh}^n X)$.

(iii)$\Rightarrow$(iv) The morphism spectrum from the mapping cone $C(f)$ to a symmetric spectrum $X$ is naturally isomorphic to the homotopy fibre of the morphism $\mathrm{Hom}(f,X)$, i.e.,

$$\mathrm{Hom}(C(f),X) \;\cong\; F(\mathrm{Hom}(f,X) : \mathrm{Hom}(B,X) \longrightarrow \mathrm{Hom}(A,X)) \;.$$

So if (iii) holds then for every injective $\Omega$-spectrum $X$ the morphism $\mathrm{Hom}(f,X)$ is a level equivalence, thus the homotopy fibre $F(\mathrm{Hom}(f,X))$ is levelwise contractible. So the isomorphic spectrum $\mathrm{Hom}(C(f),X)$ is levelwise contractible and in particular $[C(f),X] \cong \pi_0 \mathrm{Hom}(C(f),X)_0$ has only one element. So the mapping cone $C(f)$ is stably equivalent to the trivial spectrum.

(iv)$\Rightarrow$(v) If $p : C(f) \longrightarrow S^1 \wedge A$ denotes the morphism which collapses $B$ to a point then the mapping cone of $p$ is homotopy equivalent to $\Sigma B$. More precisely there is a diagram of symmetric spectra

$$\begin{tikzcd} C(f) \arrow[r] \arrow[dr, "p"'] & Z(p) \arrow[r] \arrow[d, "\simeq"'] & C(p) \arrow[d, "\simeq"] \\ & S^1 \wedge A \arrow[r, "S^1\wedge f"'] & S^1 \wedge B \end{tikzcd}$$

in which the triangle commutes and the square commutes up to homotopy. If condition (iv) holds then by the already established implication '(i)$\Longrightarrow$(ii)' the simplicial set $\mathrm{map}(C(f),X)$ is weakly contractible for every injective $\Omega$-spectrum $X$. We have a homotopy fibre sequence

$$\mathrm{map}(C(p),X) \to \mathrm{map}(Z(p),X) \to \mathrm{map}(C(f),X)$$

whose last term is weakly contractible. So the first morphism is a weak equivalence. Since it is homotopy equivalent to $\mathrm{map}(S^1 \wedge f, X)$, that morphism is a weak equivalence. Taking path components shows that $[S^1 \wedge f, X] : [S^1 \wedge B, X] \longrightarrow [S^1 \wedge A, X]$ is bijective for every injective $\Omega$-spectrum $X$, so that $S^1 \wedge f$ is a stable equivalence.

(v)$\Rightarrow$(i) Suppose that the suspension $S^1 \wedge f$ is a stable equivalence and let $X$ be an injective $\Omega$-spectrum. Then the shifted spectrum $\mathrm{sh}\, X$ is also an injective $\Omega$-spectrum, so the map $[S^1 \wedge f, \mathrm{sh}\, X] : [S^1 \wedge B, \mathrm{sh}\, X] \longrightarrow [S^1 \wedge A, \mathrm{sh}\, X]$ is bijective. By adjunction, the map $[f, \Omega(\mathrm{sh}\, X)] : [B, \Omega(\mathrm{sh}\, X)] \longrightarrow [A, \Omega(\mathrm{sh}\, X)]$ is then also bijective. Since the map $\lambda^* : X \longrightarrow \Omega(\mathrm{sh}\, X)$ is a homotopy equivalence, it follows that $f$ induces a bijection on homotopy classes of morphisms to $X$. So $f$ is a stable equivalence.

(iv)$\Leftrightarrow$(vi) We assume that $A$ and $B$ are levelwise Kan complexes. The suspension of the homotopy fibre is $\pi_*$-isomorphic, hence stably equivalent, to the mapping cone. By the equivalence of conditions (i) and (v), a symmetric spectrum is stably trivial if and only if its suspension is. So (iv) and (vi) are equivalent.

(iv)$\Leftrightarrow$(vi) If $A$ is levelwise Kan, then the adjunction counit $S^1 \wedge \Omega A \longrightarrow A$ is a $\pi_*$-isomorphism, thus a stable equivalence. So $f$ is a stable equivalence if and only if $S^1 \wedge \Omega(f)$ is. By the already established equivalence between conditions (i) and (v) this is equivalent to $\Omega f$ being a stable equivalence. $\square$

For the next proposition we recall that a commutative square of simplicial sets

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[d, "\varphi"'] & W \arrow[d, "\psi"] \\ X \arrow[r, "\beta"'] & Y \end{tikzcd}$$

is called *homotopy cartesian* if for some (hence any) factorization of the morphism $g$ as the composite of a weak equivalence $w : W \longrightarrow Z$ followed by a Kan fibration $f : Z \longrightarrow Y$ the induced morphism

$$V \xrightarrow{(\varphi, w\alpha)} X \times_Y Z$$

is a weak equivalence. The definition is in fact symmetric in the sense that the square is homotopy cartesian if and only if the square obtained by interchanging $X$ and $W$ (and the morphisms) is homotopy cartesian. So if the square is homotopy cartesian and $\psi$ (respectively $\beta$) is a weak equivalence, then so is $\varphi$ (respectively $\alpha$).

Proposition 4.4. *Consider a pullback square of symmetric spectra of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$

*in which the morphism $g$ is levelwise a Kan fibration. Then for every injective $\Omega$-spectrum $X$ the commutative square of simplicial sets*

$$\begin{tikzcd} \mathrm{map}(D,X) \arrow[r, "\mathrm{map}(g,X)"] \arrow[d, "\mathrm{map}(j,X)"'] & \mathrm{map}(B,X) \arrow[d, "\mathrm{map}(i,X)"] \\ \mathrm{map}(C,X) \arrow[r, "\mathrm{map}(f,X)"'] & \mathrm{map}(A,X) \end{tikzcd}$$

*is homotopy cartesian.*

Proof. In a first step we prove the proposition under the additional assumption that the morphism $j : C \longrightarrow D$ is a monomorphism. This implies that its basechange $i : A \longrightarrow B$ is also a monomorphism. In this situation the morphism $j \cup g : C \cup_A B \longrightarrow D$ is a $\pi_*$-isomorphism [ref], thus a weak equivalence by

Theorem 1.9. We fix an injective $\Omega$-spectrum $X$ and consider the commutative diagram of simplicial sets

$$\begin{tikzcd}
\mathrm{map}(D,X) \arrow[rr, "\mathrm{map}(g,X)"] \arrow[dr, "\mathrm{map}(f\cup i,X)"] \arrow[dd, "\mathrm{map}(j,X)"'] & & \mathrm{map}(B,X) \arrow[dd, "\mathrm{map}(i,X)"] \\
& \mathrm{map}(C\cup_A B,X) \arrow[ur] \arrow[dl] & \\
\mathrm{map}(C,X) \arrow[rr, "\mathrm{map}(f,X)"'] & & \mathrm{map}(A,X)
\end{tikzcd}$$

The right vertical map $\mathrm{map}(i,X)$ is a Kan fibration by Proposition 1.6 (i) and the lower right part of the diagram is a pullback. The morphism $\mathrm{map}(f\cup i,X)$ is a weak equivalence, so the outer commutative square is homotopy cartesian.

Now we prove the general case. We factor the morphism $j$ as the mapping cylinder inclusion $C \longrightarrow Z(j)$ followed by the projection $p : Z(j) \longrightarrow D$ which is a homotopy equivalence. Then the square decomposes as the composite of two pullback squares

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & Z(j)\times_D B \arrow[r, "\tilde{p}"] \arrow[d] & B \arrow[d, "g"] \\
C \arrow[r] & Z(j) \arrow[r, "p"'] & D
\end{tikzcd}$$

For every injective $\Omega$-spectrum $X$ the functor $\mathrm{map}(-,X)$ takes the left pullback square to a homotopy cartesian square of simplicial sets by the special case above. The projection $p$ is a level equivalence, hence so is its basechange $\tilde{p}$. So both $p$ and $\tilde{p}$ become weak equivalences after applying $\mathrm{map}(-,X)$ and the functor $\mathrm{map}(-,X)$ also takes the right pullback square to a homotopy cartesian square. The composite of two homotopy cartesian squares is homotopy cartesian, which proves the claim. $\square$

PROPOSITION 4.5. (i) *A wedge of stable equivalences is a stable equivalence.*

(ii) *A finite product of stable equivalences is a stable equivalence.*

(iii) *Consider a commutative square of morphisms of symmetric spectra*

$$\begin{tikzcd}
A \arrow[r, "\varphi_A"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "\varphi_B"'] & B'
\end{tikzcd}$$

*and let $f'' : C(\varphi_A) \longrightarrow C(\varphi_B)$ be the map induced by $f$ and $f'$ on mapping cones. Then if two of the three morphisms $f$, $f'$ and $f''$ are stable equivalences, so is the third.*

(iv) *Consider a commutative of symmetric spectra of simplicial sets*

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "j"'] & D
\end{tikzcd}$$

*If the square is a pushout, $f$ a stable equivalence and $i$ or $f$ is injective, then $g$ is a stable equivalence. If the square is a pullback, $g$ a stable equivalence and $j$ or $g$ is levelwise a Kan fibration, then $f$ is a stable equivalence.*

(v) *For every stable equivalence $f : A \longrightarrow B$ and every pointed simplicial set $K$ the morphism $K\wedge f : K\wedge A \longrightarrow K\wedge B$ is a stable equivalence. If $A$ and $B$ are levelwise Kan complexes and $K$ is finite, then the morphism $f^K : A^K \longrightarrow B^K$ is a stable equivalence.*

(vi) *Let $I$ be a filtered category. If $A : I \longrightarrow \mathcal{S}p^\Sigma$ is a functor which takes every morphism to an injective stable equivalence, then for every object $i \in I$ the canonical morphism $A(i) \longrightarrow \operatorname{colim}_I A$ is an injective stable equivalence.*

(vii) *Let $I$ be a filtered category and let $A, B : I \longrightarrow \mathcal{S}p^\Sigma$ be functors which take all morphisms in $I$ to monomorphisms of symmetric spectra. If $\tau : A \longrightarrow B$ is a natural transformation such that $\tau(i) : A(i) \longrightarrow B(i)$ is a stable equivalence for every object $i$ of $I$, then the induced morphism $\operatorname{colim}_I \tau : \operatorname{colim}_I A \longrightarrow \operatorname{colim}_I B$ on colimits is a stable equivalence.*

[is the diagonal of stable equivalences a stable equivalence ?]

Proof. (i) For every family $\{A_i\}_{i\in I}$ of symmetric spectra and every injective $\Omega$-spectrum $X$ that natural map

$$[\bigvee_{i\in I} A_i, X] \longrightarrow \prod_{i\in I} [A_i, X]$$

is bijective by the universal property of the wedge, applied to morphisms and homotopies.

(ii) Finite products are $\pi_*$-isomorphic to finite wedges (see Theorem 2.1), and $\pi_*$-isomorphisms are stable equivalences (see Theorem 1.9), so the claim follows from part (i).

(iii) We show that if $f$ and $f'$ are stable equivalences, then so if $f''$. The other cases are similar.

For every injective $\Omega$-spectrum $X$ we have a commutative diagram [explain]

$$\begin{tikzcd}
{[\Sigma A', X]} \arrow[r, "{[\Sigma\varphi_A, X]}"] \arrow[d, "{[\Sigma f', X]}"'] & {[\Sigma A, X]} \arrow[r, "{[p,X]}"] \arrow[d, "{[\Sigma f, X]}"'] & {[C(\varphi_A), X]} \arrow[r, "{[i,X]}"] \arrow[d, "{[f'', X]}"'] & {[A', X]} \arrow[r, "{[\varphi_A, X]}"] \arrow[d, "{[f', X]}"] & {[A, X]} \arrow[d, "{[f, X]}"] \\
{[\Sigma B', X]} \arrow[r, "{[\Sigma\varphi_B, X]}"'] & {[\Sigma B, X]} \arrow[r, "{[p,X]}"'] & {[C(\varphi_B), X]} \arrow[r, "{[i,X]}"'] & {[B', X]} \arrow[r, "{[\varphi_B, X]}"'] & {[B, X]}
\end{tikzcd}$$

in which both rows are long exact sequences of abelian groups. If $f$ and $f'$ are stable equivalences, then all except possibly the middle vertical maps are bijective. So the middle map is bijective by the 5-lemma. [easier to using mapping spaces into $X$?]

(iv) Let us first consider the case of a pushout square with $f$ a stable equivalence and $i$ or $f$ injective. For every injective $\Omega$-spectrum $X$ the commutative square of simplicial sets

(4.6)
$$\begin{tikzcd}
\operatorname{map}(D, X) \arrow[r, "{\operatorname{map}(g,X)}"] \arrow[d, "{\operatorname{map}(j,X)}"'] & \operatorname{map}(B, X) \arrow[d, "{\operatorname{map}(i,X)}"] \\
\operatorname{map}(C, X) \arrow[r, "{\operatorname{map}(f,X)}"'] & \operatorname{map}(A, X)
\end{tikzcd}$$

is then a pullback and at least one of the maps $\operatorname{map}(i, X)$ or $\operatorname{map}(f, X)$ is a Kan fibration by Proposition 1.6 (i). Since moreover the lower horizontal map is a weak equivalence (by Theorem 4.3 (ii)), so is the upper horizontal map. Again by Theorem 4.3 (ii) this means that $g$ is a stable equivalence.

Now we consider the case of a pullback square in which one of the morphisms $j$ or $g$ is levelwise a Kan fibration. By Proposition 4.4 the commutative square (4.6) is then homotopy cartesian. If $g$ is a stable equivalence, then the upper horizontal map is a weak equivalence (by Theorem 4.3 (ii)). This means that the lower horizontal maps induces an isomorphism of homotopy groups in positive dimensions and an injection of path components. If we replace $X$ by the injective $\Omega$-spectrum $\operatorname{sh} X$ we deduce that $\operatorname{map}(f, \operatorname{sh} X) : \operatorname{map}(C, \operatorname{sh} X) \longrightarrow \operatorname{map}(A, \operatorname{sh} X)$ induces an isomorphism on fundamental groups and so $\Omega(\operatorname{map}(f, \operatorname{sh} X)) \cong \operatorname{map}(f, \Omega(\operatorname{sh} X))$ induces a bijection on path components. Since the simplicial set $\operatorname{map}(A, X)$ is naturally weakly equivalent to $\operatorname{map}(A, \Omega(\operatorname{sh} X))$, this proves that $\operatorname{map}(f, X)$ is a weak equivalence for all injective $\Omega$-spectra $X$. Again by Theorem 4.3 this means that $f$ is a stable equivalence.

(v) The first statement follows from the adjunction bijection $[K \wedge A, X] \cong [A, X^K]$ and the fact that $X^K$ is an injective $\Omega$-spectrum whenever $K$ is. For the second statement we observe that the functor $A \mapsto A^K$ commutes with homotopy fibres and preserves the property of being levelwise Kan. So passage to homotopy fibres and the equivalence of conditions (i) and (vi) in Theorem 4.3 reduce us to showing that if $A$ is levelwise Kan and stably contractible, then so is $A^K$. We prove this by induction over the dimension of $K$. If $K$ is 0-dimensional, then $A^K$ is a finite product of copies of $A$, this stably contractible by part (ii). Now suppose that $K$ has positive dimension $n$. We do another induction on the number of non-degenerate $n$-simplices of $K$. We write $K = L \cup_{\partial\Delta[n]^+} \Delta[n]^+$ for a simplicial subset $L$ with one non-degenerate $n$-simplex less than $K$. We obtain pullback square of symmetric function spectra

$$\begin{tikzcd} A^K \arrow[r] \arrow[d] & A^L \arrow[d] \\ A^{\Delta[n]^+} \arrow[r] & A^{\partial\Delta[n]^+} \end{tikzcd}$$

in which the horizontal morphisms are restrictions, thus levelwise Kan fibrations. The spectrum $A^{\partial\Delta[n]^+}$ is stably contractible since $\partial\Delta[n]^+$ has smaller dimension. Since $\Delta[n]^+$ is weakly equivalent to $S^0$ the spectrum $A^{\Delta[n]^+}$ is level equivalent to $A^{S^0} \cong A$, thus stably contractible. In particular, the lower horizontal morphism is a stable equivalence, hence so is the upper one by part (iv). Since $A^L$ is stably contractible by induction, so is $A^K$.

(vi) For every injective $\Omega$-spectrum $X$ the simplicial set $\mathrm{map}(\mathrm{colim}_I A, X)$ is isomorphic to the inverse limit of the functor $\mathrm{map}(A, X) : I^{op} \longrightarrow \mathbf{sset}_*$. Since $A$ consists of injective stable equivalences, each morphism in the inverse system $\mathrm{map}(A, X)$ is a weak equivalence (by Theorem 4.3 (ii)) and a Kan fibration (by Proposition 1.6 (i)). Since $I$ is filtered, the map from the inverse limit $\mathrm{map}(\mathrm{colim}_I A, X)$ to each stage $\mathrm{map}(A(i), X)$ is then also a weak equivalence of simplicial set, which means that $A(i) \longrightarrow \mathrm{colim}_I A$ is a stable equivalence.

(vii) For every injective $\Omega$-spectrum $X$ the simplicial set $\mathrm{map}(\mathrm{colim}_I A, X)$ is isomorphic to the inverse limit of the functor $\mathrm{map}(A, X) : I^{op} \longrightarrow \mathbf{sset}_*$, and similarly for the functor $B$. Since $A$ and $B$ consists of injective morphisms, all morphisms in the inverse systems $\mathrm{map}(A, X)$ and $\mathrm{map}(B, X)$ are Kan fibrations (by Proposition 1.6 (i)). Filtered inverse limits of weak equivalences along Kan fibrations are again weak equivalences, so the map $\mathrm{map}(\mathrm{colim}_I B, X) \longrightarrow \mathrm{map}(\mathrm{colim}_I A, X)$ is a weak equivalence of simplicial set, which means that $\mathrm{colim}_I A \longrightarrow \mathrm{colim}_I B$ is a stable equivalence. □

Now that we mentioned many constructions which preserves stable equivalences we also mention one which does not, namely shifting (this should be contrasted with the fact that shifting does preserve $\pi_*$-isomorphisms because $\pi_{k+1}(\mathrm{sh}\, X)$ equals $\pi_k X$ as abelian groups). An example is the fundamental stable equivalence $\lambda : F_1 S^1 \longrightarrow \mathbb{S}$ of Example 4.2 which is adjoint to the identity of $S^1$. The symmetric spectrum $\mathrm{sh}(F_1 S^1)$ is isomorphic to the wedge of $F_0 S^1$ and $F_1 S^2$, while $\mathrm{sh}\,\mathbb{S} \cong F_0 S^1$; the map $\mathrm{sh}\,\lambda : \mathrm{sh}(F_1 S^1) \longrightarrow \mathrm{sh}\,\mathbb{S}$ is the projection to the wedge summand. The complementary summand $F_1 S^2 \cong S^1 \wedge F_1 S^1$ is stably equivalent, via the suspension of $\lambda$, to $S^1 \wedge \mathbb{S} \cong \Sigma^\infty S^1$, and is thus not stably contractible.

**4.1. The stable homotopy category as a localization.** We recall that a general symmetric spectrum is *not* $\pi_*$-isomorphic to an injective $\Omega$-spectrum. By Theorem I.4.44 a necessary (and in fact sufficient) condition for that is semistability. In contrast, we will now see that every symmetric spectrum is stably equivalent to an injective $\Omega$-spectrum. This is the key point in constructing a functor from the category of symmetric spectra to the stable homotopy category, thus associating a 'stable homotopy type' to every symmetric spectrum. In the language of model structures which we will discuss later, this amounts to the existence of fibrant replacements in the injective stable model structure.

THEOREM 4.7. *For every symmetric spectrum $A$ there exists a stable equivalence from $A$ to an injective $\Omega$-spectrum.*

We postpone the technical proof to the end of this section, where we give a functorial construction, using the 'small object argument'.

CONSTRUCTION 4.8. We now construct a functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$ which by Theorem 4.12 below is a localization of the category of symmetric spectra at the class of stable equivalences. First, for each symmetric spectrum $A$ we choose a stable equivalence $p_A : A \longrightarrow \gamma A$ with target an injective $\Omega$-spectrum, which is possible by Theorem 4.7. We insist that if $A$ is already an injective $\Omega$-spectrum, then $\gamma A = A$ and $p_A$ is the identity. This is not really necessary, but will make some arguments slightly easier.

There is then a preferred way to make these choices into the object part of functor: given a morphism $f : A \longrightarrow B$ of symmetric spectra, we consider the diagram of morphisms of symmetric spectra (without the dotted morphism)

(4.9)
$$\begin{tikzcd} A \arrow[r, "p_A"] \arrow[d, "f"'] & \gamma A \arrow[d, dotted, "\gamma f"] \\ B \arrow[r, "p_B"'] & \gamma B \end{tikzcd}$$

By definition of 'stable equivalence' the map $[p_A, \gamma B] : [\gamma A, \gamma B] \longrightarrow [A, \gamma B]$ is bijective. Thus there exists a unique homotopy class of morphism $\gamma f : \gamma A \longrightarrow \gamma B$ which makes the entire square commute up to homotopy.

Functoriality is a formal consequence of the uniqueness property. Since $\mathrm{Id}_{\gamma A}\, p_A = p_A$, the uniqueness of the filler guarantees that $\gamma(\mathrm{Id}_A) = \mathrm{Id}_{\gamma A}$. If $g : B \longrightarrow C$ is another morphism, then $\gamma(g)\gamma(f)p_A \simeq \gamma(g)p_B f \simeq p_C(gf)$. By uniqueness we conclude that $\gamma(g)\gamma(f) = \gamma(gf)$ as homotopy classes in $[\gamma A, \gamma C]$.

Remark 4.10. The choice $p_A : A \longrightarrow \gamma A$ of stable equivalence to an injective $\Omega$-spectrum could in fact be made functorially at the pointset level (and not just up to homotopy), see Proposition 4.21. However, if we want this extra functoriality, we cannot simultaneously arrange things so that $\gamma A = A$ if $A$ is already an injective $\Omega$-spectra. The pointset level functoriality of $\gamma$ is irrelevant for the current discussion, and so we continue without it.

The next theorem says that the functor $\gamma : \mathcal{S}p^{\Sigma} \longrightarrow \mathcal{SHC}$ is a localization of the category of symmetric spectra at the class of stable equivalence. Since the 'collection' of all categories forms a 2-category (with respect to functors as morphisms and natural transformations), one should not expect such localizations to be unique up to isomorphism, but rather only unique up to equivalence. The following definition captures the 'correct' universal property of such a 2-categorical localization. We use the following notation: if $\mathcal{C}$ and $\mathcal{D}$ are categories, then $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$ is the category whose objects are the functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations [when is this a category, i.e., small Hom-sets?].

Definition 4.11. Let $\mathcal{C}$ be a category and $W$ a class of morphisms in $\mathcal{C}$. A *localization* of $\mathcal{C}$ at the class $W$ is a functor $L : \mathcal{C} \longrightarrow \mathcal{D}$ with the following two properties:

- The functor $L$ takes all morphisms in $W$ to isomorphisms in $\mathcal{D}$.
- For every category $\mathcal{E}$, precomposition with the functor $L$ induces an equivalence of categories
$$- \circ L \;:\; \mathrm{Hom}(\mathcal{D}, \mathcal{E}) \;\longrightarrow\; \mathrm{Hom}^{W}(\mathcal{C}, \mathcal{E})$$
where the target is the full subcategory of $\mathrm{Hom}(\mathcal{C}, \mathcal{E})$ of functors which take all morphisms in $W$ to isomorphisms.

A direct consequence of the definition of 'localization' is the following. If $L : \mathcal{C} \longrightarrow \mathcal{D}$ is a localization of $\mathcal{C}$ and $W$ then for every functor $\varepsilon : \mathcal{D} \longrightarrow \mathcal{E}$ which takes $W$ to isomorphisms, there exists a functor $\bar{\varepsilon} : \mathcal{C} \longrightarrow \mathcal{E}$, unique up to natural isomorphism, such that $\bar{\varepsilon} \circ L$ is naturally isomorphic to $\varepsilon$. The existence part is the statement that $- \circ L$ as in Definition 4.11 is essentially surjective on objects (i.e., functors); the uniqueness part is the statement that $- \circ L$ is fully faithful.

The universal property of a localization $L : \mathcal{C} \longrightarrow \mathcal{D}$ makes localizations unique up to equivalence, whenever they exist. Indeed, if $L' : \mathcal{C} \longrightarrow \mathcal{D}'$ is another localization, then there exist functors $F : \mathcal{D} \longrightarrow \mathcal{D}'$ and $G : \mathcal{D}' \longrightarrow \mathcal{D}$ such that $F \circ L$ is naturally isomorphic to $L'$ and $G \circ L'$ is naturally isomorphic to $L$. Then $G \circ F \circ L$ is naturally isomorphic to $L$ and by the 'fully faithful' part of the universal property, any such natural isomorphism is of the form $\tau \circ L$ for a unique natural isomorphism $\tau : G \circ F \cong \mathrm{Id}_{\mathcal{D}}$. Similarly, $F \circ G$ is naturally isomorphic to the identity functor of $\mathcal{D}'$.

THEOREM 4.12. *The functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$ is a localization of the category of symmetric spectra at the class of stable equivalences. In particular, for every functor $\varepsilon : \mathcal{S}p^\Sigma \longrightarrow \mathcal{C}$ which takes stable equivalences to isomorphisms, then there exists a functor $\overline{\varepsilon} : \mathcal{SHC} \longrightarrow \mathcal{C}$, unique up to preferred natural isomorphism, such that $\overline{\varepsilon} \circ \gamma$ is naturally isomorphic to $\varepsilon$.*

PROOF. We start by showing that $\gamma$ takes stable equivalences to isomorphisms. By definition of $\gamma f$ we have the homotopy commutative diagram (4.9). So if $f : A \longrightarrow B$ is a stable equivalence, then $\gamma f$ is a stable equivalence between injective $\Omega$-spectra and therefore a homotopy equivalence. In other words, $\gamma f$ is an isomorphism in $\mathcal{SHC}$.

Next make some observations about functors $\varepsilon : \mathcal{S}p^\Sigma \longrightarrow \mathcal{E}$ which invert stable equivalences. For every spectrum $A$, the projection $\pi : \Delta[1]^+ \wedge A \longrightarrow A$ is a homotopy equivalence, hence a stable equivalence, so $\varepsilon(\pi)$ is an isomorphism. The two end inclusions $i_0, i_1 : A \longrightarrow \Delta[1]^+ \wedge A$ satisfy $\pi \circ i_0 = \mathrm{Id}_A = \pi \circ i_1$, so we have

$$\varepsilon(\pi) \circ \varepsilon(i_0) \ = \ \mathrm{Id}_{\varepsilon(A)} \ = \ \varepsilon(\pi) \circ \varepsilon(i_1) \ .$$

Since $\varepsilon(\pi)$ is an isomorphism, we deduce $\varepsilon(i_0) = \varepsilon(i_1)$.

Suppose now that $f, g : A \longrightarrow B$ are homotopic morphisms via some homotopy $H : \Delta[1]^+ \wedge A \longrightarrow B$. Then

$$\varepsilon(f) \ = \ \varepsilon(H) \circ \varepsilon(i_0) \ = \ \varepsilon(H) \circ \varepsilon(i_1) \ = \ \varepsilon(g) \ .$$

In other words, every functor $\varepsilon : \mathcal{S}p^\Sigma \longrightarrow \mathcal{E}$ which takes stable equivalences to isomorphisms also takes homotopic maps to the same morphisms.

Now we show that for every functor $\varepsilon : \mathcal{S}p^\Sigma \longrightarrow \mathcal{E}$ which takes stable equivalences to isomorphisms there is a functor $\overline{\varepsilon} : \mathcal{SHC} \longrightarrow \mathcal{E}$ such that $\overline{\varepsilon}\gamma$ is naturally isomorphic to $\varepsilon$. This proves that the functor $- \circ \gamma : \mathrm{Hom}(\mathcal{SHC}, \mathcal{E}) \longrightarrow \mathrm{Hom}^{\text{st. equi.}}(\mathcal{S}p^\Sigma, \mathcal{E})$ is dense (essentially surjective on objects). We simply define $\overline{\varepsilon} : \mathcal{SHC} \longrightarrow \mathcal{E}$ on objects by $\overline{\varepsilon}(A) = \varepsilon(A)$ and on morphisms via representatives by $\overline{\varepsilon}[f : A \longrightarrow B] = \varepsilon(f)$. This will automatically be a functor. If we apply the functor $\varepsilon$ to the stable equivalence $p_A : A \longrightarrow \gamma A$ we get a natural isomorphism in $\mathcal{E}$

$$\varepsilon(p_A) \ : \ \varepsilon(A) \ \longrightarrow \ \varepsilon(\gamma A) = (\overline{\varepsilon}\gamma)(A) \ .$$

It remains to show that precomposition with $\gamma$ is fully faithful. So we consider two functors $F, G : \mathcal{SHC} \longrightarrow \mathcal{E}$ and have to show that

$$- \circ \gamma \ : \ \mathrm{Nat}(F, G) \ \longrightarrow \ \mathrm{Nat}(F\gamma, G\gamma)$$

is bijective. We define the inverse map $K : \mathrm{Nat}(F\gamma, G\gamma) \longrightarrow \mathrm{Nat}(F, G)$ as follows. Given a natural transformation $\tau : F\gamma \longrightarrow G\gamma$ of functors $\mathcal{S}p^\Sigma \longrightarrow \mathcal{E}$ we define the natural transformation $K(\tau) : F \longrightarrow G$ of functors $\mathcal{SHC} \longrightarrow \mathcal{E}$ as the restriction of $\tau$ to injective $\Omega$-spectra. This makes sense because we had insisted earlier that $\gamma X = X$ and $p_X = \mathrm{Id}$ whenever $X$ is an injective $\Omega$-spectrum.

We have $K(\tau) \circ \gamma = \tau$ as natural transformations because $\gamma(\gamma X) = \gamma X$ and $p_{\gamma X} = \mathrm{Id}$ (again because $\gamma$ is the identity on injective $\Omega$-spectra). We also have $K(\varphi \circ \gamma) = \varphi$ for a natural transformation $\varphi : F \longrightarrow G$, again because $\gamma X = X$ for every injective $\Omega$-spectrum. So $K$ is indeed inverse to precomposition with $\gamma$, which finishes the proof. $\square$

The next proposition makes precise in which way the suspension of a symmetric spectrum 'is' the shift in the stable homotopy category and how homotopy cofibre and homotopy fibre sequences give rise to distinguished triangles in $\mathcal{SHC}$.

For any symmetric spectrum $A$ we have a diagram of morphisms of symmetric spectra

$$\begin{tikzcd}
S^1 \wedge A \arrow[r, "p_{S^1 \wedge A}"] \arrow[d, "S^1 \wedge p_A"'] & \gamma(S^1 \wedge A) \arrow[d, dotted, "\Phi_A"] \\
S^1 \wedge \gamma A \arrow[r, "\lambda_{\gamma A}"'] & \mathrm{sh}(\gamma A)
\end{tikzcd}$$

in which all solid arrows are stable equivalences. Since $\mathrm{sh}(\gamma A)$ is an injective $\Omega$-spectrum, there exists unique homotopy class of morphisms $\Phi_A : \gamma(S^1 \wedge A) \longrightarrow \mathrm{sh}(\gamma A)$ which makes the entire square commute up to homotopy. The morphism $\Phi_A$ is a stable equivalence between injective $\Omega$-spectra, thus an isomorphism

$$\Phi_A \ : \ \gamma(S^1 \wedge A) \ \cong \ (\gamma A)[1] \tag{4.13}$$

in the stable homotopy category.

Dually, the functor $\gamma$ commutes with taking functions out of a finite simplicial set, so in particular with loops. Indeed, if $K$ is a finite pointed simplicial set and $A$ a symmetric spectrum which is levelwise a Kan complex, then the morphism $(p_A)^K : A^K \longrightarrow (\gamma A)^K$ is a stable equivalence by part (v) of Proposition 4.5. Since $(\gamma A)^K$ is an injective $\Omega$-spectrum, there is a unique homotopy class of morphisms $\Psi_{A,K} : \gamma(A^K) \longrightarrow (\gamma A)^K$ such that $\Psi_{A,K} \circ p_{A^K}$ is homotopic to $(p_A)^K$. This morphism is a stable equivalence between injective $\Omega$-spectra, thus a homotopy equivalence. An important special case is $K = S^1$, which yields a preferred homotopy class of homotopy equivalence $\Psi_{A,S^1} : \gamma(\Omega A) \longrightarrow \Omega(\gamma A)$.

The isomorphisms $\Phi_A : \gamma(S^1 \wedge A) \longrightarrow (\gamma A)[1]$ and $\Psi_{A,K} : \gamma(A^K) \longrightarrow (\gamma A)^K$ are natural in $\mathcal{SHC}$ as functors of $A$, which is a consequence of the uniqueness properties.

Proposition 4.14. (i) *Let $f : A \longrightarrow B$ be a morphism of symmetric spectra. Then the functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$ takes the sequence*

$$A \xrightarrow{f} B \xrightarrow{i} C(f) \xrightarrow{p} S^1 \wedge A$$

*to a distinguished triangle in the stable homotopy category after identifying $\gamma(S^1 \wedge A)$ with $(\gamma A)[1]$ via $\phi_A$.*
(ii) *Let $f : A \longrightarrow B$ be a morphism of symmetric spectra which are levelwise Kan complexes. Then the functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$ takes the sequence*

$$\Omega B \xrightarrow{i} F(f) \xrightarrow{p} A \xrightarrow{f} B$$

*to a distinguished triangle in the stable homotopy category after making the identifications*

$$(\gamma(\Omega B))[1] \xrightarrow{\Psi_{B,S^1}[1]} \Omega(\gamma B)[1] \xleftarrow{\lambda^*_{\gamma B}} \gamma B \ .$$

(iii) *The functor $\gamma$ commutes with arbitrary coproducts and finite products. Thus in particular the stable homotopy category has arbitrary coproducts.*

Proof. (iii) Given a family $\{A_i\}_{i\in I}$ of symmetric spectra, then the wedge of the stable equivalences $p_i : A_i \longrightarrow \gamma(A_i)$ is a stable equivalence by Proposition 4.5 (i). Since $\gamma(\bigvee_i A_i)$ is an injective $\Omega$-spectrum, there is a unique homotopy class of morphism $g : \bigvee_i \gamma(A_i) \longrightarrow \gamma(\bigvee_i A_i)$ whose restriction with $\bigvee_i p_i$ is homotopic to the stable equivalence $p_{\bigvee A_i}$. The morphism $g$ is then also a stable equivalence, and so for every injective $\Omega$-spectrum $X$ the induced map on homotopy classes

$$[g, X] \ : \ [\gamma(\bigvee_i A_i), X] \ \longrightarrow [\bigvee_i \gamma(A_i), X]$$

is bijective. The target is isomorphic to the product $\prod_i [\gamma(A_i), X]$, which shows that the injective $\Omega$-spectrum $\gamma(\bigvee_i A_i)$ has the universal property of a coproduct of the objects $\gamma(A_i)$.

The proof that $\gamma$ preserves finite products is similar but slightly easier because products in the stable homotopy category are given by pointset level products. We consider the case of two factors. By the same reasoning as before there is a unique homotopy class of morphism $h : \gamma A \times \gamma B \longrightarrow \gamma(A \times B)$ satisfying $h(p_A \times p_B) = p_{A\times B}$, and $h$ is a stable equivalence. But now both sides are injective $\Omega$-spectra, so the stable equivalence is even a homotopy equivalence, i.e., an isomorphism in $\mathcal{SHC}$. □

**4.2. The homotopy groups of $\gamma A$.** We now have a way of associating to every symmetric spectrum an object of the stable homotopy category, via the functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$. However, the functor depends on an abstract construction which produces a stable equivalence to an injective $\Omega$-spectrum. This does not make it transparent what 'happens' to a symmetric spectrum during this passage, and it not clear how basic invariants like stable homotopy groups change in this process.

By the 'true' homotopy groups of a symmetric spectrum $A$ we mean the homotopy groups of the spectrum $\gamma A$. One can think of these true homotopy groups as 'right derived functors' of the homotopy groups, since they are the homotopy groups of a fibrant replacement (in any of the stable model structures to be introduced in Section III.2). If $A$ is semistable, then the stable equivalence $p : A \longrightarrow \gamma A$ is a $\pi_*$-isomorphism and the true homotopy groups are naturally isomorphic to the classical homotopy groups $\pi_* A$. In fact, semistable symmetric spectra are characterized by this property.

For spectra which are not semistable it would thus be interesting to describe the true homotopy groups in terms of the homotopy groups of $A$, which are often more readily computable from an explicit presentation of the symmetric spectrum. The bad news is that the true homotopy groups are *not* a functor of the classical homotopy groups, not even if one takes the $\mathcal{M}$-action into account. But the next best thing is true: there is a natural spectral sequence whose $E^2$-term depends on $\pi_* A$ as a graded $\mathcal{M}$-module and which converges to the true homotopy groups of $A$. We shall now construct this spectral sequence and discuss it in some examples.

In the next theorem and in what follows we use the notation $- \otimes_{\mathcal{M}} -$ as short hand for the tensor product over the monoid ring $\mathbb{Z}[\mathcal{M}]$.

Theorem 4.15. *There is a strongly convergent half-plane spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \pi_q A) \implies \pi_{p+q}(\gamma A) \ .$$

*The spectral sequence is natural in the symmetric spectrum $A$ with $d^r$-differential of bidegree $(-r, r-1)$. The edge homomorphism*

$$\mathbb{Z} \otimes_{\mathcal{M}} \pi_q A = E^2_{0,q} \longrightarrow \pi_q(\gamma A)$$

*is induced by the stable equivalence $p : A \longrightarrow \gamma A$.*

We will see below that the spectral sequence of Theorem 4.15 collapses in many cases, for example for semistable symmetric spectra and for free symmetric spectra (see Example 4.17), and it always collapses rationally (see Example 4.19). The spectral sequence typically does not collapse for semifree symmetric spectra, see Example 4.20.

We need a preparatory Lemma.

Lemma 4.16. *For every symmetric spectrum $A$ there is a morphism of symmetric spectra $P \longrightarrow A$ with the following properties:*

(i) *the induced map $\pi_* P \longrightarrow \pi_* A$ is surjective;*
(ii) *for all $p \geq 1$ the groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \pi_* P)$ are trivial;*
(iii) *the map $\mathbb{Z} \otimes_{\mathcal{M}} (\pi_* P) \longrightarrow \pi_*(\gamma P)$ induced by the stable equivalence $p_P : P \longrightarrow \gamma P$ is bijective.*

Proof. For each $k \in \mathbb{Z}$ we choose a set of generators of $\pi_k A$ as an $\mathcal{M}$-module, represent each generator by a pointed map $S^{k+n} \longrightarrow A_n$ for large enough $n$ and consider the adjoint morphism $F_n S^{k+n} \longrightarrow A$. We take $P$ as the wedge of these spectra $F_n S^{k+n}$ for varying $k$ and varying generators, with the induced map to $A$, which is then surjective on homotopy groups. The Tor groups of this wedge vanish by Lemma 4.26 (ii), the isomorphism $\pi_*(F_n S^{k+n}) \cong \mathcal{P}_n \otimes \pi^s_{*-k}$ (see (4.33) of Chapter I) and because homotopy takes wedges to sums.

To see that property (iii) holds we can similarly restrict to a single wedge summand of the form $F_n S^{k+n}$. For $k \geq 0$ the morphism $\lambda : F_n S^{k+n} \longrightarrow F_0 S^k = \Sigma^\infty S^k$ is a stable equivalence (compare Example 4.2) with target a semistable symmetric spectrum. Since the target is semistable, its classical homotopy groups are the true homotopy groups and we may show that $\lambda$ induces an isomorphism $\mathbb{Z} \otimes_{\mathcal{M}} (\pi_* F_n S^{k+n}) \longrightarrow \pi_*(\Sigma^\infty S^k)$ By (4.33) of Chapter I, the effect of the morphism $\lambda$ on homotopy groups is isomorphic to the morphism of $\mathcal{M}$-modules

$$\mathcal{P}_n \otimes \pi^s_* \longrightarrow \mathcal{P}_0 \otimes \pi^s_* \cong \pi^s_*$$

induced by the surjection $\pi : \mathcal{P}_n \longrightarrow \mathcal{P}_0$ adjoint to the preferred generator of $\mathcal{P}_0 \cong \mathbb{Z}$. Since $\mathbb{Z} \otimes_{\mathcal{M}} \pi : \mathbb{Z} \otimes_{\mathcal{M}} \mathcal{P}_n \longrightarrow \mathbb{Z} \otimes_{\mathcal{M}} \mathcal{P}_0$ is an isomorphism, this proves property (iii) $\square$

Proof of Theorem 4.15. We inductively define symmetric spectra $A_n$ and $P_n$ starting with $A_0 = A$. In each step we choose a level fibration $f_n : P_n \longrightarrow A_n$ with the properties of Lemma 4.16, define $A_{n+1}$ as the fibre of $f_n$, denote by $i_n : A_{n+1} \longrightarrow P_n$ the inclusion, and iterate the construction. Then the homotopy groups of the sequence of symmetric spectra

$$\cdots \longrightarrow P_{n+1} \xrightarrow{i_n f_{n+1}} P_n \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{f_0} A_0 = A$$

give a resolution of $\pi_* A$ by tame $\mathcal{M}$-modules which are $\mathbb{Z} \otimes_{\mathcal{M}} -$ acyclic. Since the strict fibre $A_{n+1}$ is level equivalent to the homotopy fibre $F(f_n)$ , Proposition 4.14 (ii) provides a distinguished triangle in the stable homotopy category

$$\gamma(A_{n+1}) \xrightarrow{\gamma(i_n)} \gamma(P_n) \xrightarrow{\gamma(f_n)} \gamma(A_n) \longrightarrow \gamma(A_{n+1})[1]$$

which gives rise to a long exact sequence of homotopy groups. These homotopy groups thus assemble into an exact couple with

$$E^1_{p,q} = \pi_q(\gamma(P_p)) \quad \text{and} \quad D^1_{p,q} = \pi_q(\gamma(A_p))$$

and morphisms

$$\begin{array}{llll} j & : & D^1_{p+1,q} \longrightarrow E^1_{p,q} & \text{induced by } \gamma(i_p) : \gamma(A_{p+1}) \longrightarrow \gamma(P_p), \\ k & : & E^1_{p,q} \longrightarrow D^1_{p,q} & \text{induced by } \gamma(f_p) : \gamma(P_p) \longrightarrow \gamma(A_p), \\ i & : & D^1_{p,q} \longrightarrow D^1_{p+1,q-1} & \end{array}$$

given by the boundary map $\pi_q(\gamma(A_p)) \longrightarrow \pi_{q-1}(\gamma(A_{p+1}))$ of the distinguished triangle.

By property (iii) of Lemma 4.16 we have

$$E^1_{p,q} = \pi_q(\gamma(P_p)) \;\cong\; \mathbb{Z} \otimes_{\mathcal{M}} (\pi_q(P_p)) \; ;$$

under this isomorphism, the differential $d^1 = jk : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ becomes the map obtained by applying $\mathbb{Z} \otimes_{\mathcal{M}} -$ to the above resolution of $\pi_* A$. Since the homotopy groups of the spectra $P_p$ are acyclic for the functor $\mathbb{Z} \otimes_{\mathcal{M}} -$, the $E^2$-term calculates the Tor groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, \pi_q A)$.

It remains to discuss convergence of the spectral sequence. The $p$th filtration subgroup $F^p$ of the abutment $\pi_*(\gamma A)$ is the kernel of the map

$$i^p \;:\; \pi_*(\gamma A) = D^1_{0,*} \;\longrightarrow\; D^1_{p,*-p} = \pi_{*-p}(\gamma(A_p)) \; .$$

To prove that the spectral sequence converges to the homotopy groups of $\gamma A$ we show that the filtration is exhaustive, i.e., $\pi_q(\gamma A) = \bigcup_p F^p_q$.

By construction, the connecting maps $A_k \longrightarrow \Sigma A_{k+1}$ induce the trivial map on homotopy groups, so the mapping telescope of the sequence

$$A = A_0 \;\longrightarrow\; \Sigma A_1 \;\longrightarrow\; \Sigma^2 A_2 \;\longrightarrow\; \cdots$$

has trivial homotopy groups and is thus stably contractible. A mapping telescope of stable equivalences is a stable equivalence, so the mapping telescope of the sequence

$$\gamma(A_0) \;\longrightarrow\; \gamma(A_1)[1] \;\longrightarrow\; \gamma(A_2)[2] \;\longrightarrow\; \cdots$$

of stably fibrant replacements is also stably contractible. But this is a mapping telescope of semistable spectra, thus itself semistable. Thus the homotopy groups of the mapping telescope of the sequence $\gamma(A_p)[p]$, which are isomorphic to the colimit of homotopy groups, are trivial. Since each instance of the map $i : D^1_{p,q} \longrightarrow D^1_{p+1,q-1}$ is induced by a connecting homomorphism $\gamma(A_k) \longrightarrow \gamma(A_{k+1})[1]$ this shows that the kernels of the maps $i^p$ exhaust all of $\pi_*(\gamma A)$. The spectral sequence is concentrated in a half-plane and has *exiting differentials* in the sense of [**5**, II.6], so it is strongly convergent. □

Example 4.17 (Semistable and free symmetric spectra). When $X$ is a semistable symmetric spectrum or a free symmetric spectrum, then the higher Tor groups for the homotopy of $X$ vanish by part (ii) of Lemma I.4.26. Thus in the spectral sequence of Theorem 4.15 we have $E^2_{p,q} = 0$ for $p \neq 0$, and so the edge homomorphism

$$\mathbb{Z} \otimes_{\mathcal{M}} (\pi_* X) \;\longrightarrow \pi_*(\gamma X)$$

is an isomorphism. For the free symmetric spectrum generated by a pointed space (or simplicial set) $K$ in level $m$, the map $K \wedge \lambda : K \wedge F_n S^n \longrightarrow K \wedge \mathbb{S} \cong \Sigma^\infty K$ is a stable equivalence, hence so is its $n$-fold loop and thus the composite

$$F_n K \longrightarrow \Omega^n(K \wedge F_n S^n) \longrightarrow \Omega^n(\Sigma^\infty K)$$

since the first map is a $\pi_*$-isomorphism. So the free symmetric spectrum $F_n K$ represents the same stable homotopy type as $\gamma(\Sigma^\infty K)[-n]$.

Example 4.18 (Eilenberg-Mac Lane spectra). In Example I.4.27 we associate an Eilenberg-Mac Lane spectrum $HW$ to every tame $\mathcal{M}$-module $W$. The homotopy groups of $HW$ are concentrated in dimension 0, where we get the module $W$ back. So the spectral sequence of Theorem 4.15 for $HW$ collapses onto the axis $q = 0$ to isomorphisms

$$\pi_p(\gamma(HW)) \cong \mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, W) .$$

In particular, the true homotopy groups of $HW$ need not be concentrated in dimension 0. One can show that $HW$ is in fact stably equivalent to the product of the Eilenberg-Mac Lane spectra associated to the groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, W)$, shifted up $p$ dimensions.

Here is an example which shows that for non-trivial $W$ the Eilenberg-Mac Lane spectrum $HW$ can be stably contractible: we let $W$ be the kernel of a surjection $\mathcal{P}_n \longrightarrow \mathbb{Z}$. Lemma I.4.26 and the long exact sequence of Tor groups show that the groups $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Z}, W)$ vanish for all $p \geq 0$. Thus the homotopy groups of $\gamma(HW)$ are trivial, i.e., $HW$ is stably contractible.

Example 4.19 (Rational collapse). We claim that for every tame $\mathcal{M}$-module $W$ and all $p \geq 1$, we have $\mathrm{Tor}_p^{\mathbb{Z}[\mathcal{M}]}(\mathbb{Q}, W) = 0$. So the spectral sequence of Theorem 4.15 always collapses rationally and the edge homomorphism is a rational isomorphism

$$\mathbb{Q} \otimes_{\mathcal{M}} (\pi_* X) \longrightarrow \mathbb{Q} \otimes \pi_*(\gamma X) .$$

The rational vanishing of higher Tor groups is special for *tame* $\mathcal{M}$-modules.

To prove the claim we consider a monomorphism $i : V \longrightarrow W$ of tame $\mathcal{M}$-modules and show that the kernel of the map $\mathbb{Z} \otimes_{\mathcal{M}} i : \mathbb{Z} \otimes_{\mathcal{M}} V \longrightarrow \mathbb{Z} \otimes_{\mathcal{M}} W$ is a torsion group. The inclusions $W^{(n)} \longrightarrow W$ induce an isomorphism

$$\mathrm{colim}_n \ \mathbb{Z} \otimes_{\Sigma_n} W^{(n)} \xrightarrow{\cong} \mathbb{Z} \otimes_{\mathcal{M}} W .$$

For every $n \geq 0$, the kernel of $\mathbb{Z} \otimes_{\Sigma_n} i^{(n)} : \mathbb{Z} \otimes_{\Sigma_n} V^{(n)} \longrightarrow \mathbb{Z} \otimes_{\Sigma_n} W^{(n)}$ is annihilated by the order of the group $\Sigma_n$. Since the kernel of $\mathbb{Z} \otimes_{\mathcal{M}} i$ is the colimit of the kernels of the maps $\mathbb{Z} \otimes_{\Sigma_n} i^{(n)}$, it is torsion. Thus the functor $\mathbb{Q} \otimes_{\mathcal{M}} -$ is exact on short exact sequences of tame $\mathcal{M}$-modules and the higher Tor groups vanish as claimed.

Example 4.20 (Semifree symmetric spectra). For semifree symmetric spectra (see Example 2.13) the spectral sequence of Theorem 4.15 typically does not degenerate. As an example we consider the semifree symmetric spectrum $G_2 S^2$, where $S^2$ is a $\Sigma_2$-space by coordinate permutations.

We first identify the stable equivalence type of $G_2 S^2$. The spectrum $G_2 S^2$ is isomorphic to the quotient spectrum of $\Sigma_2$ permuting the smash factors of $(F_1 S^1)^{\wedge 2}$. Since the $\Sigma_2$-action on $(F_1 S^1)^{\wedge 2}$ is free [not yet shown], the map

$$E\Sigma_2^+ \wedge_{\Sigma_2} (F_1 S^1)^{(2)} \longrightarrow (F_1 S^1)^{\wedge 2}/\Sigma_2 = G_2 S^2$$

which collapses $E\Sigma_2$ to a point is a level equivalence. On the other hand, the stable equivalence $\lambda^{(2)} : (F_1 S^1)^{(2)} \longrightarrow \mathbb{S}$ is $\Sigma_2$-equivariant, so it induces a stable equivalence

$$E\Sigma_2^+ \wedge_{\Sigma_2} (F_1 S^1)^{(2)} \longrightarrow E\Sigma_2^+ \wedge_{\Sigma_2} \mathbb{S} = \Sigma^\infty B\Sigma_2^+$$

on homotopy orbit spectra. Altogether we conclude that $G_2 S^2$ is stably equivalent to $\Sigma^\infty B\Sigma_2^+$.

The spectral sequence of Theorem 4.15 for $G_2 S^2$ has as $E^2$-term the Tor groups of $\pi_*(G_2 S^2)$. According to (4.32) of Chapter I these homotopy groups are isomorphic to $\mathcal{P}_2 \otimes_{\Sigma_2} (\pi^{\mathrm{s}}_{*+2} S^2)(\mathrm{sgn})$. The sign representation cancels the sign action induced by the coordinate flip of $S^2$, so we have an isomorphism of $\mathcal{M}$-modules $\pi_q(G_2S^2) \cong \mathcal{P}_2 \otimes_{\Sigma_2} \pi_q^{\mathrm{s}} S^0$, this time with trivial action on the stable homotopy groups of spheres. Using part (iii) of Lemma I.4.26, the spectral sequence of Theorem 4.15 for $G_2S^2$ takes the form

$$E^2_{p,q} \;\cong\; G_p(\Sigma_2; \pi_q^{\mathrm{s}} S^0) \;\Longrightarrow\; \pi^{\mathrm{s}}_{p+q}(B\Sigma_2^+) \; .$$

This spectral sequence has non-trivial differentials and it seems likely that it coincides with the Atiyah-Hirzebruch spectral sequence for the stable homotopy of the space $B\Sigma_2^+$. [In general $G_mL$ is stably equivalent to the semistable symmetric spectrum $L \wedge_{\Sigma_n} \Omega^n(\mathcal{S}|\mathbb{S}|)$]

**4.3. Proof of Theorem 4.7.** We have to construct, for every symmetric spectrum $A$, a stable equivalence $p : A \longrightarrow \gamma A$ such that the target $\gamma A$ is an injective $\Omega$-spectrum. We divide this construction into two steps.

PROPOSITION 4.21. (i) *There exists a endofunctor $(-)^{inj}$ on the category of symmetric spectra and a natural level equivalence $A \longrightarrow A^{inj}$ such that $A^{inj}$ is an injective spectrum.*
(ii) *There exists a endofunctor $Q$ on the category of symmetric spectra and a natural stable equivalence $A \longrightarrow QA$ such that $QA$ is an $\Omega$-spectrum.*

We get Theorem 4.7 by setting $\gamma A = (QA)^{\mathrm{inj}}$ and as morphism $p_A : A \longrightarrow \gamma A$ the composite of the stable equivalence of part (ii) of Proposition 4.21 with the level equivalence, for the spectrum $QA$, of part (i).

In both cases we use the *small object argument* (see Theorem 1.6 of Appendix) with respect to a certain class of morphisms of symmetric spectra. As usual with small object arguments we have to limit the size of objects. We call a symmetric spectrum of simplicial sets *countable* if the cardinality of the disjoint union of all simplices in all levels is countable.

LEMMA 4.22. *A morphism of symmetric spectra is an injective fibration if and only if it has the right lifting property with respect to all injective level equivalences between countable symmetric spectra.*

PROOF. Suppose that $f : X \longrightarrow Y$ has the right lifting property with respect to all injective level equivalences with countable target (and hence source). Consider a lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which $i : A \longrightarrow B$ is an injective level equivalence, with no restriction on the cardinality of $B$.

We denote by $\mathcal{P}$ the set of 'partial lifts': an element of $\mathcal{P}$ is a pair $(U, h)$ consisting of a symmetric subspectrum $U$ of $B$ which contains the image of $A$ and such that the inclusion $U \longrightarrow B$ (and hence the morphism $A \longrightarrow U$) is a level equivalence and a morphism $h : U \longrightarrow X$ making the following diagram commute

$$\begin{tikzcd} A \arrow[rr] \arrow[d, "i"'] & & X \arrow[d, "f"] \\ U \arrow[r, "\text{incl.}"'] \arrow[urr, "h"] & B \arrow[r] & Y \end{tikzcd}$$

The set $\mathcal{P}$ can be partially ordered by declaring $(U, h) \leq (U', h')$ if $U$ is contained in $U'$ and $h'$ extends $h$. Then every chain in $\mathcal{P}$ has an upper bound, namely the union of all the subspectra $U$ with the common extension of the morphisms $h$. So by Zorn's lemma, the set $\mathcal{P}$ has a maximal element $(V, k)$. We show that $V = B$, so $k$ provides the required lifting showing that $f$ is an injective fibration.

We argue by contradiction and suppose that $V$ is strictly smaller than $B$. Then we can find a countable subspectrum $W$ of $B$ which is *not* contained in $V$ and such that the morphism $V \cap W \longrightarrow W$ is a level equivalence [justify; see Lemmas 5.1.6 and 5.1.7 of [**25**]]. Since $W$ is countable, $f$ has the right lifting property with respect to the inclusion $V \cap W \longrightarrow W$. We have a pushout square

$$\begin{tikzcd} V \cap W \arrow[r] \arrow[d] & V \arrow[d] \\ W \arrow[r] & V \cup W \end{tikzcd}$$

so $f$ also has the right lifting property with respect to the inclusion $V \longrightarrow V \cup W$. But that means that the morphism $k : V \longrightarrow X$ can be extended to $V \cup W$, which contradicts the assumption that $(V, k)$ is a maximal element in the set $\mathcal{P}$ extensions. □

Proof of part (i) of Proposition 4.21. Let $I$ be a set containing one morphism of each isomorphism class of injective level equivalences $i : A \longrightarrow B$ for which $B$ is a countable symmetric spectrum. The class of injective level equivalences of symmetric spectra is closed under wedges, cobase change and composition, possibly transfinite. So every $I$-cell complex is an injective level equivalence.

We apply the small object argument (see Theorem 1.6) to the unique morphism from a given symmetric spectrum $A$ to the trivial spectrum. We obtain a functor $A \mapsto A^{inj}$ together with a natural transformation $j : A \longrightarrow A^{inj}$ which is an $I$-cell complex, hence an injective level equivalence. Moreover, the morphism from $A^{inj}$ to the trivial spectrum is $I$-injective. Lemma 4.22 shows that $A^{inj}$ is an injective spectrum. □

Proof of part (ii) of Proposition 4.21. We use the small object argument with respect to a certain set $J$ of injective stable equivalences. First we let $\lambda_n : F_{n+1}S^1 \longrightarrow F_n S^0$ denote the morphism which is adjoint to the wedge summand inclusion $S^1 \longrightarrow (F_n S^0)_{n+1} = \Sigma^+_{n+1} \wedge S^1$ indexed by the identity element. The morphism $\lambda_n$ factors through the mapping cylinder as $\lambda_n = r_n c_n$ where $c_n : F_{n+1}S^1 \longrightarrow Z(\lambda_n)$ is the 'front' mapping cylinder inclusion and $r_n : Z(\lambda_n) \longrightarrow F_n S^0$ is the projection, which is a homotopy equivalence. We then define $K$ as the set of all pushout product maps

$$i_m \wedge c_n \;:\; \Delta[m]^+ \wedge F_{n+1}S^1 \cup_{\partial\Delta[m]^+ \wedge F_{n+1}S^1} \partial\Delta[m]^+ \wedge Z(\lambda_n) \;\longrightarrow\; \Delta[m]^+ \wedge Z(\lambda_n)$$

for $n, m \geq 0$, where $i_m : \partial\Delta[m] \longrightarrow \Delta[m]$ is the inclusion. We let $FI_\Lambda$ be the set of all morphisms $F_n\Lambda^k[m]^+ \longrightarrow F_n\Delta[m]^+$ induced by the horn inclusions for $n, m \geq 0$ and $0 \leq k \leq m$.

By adjointness, a symmetric spectrum $X$ has the right lifting property with respect to the set $FI_\Lambda$ if and only if for all $n \geq 0$ the simplicial set $X_n$ has the right lifting property for all inclusions of horns into simplices, i.e., if $X_n$ is a Kan simplicial set. By adjointness, a symmetric spectrum $X$ has the right lifting property with respect to the set $K$ if and only if for all $n \geq 0$ the map of simplicial sets $\mathrm{map}(c_n, X) : \mathrm{map}(Z(\lambda_n), X) \longrightarrow \mathrm{map}(F_{n+1}S^1, X) \cong \Omega X_{n+1}$ has the right lifting property for all inclusions of boundaries into simplices, which is equivalent to $\mathrm{map}(c_n, X)$ being an acyclic Kan fibration of simplicial set. Since the mapping cylinder $Z(\lambda_n)$ is homotopy equivalent to $F_n S^0$, the simplicial set $\mathrm{map}(Z(\lambda_n), X)$ is homotopy equivalent to $\mathrm{map}(F_n S^0, X) \cong X_n$. So altogether the right lifting property with respect to the set $K$ implies that the map $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence of simplicial sets.

So if $X$ has the right lifting property with respect to the union $J = K \cup FI_\Lambda$ then $X$ is levelwise Kan and $\tilde{\sigma}_n : X_n \longrightarrow \Omega X_{n+1}$ is a weak equivalence, so $X$ is an $\Omega$-spectrum.

Now we apply the small object argument (see Theorem 1.6) for the set $J$ to the unique morphism from a given symmetric spectrum $A$ to the trivial spectrum. We obtain a functor $A \mapsto QA$ together with a natural transformation $j : A \longrightarrow QA$ which is an $J$-cell complex and such that $QA$ is $J$-injective. By the above, this means that $QA$ is an $\Omega$-spectrum. By Proposition 4.5 the class of injective stable equivalences of symmetric spectra is closed under wedges, cobase change and composition, possibly transfinite. So every $J$-cell complex such as $j : A \longrightarrow QA$ is an injective stable equivalence, which finishes the proof. □

In Proposition 1.8 we proved that every level equivalence between injective $\Omega$-spectra is a homotopy equivalence and that every $\pi_*$-isomorphism between $\Omega$-spectra is a level equivalence. Along similar lines we have

Lemma 4.23. *Every stable equivalence between semistable symmetric spectra is a $\pi_*$-isomorphism.*

Proof. Suppose $f : A \longrightarrow B$ is a stable equivalence between semistable spectra. We first treat the special case where $A$ and $B$ are $\Omega$-spectra and show that then $f$ is even a level equivalence. In the commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A^{inj} \arrow[d, "f^{\mathrm{inj}}"] \\ B \arrow[r] & B^{inj} \end{tikzcd}$$

the horizontal morphisms are level equivalences with injective targets of Proposition 4.21 (i). Since $A$ and $B$ are $\Omega$-spectra, so are $A^{\mathrm{inj}}$ and $B^{\mathrm{inj}}$. Thus the spectra $A^{\mathrm{inj}}$ and $B^{\mathrm{inj}}$ are injective $\Omega$-spectra. The morphism $f^{\mathrm{inj}}$ is a stable equivalence between injective $\Omega$-spectra, hence a homotopy equivalence. In particular, all three morphisms except possibly $f$ are $\pi_*$-isomorphisms, hence $f$ is a $\pi_*$-isomorphism.

Now we treat the general case where $A$ and $B$ are arbitrary semistable symmetric spectra. In the commutative square

$$\begin{tikzcd} A \arrow[r, "\lambda_A^\infty"] \arrow[d, "f"'] & R^\infty(\mathcal{S}|A|) \arrow[d, "R^\infty(\mathcal{S}|f|)"] \\ B \arrow[r, "\lambda_B^\infty"'] & R^\infty(\mathcal{S}|B|) \end{tikzcd}$$

the horizontal morphisms are $\pi_*$-isomorphisms and the spectra $R^\infty A$ and $R^\infty B$ are $\Omega$-spectra, by parts (iv) and (v) of Theorem I.4.44. Since $\pi_*$-isomorphisms are stable equivalences and $f$ is one, the morphism $R^\infty f$ is a stable equivalence. By the first part $R^\infty f$ is a $\pi_*$-isomorphism, and hence so is $f$. □

## 5. Derived smash product

The main result of this section is that the pointset level smash product of symmetric spectrum (see Section I.3) descends to a closed symmetric monoidal product on the stable homotopy category. Recall that $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$ denotes the universal functor from symmetric spectra to the stable homotopy category which inverts stable equivalences (see Theorem 4.12).

Theorem 5.1. *The smash product of symmetric spectra has a total left derived functor. More precisely, there exists a functor*

$$\wedge^L \ : \ \mathcal{SHC} \times \mathcal{SHC} \ \longrightarrow \ \mathcal{SHC}$$

*and a natural transformation*

$$\psi_{A,B} \ : \ (\gamma A) \wedge^L (\gamma B) \ \longrightarrow \ \gamma(A \wedge B)$$

*with the following universal property. If $F : \mathcal{SHC} \times \mathcal{SHC} \longrightarrow \mathcal{SHC}$ is any functor and $\beta : F(\gamma A, \gamma B) \longrightarrow \gamma(A \wedge B)$ a natural transformation, then there exists a unique natural transformation $\kappa : F(X,Y) \longrightarrow X \wedge^L Y$ such that $\psi \circ \kappa(\gamma \times \gamma) = \beta$.*

We refer to the functor $\wedge^L$ as the *derived smash product*. We postpone the proof of Theorem 5.1 to the end of this section. What makes it work is, roughly speaking, that every symmetric spectrum $A$ admits a 'flat resolution', i.e., a level equivalence $A^\flat \longrightarrow A$ from a symmetric spectrum such that smashing with $A^\flat$ preserves level and stable equivalences. The proof will also show that if $A$ or $B$ is flat, than the universal natural transformation $\psi_{A,B} : (\gamma A) \wedge^L (\gamma B) \longrightarrow \gamma(A \wedge B)$ is an isomorphism in the stable homotopy category. In other words, as soon as one of the factors is flat, the pointset level smash product has the 'correct' stable homotopy type.

Theorem 5.2. *For every injective $\Omega$-spectrum $Z$ the functor* $\mathrm{Hom}(-, Z)$ *has a total right derived functor. More precisely, there exists a functor*

$$\mathrm{RHom}(-, Z) \ : \ \mathcal{SHC}^{op} \ \longrightarrow \ \mathcal{SHC}$$

*and a natural transformation*

$$\phi \ : \ \gamma(\mathrm{Hom}(A, Z)) \ \longrightarrow \ \mathrm{RHom}(\gamma A, Z)$$

*of functors from $(\mathcal{S}p^{\Sigma})^{op}$ to $\mathcal{SHC}$ with the following universal property. If $G : \mathcal{SHC}^{op} \longrightarrow \mathcal{SHC}$ is any functor and $\alpha : \gamma(\mathrm{Hom}(A, Z)) \longrightarrow G(\gamma A)$ a natural transformation, then there exists a unique natural transformation $\lambda : \mathrm{RHom}(-, Z) \longrightarrow G$ such that $\lambda(\gamma)\phi = \alpha$.*

We draw an important corollary from Theorem 5.1.

Theorem 5.3. *The stable homotopy category is closed symmetric monoidal with respect to the derived smash product. The unit object is $\gamma\mathbb{S}$, the stably equivalent replacement of the sphere spectrum.*

[check: there is a unique way to choose natural coherence isomorphisms such that $\psi : \gamma A \wedge^L \gamma B \longrightarrow \gamma(A \wedge B)$ and the identity of $\gamma\mathbb{S}$ make $\gamma : \mathcal{S}p^{\Sigma} \longrightarrow \mathcal{SHC}$ into a lax symmetric monoidal functor]

Proof. We have to construct coherence isomorphisms for the derived smash product; of course, these don't just come out of the blue, but they are 'left derived' from the coherence isomorphisms for the pointset level smash product. We start with the associativity. We consider the functor

$$\wedge^L \circ (\wedge^L \times \mathrm{Id}) \; : \; \mathcal{SHC} \times \mathcal{SHC} \times \mathcal{SHC} \;\longrightarrow\; \mathcal{SHC}$$

together with the natural transformation

$$(\gamma A \wedge^L \gamma B) \wedge^L \gamma C \xrightarrow{\psi_{A,B} \wedge^L \mathrm{Id}} \gamma(A \wedge B) \wedge^L \gamma C \xrightarrow{\psi_{A\wedge B,C}} \gamma((A \wedge B) \wedge C) \xrightarrow{\gamma(\alpha_{A,B,C})} \gamma(A \wedge (B \wedge C)) \; .$$

The universal property of the pair $(\wedge^L, \psi)$ implies that the above pair is a total left derived functor of $\wedge \circ (\mathrm{Id} \times \wedge) : (\mathcal{S}p^{\Sigma})^3 \longrightarrow \mathcal{S}p^{\Sigma}$.

On the other hand there is the functor

$$\wedge^L \circ (\mathrm{Id} \times \wedge^L) \; : \; \mathcal{SHC} \times \mathcal{SHC} \times \mathcal{SHC} \;\longrightarrow\; \mathcal{SHC}$$

together with the natural transformation

$$\gamma A \wedge^L (\gamma B \wedge^L \gamma C) \xrightarrow{\mathrm{Id} \wedge^L \psi_{B,C}} \gamma A \wedge^L \gamma(B \wedge C) \xrightarrow{\psi_{A,B\wedge C}} \gamma(A \wedge (B \wedge C)) \; .$$

The universal property of the pair $(\wedge^L, \psi)$ similarly implies that this pair is another total left derived functor of $\wedge \circ (\mathrm{Id} \times \wedge)$. By the uniqueness of universal objects there is a preferred natural isomorphism

$$\bar{\alpha}_{X,Y,Z} \; : \; (X \wedge^L Y) \wedge^L Z \;\longrightarrow\; X \wedge^L (Y \wedge^L Z)$$

of functors from $\mathcal{SHC}^3$ to $\mathcal{SHC}$ such that for all symmetric spectra $A, B$ and $C$ the diagram

$$\begin{tikzcd}
(\gamma A \wedge^L \gamma B) \wedge^L \gamma C \arrow[r, "\psi_{A,B}\wedge^L \mathrm{Id}"] \arrow[d, "\bar{\alpha}_{\gamma A,\gamma B,\gamma C}"'] & \gamma(A \wedge B) \wedge^L \gamma C \arrow[r, "\psi_{A\wedge B,C}"] & \gamma((A \wedge B) \wedge C) \arrow[d, "\gamma(\alpha_{A,B,C})"] \\
\gamma A \wedge^L (\gamma B \wedge^L \gamma C) \arrow[r, "\mathrm{Id}\wedge \psi_{B,C}"'] & \gamma A \wedge^L \gamma(B \wedge C) \arrow[r, "\psi_{A,B\wedge C}"'] & \gamma(A \wedge (B \wedge C))
\end{tikzcd}$$

commutes. Similarly, the universal properties of total left derived functors make sure that the pentagon axiom for the pointset level smash product of symmetric spectra is inherited by the derived smash product.

The construction of the symmetry isomorphism $\bar{\tau} : X \wedge^L Y \longrightarrow Y \wedge^L X$ is completely analogous, and there is a unique such natural isomorphism such that for all symmetric spectra $A$ and $B$ the

$$\begin{tikzcd}
\gamma A \wedge^L \gamma B \arrow[r, "\psi_{A,B}"] \arrow[d, "\bar{\tau}_{\gamma A,\gamma B}"'] & \gamma(A \wedge B) \arrow[d, "\gamma(\tau_{A,B})"] \\
\gamma B \wedge^L \gamma A \arrow[r, "\psi_{B,A}"'] & \gamma(B \wedge A)
\end{tikzcd}$$

commutes. [define unit isomorphism $\bar{r}_X : X \wedge^L \gamma(\mathbb{S}) \cong X$, coherence diagrams] □

Now that we constructed the derived smash product (modulo the proof of Theorem 5.1), we can consider monoid objects in the stable homotopy category with respect to the derived smash product. For us a *homotopy ring spectrum* or *ring spectrum up to homotopy* is an injective $\Omega$-spectrum $S$ together with morphisms $\mu : S \wedge^L S \longrightarrow S$ and $\iota : \gamma\mathbb{S} \longrightarrow S$ in the stable homotopy category which are associative and unital in the sense that the following diagrams commute

$$\begin{tikzcd}
(S \wedge^L S) \wedge^L S \arrow[r, "\bar{\alpha}_{S,S,S}"] \arrow[d, "\mu\wedge^L \mathrm{Id}"'] & S \wedge^L (S \wedge^L S) \arrow[r, "\mathrm{Id}\wedge^L\mu"] & S \wedge^L S \arrow[d, "\mu"] \\
S \wedge^L S \arrow[rr, "\mu"'] & & S
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{S} \wedge^L S \arrow[r, "\iota\wedge^L \mathrm{Id}"] \arrow[dr, "\bar{l}_S"'] & S \wedge^L S \arrow[d, "\mu"] & S \wedge^L \mathbb{S} \arrow[l, "\mathrm{Id}\wedge^L\iota"'] \arrow[dl, "\bar{r}_S"] \\
& S &
\end{tikzcd}$$

A homotopy ring spectrum $(S, \mu, \iota)$ is *homotopy commutative* if the multiplication is unchanged when composed with the symmetric isomorphism, i.e., if the relation $\mu \circ \bar{\tau}_{S,S} = \mu$ holds in the stable homotopy category.

The definition of the derived smash product was such that the universal functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$ is lax symmetric monoidal (with respect to the universal transformation $\psi : \wedge^L \circ (\gamma \times \gamma) \longrightarrow \gamma \circ \wedge$). A formal consequence is that $\gamma$ takes symmetric ring spectra to homotopy ring spectra. Indeed, if $(R, \mu : R \wedge R \longrightarrow R, \iota : \mathbb{S} \longrightarrow R)$ is a symmetric ring spectrum, then $\gamma R$ becomes a ring spectrum up to homotopy with respect to the multiplication map

$$(\gamma R) \wedge^L (\gamma R) \xrightarrow{\psi_{R,R}} \gamma(R \wedge R) \xrightarrow{\gamma(\mu)} \gamma R$$

and the unit map $\gamma(\iota) : \gamma\mathbb{S} \longrightarrow \gamma R$.

The converse is far from being true. More precisely, given a ring spectrum up to homotopy $S$ one can ask if there is a symmetric ring spectrum $R$ such that $\gamma R$ is isomorphic to $S$ as a homotopy ring spectrum. There is an infinite sequence of coherence obstructions for the associativity to get a positive answer. [Illustrate the pentagon coherence condition?] The question of when a homotopy commutative homotopy ring spectrum is represented by a commutative symmetric ring spectrum is even more subtle. We hope to get back to this later, and discuss some of the obstruction theories available to tackle such 'rigidification' questions.

A concrete example is the mod-$p$ Moore spectrum $S\mathbb{Z}/p$ for a prime $p \geq 5$. Indeed, $S\mathbb{Z}/p$ has a homotopy associative and homotopy commutative multiplication in the stable homotopy category for $p \geq 5$, but there is no symmetric ring spectrum whose underlying spectrum is a mod-$p$ Moore spectrum.

[derived smash product commutes with suspension and preserves distinguished triangles]

**5.1. Flat symmetric spectra.** To motivate the following definition, recall that a module over a commutative ring is called flat if tensoring with it preserves monomorphisms.

Definition 5.4. A symmetric spectrum $A$ of simplicial sets is *flat* if the functor $A \wedge -$ preserves monomorphisms.

There is a corresponding notion of flatness for symmetric spectra of topological spaces [reference], but simply requiring that $A \wedge -$ preserves monomorphisms is not the right condition in the topological context. We will show in Chapter III that flat symmetric spectra are the cofibrant objects in suitable 'flat model structures'.

Example 5.5. For every $m \geq 0$ and every pointed $\Sigma_m$-simplicial set $L$ the semifree symmetric spectrum $G_m L$ is flat and moreover, the functor $G_m L \wedge -$ preserves level equivalences. As special cases, this applies to free symmetric spectra $F_n K$ and suspension spectra $\Sigma^\infty K$.

Indeed, if $X$ is another symmetric spectrum then $G_m L \wedge X$ is isomorphic to the twisted smash product $L \rhd_m X$ (see Proposition I.3.5) and so it is trivial in levels below $m$ and we have a natural isomorphism

$$(G_m L \wedge X)_{m+n} \cong \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n \ , \tag{5.6}$$

As a pointed simplicial set, the right hand side is a wedge of $\binom{m+n}{m}$ copies of $L \wedge X_n$. So if $f : X \longrightarrow Y$ is a monomorphism respectively a level equivalence of symmetric spectra of simplicial sets, then so is $\mathrm{Id} \wedge f : G_m L \wedge X \longrightarrow G_m L \wedge Y$.

Some other properties of flat spectra are fairly straightforward from the definition:

PROPOSITION 5.7. (i) *A wedge of flat symmetric spectra is flat.*
(ii) *A filtered colimit of flat symmetric spectra is flat.*
(iii) *The smash product of two flat symmetric spectra is flat.*

PROOF. Properties (i) and (ii) follow from the two facts that the smash product commutes with colimits and that a filtered colimit or a wedge of monomorphisms is a monomorphism.

(iii) Let $A$ and be $B$ flat symmetric spectra of simplicial sets. If $f : X \longrightarrow Y$ is a monomorphism, then $\mathrm{Id} \wedge f : B \wedge X \longrightarrow B \wedge Y$ is a monomorphism since $B$ is flat; then $\mathrm{Id} \wedge \mathrm{Id} \wedge f : A \wedge B \wedge X \longrightarrow A \wedge B \wedge Y$ is a monomorphism since $A$ is flat (where we have implicitly used the associativity isomorphisms). Thus $A \wedge B$ is flat. □

An example of a non-flat symmetric spectrum is $\bar{\mathbb{S}}$, the subspectrum of the sphere spectrum given by

$$\bar{\mathbb{S}}_n \;=\; \begin{cases} * & \text{for } n = 0 \\ S^n & \text{for } n \geq 1. \end{cases} \tag{5.8}$$

So the difference between $\bar{\mathbb{S}}$ and $\mathbb{S}$ is only one missing vertex in level 0, but that missing vertex makes a huge difference for the flatness property. Indeed, since $\bar{\mathbb{S}}$ is trivial in level 0 we have

$$(\bar{\mathbb{S}} \wedge \bar{\mathbb{S}})_2 \;=\; \Sigma_2^+ \wedge \bar{\mathbb{S}}_1 \wedge \bar{\mathbb{S}}_1 \;=\; \Sigma_2^+ \wedge S^2$$

while $(\bar{\mathbb{S}} \wedge \mathbb{S})_2 \cong \bar{\mathbb{S}}_2 = S^2$. So $\bar{\mathbb{S}} \wedge -$ does not take the inclusion $\bar{\mathbb{S}} \longrightarrow \mathbb{S}$ to a monomorphism, hence $\bar{\mathbb{S}}$ is not flat.

Now we develop a convenient criterion for recognizing flat symmetric spectra which involves latching spaces.

DEFINITION 5.9. The *nth latching space* $L_n A$ of a symmetric spectrum $A$ is the $\Sigma_n$-simplicial set $(A \wedge \bar{\mathbb{S}})_n$ where $\bar{\mathbb{S}}$ is the subspectrum of the sphere spectrum defined in (5.8). The $n$th level of the morphism $\mathrm{Id} \wedge i : A \wedge \bar{\mathbb{S}} \longrightarrow A \wedge \mathbb{S} \cong A$, for $i : \bar{\mathbb{S}} \longrightarrow \mathbb{S}$ the inclusion, provides a natural map of pointed $\Sigma_n$-spaces $\nu_n : L_n A \longrightarrow A_n$.

Since the latching spaces play important roles in what follows, we make their definition more explicit. Specializing the construction of the smash product (compare Section I.3) to $A \wedge \bar{\mathbb{S}}$ displays $L_n A$ as the coequalizer, in the category of pointed $\Sigma_n$-simplicial sets, of two maps

$$\bigvee_{p=0}^{n-2} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_1 \times \Sigma_{n-p-1}} A_p \wedge S^1 \wedge S^{n-p-1} \;\longrightarrow\; \bigvee_{p=0}^{n-1} \Sigma_n^+ \wedge_{\Sigma_p \times \Sigma_{n-p}} A_p \wedge S^{n-p} \; .$$

(in the target we have discarded the wedge summand which would contribute $A_n \wedge \bar{\mathbb{S}}_0$, since that is just a point, and similarly in the source). One of the maps takes the wedge summand indexed by $p$ to the wedge summand indexed by $p+1$ using the map

$$\sigma_p \wedge \mathrm{Id} \;:\; A_p \wedge S^1 \wedge S^{n-p-1} \;\longrightarrow\; A_{p+1} \wedge S^{n-p-1}$$

and inducing up. The other map takes the wedge summand indexed by $p$ to the wedge summand indexed by $p$ using the canonical isomorphism

$$A_p \wedge S^1 \wedge S^{n-p-1} \;\xrightarrow{\cong}\; A_p \wedge S^{n-p}$$

and inducing up.

For example, $L_0 A$ is a one-point simplicial set, $L_1 A = A_0 \wedge S^1$ and $L_2 A$ is the pushout of the diagram

$$A_0 \wedge S^2 \;\xleftarrow{\text{act on } S^2}\; \Sigma_2^+ \wedge A_0 \wedge S^2 \;\xrightarrow{\mathrm{Id} \wedge \sigma_0 \wedge \mathrm{Id}}\; \Sigma_2^+ \wedge A_1 \wedge S^1 \; .$$

Thus $L_2A$ is the quotient of $\Sigma_2^+ \wedge A_1 \wedge S^1$ by the equivalence relation generated by

$$\gamma \wedge \sigma_0(a \wedge x) \wedge y \;\sim\; (\gamma(1,2)) \wedge \sigma_0(a \wedge y) \wedge x$$

for $a \in A_0$ and $x, y \in S^1$. In general, $L_nA$ is a quotient of $\Sigma_n^+ \wedge_{\Sigma_{n-1}} A_{n-1} \wedge S^1$ by a suitable equivalence relation.

PROPOSITION 5.10. *A symmetric spectrum of simplicial sets $A$ is flat if and only if for every $n \geq 1$ the map of $\Sigma_n$-simplicial sets $\nu_n : L_nA \longrightarrow A_n$ is injective.*

The proof of Proposition 5.10 uses a certain natural 'filtration' for symmetric spectra which shows how a general symmetric spectrum is built from semifree ones. We have put the term 'filtration' in quotes since in general this only is a natural sequence of symmetric spectra and morphisms with colimit the given spectrum. In the special case of flat symmetric spectra, the morphisms are injective.

For any integer $k$ we denote by $\mathbb{S}^{[k]}$ the sphere spectrum truncated below level $k$, i.e., the symmetric subspectrum of $\mathbb{S}$ with level

$$(\mathbb{S}^{[k]})_n \;=\; \begin{cases} * & \text{for } n < k \\ S^n & \text{for } n \geq k. \end{cases}$$

For example we have $\mathbb{S}^{[1]} = \bar{\mathbb{S}}$ and $\mathbb{S}^{[k]} = \mathbb{S}$ for all $k \leq 0$. Note that two consecutive truncated sphere spectra are related by the equation $S^1 \wedge \mathbb{S}^{[k]} = \mathrm{sh}(\mathbb{S}^{[1+k]})$. Given a symmetric spectrum $A$ we define a sequence $F^mA$ of symmetric spectra by

$$(F^mA)_n \;=\; (\mathbb{S}^{[n-m]} \wedge A)_n$$

as a $\Sigma_n$-space. The structure map $(F^mA)_n \wedge S^1 \longrightarrow (F^mA)_{n+1}$ is given by the composite

$$\begin{aligned}(\mathbb{S}^{[n-m]} \wedge A)_n \wedge S^1 \;&\xrightarrow{\text{twist}}\; S^1 \wedge (\mathbb{S}^{[n-m]} \wedge A)_n \;=\; (S^1 \wedge \mathbb{S}^{[n-m]} \wedge A)_n \;=\; (\mathrm{sh}(\mathbb{S}^{[1+n-m]}) \wedge A)_n \\ &\xrightarrow{\xi_{\mathbb{S}^{[1+n-m]},A}} (\mathrm{sh}(\mathbb{S}^{[1+n-m]} \wedge A))_n \;=\; (\mathbb{S}^{[1+n-m]} \wedge A)_{1+n} \;\xrightarrow{\chi_{1,n}} (\mathbb{S}^{[n+1-m]} \wedge A)_{n+1}\ .\end{aligned}$$

For example we have

$$F^0A = \Sigma^\infty A\ , \quad (F^{n-1}A)_n = L_nA \quad \text{and} \quad (F^mA)_n = A_n \text{ for } m \geq n.$$

In general the spaces of $(F^mA)_n$ for $m < n$ are a kind of 'generalized latching objects'. The inclusions $\mathbb{S}^{[k+1]} \longrightarrow \mathbb{S}^{[k]}$ induce morphisms $j_m : F^{m-1}A \longrightarrow F^mA$ and in a fixed level $n$, the system stabilizes to $(A \wedge \mathbb{S})_n$ which is isomorphic to $A_n$. So the colimit of the sequence of symmetric spectra $F^mA$ over the morphisms $j_m$ is isomorphic to $A$.

PROPOSITION 5.11. *For every symmetric spectrum $A$ and every $m \geq 0$ the commutative square*

(5.12)
$$\begin{tikzcd} G_mL_mA \arrow[r, "G_m\nu_m"] \arrow[d] & G_mA_m \arrow[d] \\ F^{m-1}A \arrow[r, "j_m"'] & F^mA \end{tikzcd}$$

*is a pushout square, where the vertical morphisms are adjoint to the identity $L_mA = (F^{m-1}A)_m$ respectively the isomorphism $(F^mA)_m \cong A_m$.*

PROOF. The commutative square of symmetric spectra

$$\begin{tikzcd} S^k \triangleright_k \bar{\mathbb{S}} \arrow[r] \arrow[d] & S^k \triangleright_k \mathbb{S} = G_kS^k \arrow[d] \\ \mathbb{S}^{[k+1]} \arrow[r] & \mathbb{S}^{[k]} \end{tikzcd}$$

is a pushout, where $\triangleright_k$ is the twisted tensor product, see Example I.2.20. So after smashing with $A$ we obtain a pushout square

$$\begin{tikzcd} S^k \triangleright_k (\bar{\mathbb{S}} \wedge A) \arrow[r] \arrow[d] & S^k \triangleright_k A \arrow[d] \\ \mathbb{S}^{[k+1]} \wedge A \arrow[r] & \mathbb{S}^{[k]} \wedge A \end{tikzcd}$$

Evaluating at level $k+m$ we obtain a pushout square of $\Sigma_{k+n}$-spaces (or simplicial sets)

$$\begin{tikzcd} \Sigma^+_{k+m} \wedge_{\Sigma_k \times \Sigma_m} S^k \wedge L_m A \arrow[r] \arrow[d] & \Sigma^+_{k+m} \wedge_{\Sigma_k \times \Sigma_m} S^k \wedge A_m \arrow[d] \\ (F^{m-1}A)_{k+m} \arrow[r] & (F^m A)_{k+m} \end{tikzcd}$$

which is precisely the part of the square (5.12) in level $k+m$. $\square$

Proof of Proposition 5.10. One direction is essentially clear by definition: the inclusion $i : \bar{\mathbb{S}} \longrightarrow \mathbb{S}$ is certainly a monomorphism, so if $A$ is flat, then $\mathrm{Id} \wedge i : A \wedge \bar{\mathbb{S}} \longrightarrow A \wedge \mathbb{S}$ is a monomorphism, hence each level $\nu_n : L_n A \longrightarrow A_n$ is injective.

[Alternative: define 'flat cofibrations'; show they are preserves under cobase change, composition; show inductively that $F^{m-1}A \longrightarrow F^m A$ is a flat cofibration] Now suppose conversely that the maps $L_n A \longrightarrow A_n$ are injective and let $f : X \longrightarrow Y$ be a monomorphism of symmetric spectra of simplicial sets. We use the filtration of $A$ by the spectra $F^m A$ and show inductively that the map $\mathrm{Id} \wedge f : F^m A \wedge X \longrightarrow F^m A \wedge Y$ is a monomorphism. Since $A$ is the colimit of the $F^m A$, the smash product commutes with colimits and the sequential colimit of monomorphisms is a monomorphism, this proves that $\mathrm{Id} \wedge f : A \wedge X \longrightarrow A \wedge Y$ is a monomorphism, i.e., $A$ is flat.

For the inductive step we use the pushout square of Proposition 5.11. Since smashing is a left adjoint the spectrum $F^m A \wedge X$ is (isomorphic to) the pushout of the upper row in the commutative diagram

$$\begin{tikzcd} F^{m-1}A \wedge X \arrow[d, "\mathrm{Id} \wedge f"'] & G_m L_m A \wedge X \arrow[l] \arrow[r, "G_m \nu_m \wedge \mathrm{Id}"] \arrow[d, "\mathrm{Id} \wedge f"'] & G_m A_m \wedge X \arrow[d, "\mathrm{Id} \wedge f"] \\ F^{m-1}A \wedge Y & G_m L_m A \wedge Y \arrow[l] \arrow[r, "G_m \nu_m \wedge \mathrm{Id}"'] & G_m A_m \wedge Y \end{tikzcd} \tag{5.13}$$

and $F^m A \wedge Y$ is the pushout of the lower row. The left vertical morphism is injective by induction hypothesis, and we claim that in addition the pushout product map of the right square in (5.13)

$$G_m \nu_m \wedge f \;:\; G_m L_m A \wedge Y \cup_{G_m L_m A \wedge X} G_m A_m \wedge X \;\longrightarrow\; G_m A_m \wedge Y$$

is injective. We can verify this claim levelwise and we can use the description of the levels of a smash product $G_m L \wedge X$, for $L$ any $\Sigma_m$-simplicial set,

$$(G_m L \wedge X)_{m+n} \;\cong\; \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n$$

of Proposition I.3.5. The claim in level $m+n$ then follows from the fact that the map of pointed simplicial sets

$$\nu_m \wedge f_n \;:\; L_m A \wedge Y_n \cup_{L_m A \wedge X_n} A_m \wedge X_n \;\longrightarrow\; A_m \wedge Y_n$$

is injective since both $\nu_m : L_m A \longrightarrow A_m$ and $f_n : X_n \longrightarrow Y_n$ are injective.

It is a general fact about simplicial sets (actually about set) that given a commutative diagram

$$\begin{tikzcd} A \arrow[d] & B \arrow[l] \arrow[r] \arrow[d] & C \arrow[d] \\ A' & B' \arrow[l] \arrow[r] & C' \end{tikzcd}$$

in which the map from $A \longrightarrow A'$ and the map $B' \cup_B C \longrightarrow C'$ are injective, then the induced map on pushouts $A \cup_B C \longrightarrow A' \cup_{B'} C'$ is injective.

If we apply this levelwise to the diagram (5.13) we can conclude that the induced morphism on pushouts $F^m A \wedge X \longrightarrow F^m A \wedge Y$ is also a monomorphism, which finishes the proof. □

Proposition 5.14. *Let $A$ be a flat symmetric spectrum.*

(i) *Smashing with $A$ preserves level equivalences, $\pi_*$-isomorphisms and stable equivalences.*
(ii) *For every injective spectrum $X$ the symmetric spectrum* $\mathrm{Hom}(A, X)$ *is injective.*

Proof. We first prove the statement in (i) referring to level equivalences. For a level equivalence $f : X \longrightarrow Y$ we show by induction that the morphisms $f \wedge \mathrm{Id} : X \wedge F^m A \longrightarrow Y \wedge F^m A$ are level equivalence for all $m \geq 0$. For the inductive step we use the pushout square of Proposition 5.11. Since smashing is a left adjoint the spectrum $F^m A \wedge X$ is (isomorphic to) the pushout of the upper row in the commutative diagram

$$\begin{tikzcd}
F^{m-1}A \wedge X \arrow[d, "\mathrm{Id} \wedge f"'] & L_m A \rhd_m X \arrow[l] \arrow[r, "\nu_m \rhd_m \mathrm{Id}"] \arrow[d, "\mathrm{Id} \rhd_m f"] & A_m \rhd_m X \arrow[d, "\mathrm{Id} \rhd_m f"] \\
F^{m-1}A \wedge Y & L_m A \rhd_m Y \arrow[l] \arrow[r, "\nu_m \rhd_m \mathrm{Id}"'] & A_m \rhd_m Y
\end{tikzcd}$$

and $F^m A \wedge Y$ is the pushout of the lower row. The left vertical morphism is a level equivalence by induction hypothesis and the other two vertical morphism are level equivalences by Example 5.5. Because $(L \rhd_m X)_{m+n} = \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} L \wedge X_n$ the functor $- \rhd_m X$ takes injective maps of $\Sigma_m$-simplicial sets to monomorphisms of symmetric spectra, and similarly for $Y$. Since $A$ is flat the map $\nu_m : L_m A \longrightarrow A_m$ is injective, and so the two horizontal morphisms labeled $\nu_m \rhd_m \mathrm{Id}$ are monomorphisms. The gluing lemma for simplicial sets allows us to conclude that the induced morphism on pushouts $F^m A \wedge X \longrightarrow F^m A \wedge Y$ is also a level equivalence, which finishes the proof.

We now prove (i) for $\pi_*$-isomorphisms. Smash product with $A$ commutes with the mapping cone construction, so using the long exact sequence of homotopy groups of Proposition I.4.7 (i) it suffices to show that if a symmetric spectrum $C$ has trivial stable homotopy groups, then so does $A \wedge C$. By [...] the symmetric spectrum $L \wedge C$ has trivial stable homotopy groups for every pointed simplicial set $L$. The isomorphism

$$\pi_k(L \rhd_m C) \;\cong\; \mathbb{Z}[\mathcal{M}]^{+m} \otimes_{\Sigma_m \times \mathcal{M}} \pi_{k+m}(L \wedge C)$$

(see (4.29) in Chapter I) combined with the isomorphism $L \rhd_m C \cong G_m L \wedge C$ shows that the homotopy groups of $G_m L \wedge C$ are trivial. In other words, the claim holds for semifree symmetric spectra.

If $A$ is an arbitrary flat symmetric spectrum we again use induction and show that $F^m A \wedge C$ has trivial homotopy groups for all $m \geq 0$. Since homotopy groups commute with filtered colimits this show that the groups $\pi_*(A \wedge C)$ vanish and it finished the argument. In the inductive step we use the pushout square of Proposition 5.11. Since $A$ is flat the morphism $\nu_m : L_m A \longrightarrow A_m$ is injective, hence $j_m \wedge \mathrm{Id} : F^{m-1} A \wedge C \longrightarrow F^m A \wedge C$ is a monomorphism. We know by induction that the homotopy groups of $F^{m-1} A \wedge C$ vanish, and we know that the homotopy groups of

$$(F^m A \wedge C)/(F^{m-1} A \wedge C) \;\cong\; G_m(A_m/L_m A) \wedge C$$

vanish by the special case. So the long exact sequence of homotopy groups shows that the homotopy groups of $F^m A \wedge C$ are trivial.

Now we prove statement (ii). Given an level equivalence $i : K \longrightarrow L$ of symmetric spectra which is also a monomorphism and morphism $g : K \longrightarrow \mathrm{Hom}(A, X)$ we have to produce an extension $\bar{g} : L \longrightarrow \mathrm{Hom}(A, X)$ satisfying $\bar{g} \circ f = g$. By the definition of flatness and part (i) the morphism $i \wedge \mathrm{Id} : K \wedge A \longrightarrow L \wedge A$ is an injective level equivalence. Since $X$ is injective, the adjoint $G : K \wedge A \longrightarrow X$ if $g$ has an extension $\bar{G} : L \wedge A \longrightarrow X$ satisfying $\bar{G}(f \wedge \mathrm{Id}) = G$. The adjoint $L \longrightarrow \mathrm{Hom}(A, X)$ of $\bar{G}$ is then the required extension of $g$.

Now we prove the part of (i) referring to stable equivalences. If $Z$ is an injective $\Omega$-spectrum, then $\mathrm{Hom}(A, X)$ is injective by part (ii) and an $\Omega$-spectrum by [ref.] So for every stable equivalence $f : X \longrightarrow Y$

the induced map $[f, \mathrm{Hom}(A,Z)] : [Y, \mathrm{Hom}(A,Z)] \longrightarrow [X, \mathrm{Hom}(A,Z)]$ is bijective. By adjunction, $[f \wedge A, Z] : [Y \wedge A, Z] \longrightarrow [X \wedge A, Z]$ is bijective. Since this holds for all injective $\Omega$-spectra $Z$, the morphism $f \wedge \mathrm{Id} : X \wedge A \longrightarrow Y \wedge A$ is a stable equivalence. □

Example 5.15. Here is an example which shows that smashing with an arbitrary symmetric spectrum does not preserve level equivalences. Let $X$ be the symmetric spectrum with $X_0 = S^0$, $X_1 = CS^1$ and $X_n = *$ for $n \geq 2$. Here $CS^1 = S^1 \wedge \Delta[1]$ is the cone on $S^1$, where $\Delta[1]$ is pointed at the 0th vertex. The only nontrivial structure map $\sigma_0 : X_0 \wedge S^1 \longrightarrow X_1$ is the cone inclusion $S^1 \longrightarrow CS^1$. Let $Y$ be the symmetric spectrum with $Y_0 = S^0$ and $Y_n = *$ for $n \geq 1$. Then the unique morphism $f : X \longrightarrow Y$ which is the identity in level 0 is a level equivalence, but we claim that $f \wedge \bar{\mathbb{S}} : X \wedge \bar{\mathbb{S}} \longrightarrow Y \wedge \bar{\mathbb{S}}$ is not a level equivalence. Indeed, in level 2 we have

$$(X \wedge \bar{\mathbb{S}})_2 \;=\; L_2 X \;\cong\; \mathrm{pushout}(S^2 \xleftarrow{\mathrm{act}} \Sigma_2^+ \wedge S^2 \xrightarrow{i \wedge S^1} \Sigma_2^+ \wedge (CS^1) \wedge S^1)$$

which is the suspension of the double cone on $S^1$, i.e., a simplicial 3-sphere. In contrast,

$$(Y \wedge \bar{\mathbb{S}})_2 \;=\; L_2 Y \;\cong\; \mathrm{pushout}(S^2 \xleftarrow{\mathrm{act}} \Sigma_2^+ \wedge S^2 \to *)$$

is a point, so $f \wedge \bar{\mathbb{S}}$ is not a weak equivalence in level 2. [give an examples where $\pi_*$-isos or stable equivalences are not preserved]

Proposition 5.16 (Flat resolutions). *There exists a functor $(-)^\flat : \mathcal{S}p^\Sigma \longrightarrow \mathcal{S}p^\Sigma$ with values in flat symmetric spectra and a natural level equivalence $A^\flat \longrightarrow A$. Moreover, there is an isomorphism $(K \wedge A)^\flat \cong K \wedge A^\flat$ which is natural in pointed simplicial sets $K$ and symmetric spectra $A$.*

Proof. Given a symmetric spectrum $A$ we construct the 'flat resolution' $A^\flat$ and the level equivalence $r : A^\flat \longrightarrow A$ level by level, starting with $A^\flat_0 = A_0$ and $r_0 = \mathrm{Id}$. Suppose now that $A^\flat$ and $r$ have been constructed up to level $n-1$. The definition of the $n$th latching only involves the data of a symmetric spectrum in levels strictly smaller than $n$. So we have a latching object $L_n A^\flat$ and the morphism $r$ induces a $\Sigma_n$-equivariant map $L_n r : L_n A^\flat \longrightarrow L_n A$. We define $A^\flat_n$ as the pointed mapping cylinder of the composite map

$$L_n A^\flat \xrightarrow{L_n r} L_n A \xrightarrow{\nu_n} A_n \; .$$

This inherits a $\Sigma_n$-action from the actions on $L_n A^\flat$ and $A_n$, and the trivial action on the cylinder coordinate. The structure map

$$\sigma_{n-1} \;:\; A^\flat_{n-1} \wedge S^1 \;\longrightarrow\; A^\flat_n = Z(\nu_n \circ L_n r : L_n A^\flat \longrightarrow A_n)$$

is the composite of the map

$$A^\flat_{n-1} \wedge S^1 \;\xrightarrow{1 \wedge \mathrm{Id}}\; \Sigma_n^+ \wedge_{\Sigma_{n-1} \times \Sigma_1} A^\flat_{n-1} \wedge S^1 \;\xrightarrow{\mathrm{proj}}\; L_n A^\flat$$

with the inclusion into the mapping cylinder. The $n$th level of the morphism $r$ is the projection $A^\flat_n = Z(\nu_n \circ L_n r) \longrightarrow A_n$ of the mapping cylinder onto the target; this is a homotopy equivalence so in particular a weak equivalence.

After the dust settles we have constructed a symmetric spectrum $A^\flat$ and a morphism $r : A^\flat \longrightarrow A$ which is levelwise a simplicial homotopy equivalence, so altogether a level equivalence (but in general *not* a homotopy equivalence of symmetric spectra). The construction is made so that the map $\nu_n : L_n A^\flat \longrightarrow A^\flat_n$ is the mapping cylinder inclusion, thus injective. So by the criterion of Proposition 5.10 the symmetric spectrum $A^\flat$ is indeed flat. □

Theorem 5.17. *Let $X$ and $Y$ be two semistable spectra one of which is flat. Then the smash product $X \wedge Y$ is semistable.*

Proof. Suppose $X$ is flat and semistable. We first prove the proposition when $Y$ has a special form, namely $Y = \Omega^n L'(\Sigma^\infty K)$ for a pointed simplicial set $K$, where $L'$ is a level fibrant replacement functor. Smashing with a flat symmetric spectrum preserves level equivalences (Proposition 5.14) so $X \wedge L'(\Sigma^\infty K)$ is level equivalent to $X \wedge \Sigma^\infty K$, which is isomorphic to the symmetric spectrum $K \wedge X$ and thus semistable by Example I.4.52.

The counit of the adjunction between loop and suspension is a $\pi_*$-isomorphism $\varepsilon : S^n \wedge \Omega^n L'(\Sigma^\infty K) \longrightarrow L'(\Sigma^\infty K)$, so by Proposition 5.14 the map

$$\mathrm{Id} \wedge \varepsilon \ : \ X \wedge S^n \wedge \Omega^n L'(\Sigma^\infty K) \ \longrightarrow \ X \wedge L'(\Sigma^\infty K)$$

is a $\pi_*$-isomorphism. Since the target is semistable, so is $X \wedge S^n \wedge \Omega^n L'(\Sigma^\infty K)$. A symmetric spectrum is semistable if and only if its suspension is, so we conclude that $X \wedge \Omega^n L'(\Sigma^\infty K)$ is semistable.

To prove the general case we use Shipley's detection functor [**56**, Sec. 3] which associates to every symmetric spectrum $Y$ the semistable symmetric spectrum $DY$. Here $DY$ is the homotopy colimit of a functor $\mathcal{D}_Y : I \longrightarrow \mathcal{S}p^\Sigma$ from the category $I$ to the category of symmetric spectra with values $\mathcal{D}_Y(\mathbf{n}) = \Omega^n L'(\Sigma^\infty Y_n)$. By the above the symmetric spectrum $X \wedge \mathcal{D}_Y(\mathbf{n}) = X \wedge \Omega^n L'(\Sigma^\infty Y_n)$ is semistable for each $n \geq 0$. Hence the homotopy colimit of the functor $X \wedge \mathcal{D}_Y : I \longrightarrow \mathcal{S}p^\Sigma$, which is isomorphic to $X \wedge DY$, is semistable by Example 4.53.

By [**56**, Cor. 3.1.7] the semistable spectrum $Y$ is related by a chain of $\pi_*$-isomorphisms to the symmetric spectrum $DY$. By Proposition 5.14 $X \wedge Y$ is thus related by a chain of $\pi_*$-isomorphisms to the symmetric spectrum $X \wedge DY$, which we just recognized as semistable. Hence $X \wedge Y$ is semistable, which finishes the proof. $\square$

We recall from Example I.2.39 that every $\Gamma$-space $A$ of simplicial sets can be extended to a simplicial functor from the category of pointed simplicial sets. The extended functor then comes with a natural associative and unital assembly map $A(K) \wedge L \longrightarrow A(K \wedge L)$. Evaluating $A$ on simplicial spheres give symmetric spectrum $A(\mathbb{S})$. More generally, we can evaluate a $\Gamma$-space $A$ on any symmetric spectrum and obtain a new symmetric spectrum $A(X)$ by $A(X)_n = A(X_n)$ with structure maps

$$A(X_n) \wedge S^1 \xrightarrow{\text{assembly}} A(X_n \wedge S^1) \xrightarrow{\sigma_n} A(X_{n+1}) \ .$$

In this situation the collection of assembly maps

$$A(S^n) \wedge X_m \xrightarrow{\text{assembly}} A(S^n \wedge X_m) \xrightarrow{A(\text{twist})} A(X_m \wedge S^n) \xrightarrow{\sigma^n} A(X_{m+n}) \xrightarrow{A(\chi_{m,n})} A(X_{n+m})$$

form a bimorphism and thus assemble into a morphism of symmetric spectra

$$A(\mathbb{S}) \wedge X \ \longrightarrow \ A(X)$$

which we also refer to as the assembly map.

Proposition 5.18. *Let $A$ be is a $\Gamma$-space of simplicial sets and $X$ a symmetric spectrum of simplicial sets.*

(i) *The symmetric spectrum $A(\mathbb{S})$ obtained by evaluating $A$ on spheres is flat.*
[is $A(X)$ flat if $X$ is flat?]
(ii) *The assembly morphism $A(\mathbb{S}) \wedge X \longrightarrow A(X)$ is a $\pi_*$-isomorphism.*
(iii) *If $X$ is semistable then the symmetric spectrum spectrum $A(X)$ is again semistable.*

Proof. (i) For every $\Gamma$-space of simplicial sets $X$ and $n \geq 0$ we define a new $\Gamma$-space $L_n X$ by

$$(L_n X)(K) \ = \ \mathrm{colim}_{A \neq \subset \{1,\dots,n\}} X(K^A) \wedge K^{cA}$$

for pointed sets $K$, where $cA = \{1, \dots, n\} - A$ is the complement of $A$. The assembly maps assemble into a morphism of $\Gamma$-spaces $\lambda_n : L_n X \longrightarrow X(-^{\wedge n})$. We then have $X(\mathbb{S})_n = X(S^n) = X((S^1)^{\wedge n})$ and $L_n(X(\mathbb{S})) = (L_n X)(S^1)$ and the map $\lambda_n : L_n(X(\mathbb{S})) \longrightarrow X(S^n)$ is given by evaluation the transformation $\lambda_n : L_n X \longrightarrow X(-^{\wedge n})$ at $S^1$.

So it suffices to show that for every $\Gamma$-space the morphism $L_n X \longrightarrow X(-^{\wedge n})$ is injective.

For all pointed simplicial sets $A$ and $B$ and all $\Gamma$-spaces $X$ the natural map

$$X(A) \wedge B \cup_{A \wedge X(1^+) \wedge B} A \wedge X(B) \ \longrightarrow \ X(A \wedge B)$$

induced by the assembly maps is injective. [...] For $A = B = K$ this is the above.

(ii) We choose a strict (i.e., objectwise) weak equivalence of $\Gamma$-spaces $A^c \longrightarrow A$ such that $A^c$ is cofibrant in the strict Quillen model structure [ref.] Then for every simplicial set $K$, the map $A^c(K) \longrightarrow A(K)$ is a weak equivalence and thus the morphisms of symmetric spectra $A^c(\mathbb{S}) \longrightarrow A(\mathbb{S})$ and $A^c(X) \longrightarrow A(X)$ are level equivalences. In the commutative square

$$\begin{tikzcd} A^c(\mathbb{S}) \wedge X \arrow[r] \arrow[d] & A^c(X) \arrow[d] \\ A(\mathbb{S}) \wedge X \arrow[r] & A(X) \end{tikzcd}$$

the left vertical morphism is a level equivalence since $A^c(\mathbb{S})$ and $A(\mathbb{S})$ are flat. So in order to show that the lower assembly map is a $\pi_*$-isomorphism we can show that the upper assembly map is one; in other words, we can assume without loss of generality that the $\Gamma$-space $A$ is Q-cofibrant.

Now use induction over the skeleta of the $\Gamma$-space. This reduced to the special case $A = \Gamma^n$ of a representable $\Gamma$-space where the assembly map becomes $(\mathbb{S} \times \cdots \times \mathbb{S}) \wedge X = \Gamma^n(\mathbb{S}) \wedge X \longrightarrow \Gamma^n(X) = X^n$ which is a $\pi_*$-isomorphism.

(iii) The symmetric spectrum $A(\mathbb{S})$ is flat (by Proposition 5.7 (ii)) and semistable, so by Theorem 5.17 the smash product $A(\mathbb{S}) \wedge X$ is semistable. By Proposition 5.18 the assembly map $A(\mathbb{S}) \wedge X \longrightarrow A(X)$ is a $\pi_*$-isomorphism, hence its target is also semistable. $\square$

EXAMPLE 5.19. Let $A$ be an abelian group. We consider the $\Gamma$-space $HA$ which assigns to a finite pointed set $k^+$ the simplicial abelian group $A \otimes \tilde{\mathbb{Z}}[k^+]$. Then $HA(\mathbb{S})$ equals the Eilenberg-Mac Lane spectrum $HA$ as defined in Example I.2.7. Proposition 5.18 shows that the Eilenberg-Mac Lane spectrum $HA$ is flat and the assembly morphism

$$HA \wedge X \ \longrightarrow\ HA(X)$$

is a $\pi_*$-isomorphism for every symmetric spectrum of simplicial sets $X$. The $n$th level of the target spectrum is $A \otimes \tilde{\mathbb{Z}}[X_n]$ whose homotopy groups are the reduced homology groups of the pointed simplicial set $X_n$ with coefficients in $A$. So we get natural isomorphisms of abelian groups

$$\pi_k(HA \wedge X) \ \cong\ \mathrm{colim}_n\ \tilde{H}_{k+n}(X; A)\ . \tag{5.20}$$

We still owe the *proof of Theorem 5.1*, i.e., we have to construct the derived smash product. Given injective $\Omega$-spectra $X$ and $Y$ define the derived smash product on objects by

$$X \wedge^L Y \ =\ \gamma(X^\flat \wedge Y)$$

where $r : X^\flat \longrightarrow X$ is the 'flat resolution' of Proposition 5.16, i.e, a functorial level equivalence with flat source. We could also use flat resolutions of both factors, but that makes no real difference since the morphism $X^\flat \wedge Y^\flat \longrightarrow X^\flat \wedge Y$ is a level equivalence by Proposition 5.14 (i), and so becomes an isomorphism in the stable homotopy category after applying the universal functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$.

This construction is evidently functorial in morphisms (i.e., homotopy classes) $Y \longrightarrow Y'$ since smashing commutes with $\Delta[1]^+ \wedge -$ and hence preserves the homotopy relation. Functoriality in $X$ need an extra argument. Since the flat resolutions of Proposition 5.16 commutes with smash product with $\Delta[1]^+$ it induces a well-defined map $(-)^\flat : [X, X'] \longrightarrow [X^\flat, (X')^\flat]$ on homotopy classes of morphisms, and then we can define $[f] \wedge^L Y : X \wedge^L Y \longrightarrow X' \wedge^L Y$ to be $[f^\flat \wedge Y]$.

Having defined a functor $\wedge^L : \mathcal{SHC} \times \mathcal{SHC} \longrightarrow \mathcal{SHC}$ we now need universal natural transformation $\psi_{A,B} : (\gamma A) \wedge^L (\gamma B) \longrightarrow \gamma(A \wedge B)$ in symmetric spectra $A$ and $B$. We have a stable equivalence $(p_A)^\flat : A^\flat \longrightarrow (\gamma A)^\flat$ obtained by taking flat resolutions of the stable equivalence $p_A : A \longrightarrow \gamma A$. Since source and target of $(p_A)^\flat$ are flat, the map $(p_A)^\flat \wedge p_B : A^\flat \wedge B \longrightarrow (\gamma A)^\flat \wedge \gamma B$ is again a stable equivalence. So this last map become an isomorphism in the stable homotopy category after applying the universal functor $\gamma : \mathcal{S}p^\Sigma \longrightarrow \mathcal{SHC}$. Now we define $\psi_{A,B}$ as the composite

$$(\gamma A) \wedge^L (\gamma B) = \gamma\left((\gamma A)^\flat \wedge \gamma B\right) \xrightarrow{\gamma((p_A)^\flat \wedge p_B)^{-1}} \gamma(A^\flat \wedge B) \xrightarrow{\gamma(r_A \wedge \mathrm{Id})} \gamma(A \wedge B)\ .$$

If $A$ or $B$ is flat, then the morphism $r_A \wedge \mathrm{Id} : A^\flat \wedge B \longrightarrow A \wedge B$ is a level equivalence [justify for $A$ flat], so it becomes an isomorphism after applying $\gamma$. Thus if at least one of $A$ or $B$ is flat, then $\psi_{A,B}$ is an isomorphism in the stable homotopy category.

Having defined the functor $\wedge^L$ and the transformation $\psi : \wedge^L \circ (\gamma \times \gamma) \longrightarrow \gamma \circ \wedge$, it remains to show that the pair $(\wedge^L, \psi)$ has the universal property of a total left derived functor. So we consider another functor $F : \mathcal{SHC} \times \mathcal{SHC} \longrightarrow \mathcal{SHC}$ and a natural transformation $\beta : F(\gamma A, \gamma B) \longrightarrow \gamma(A \wedge B)$. We define a natural transformation $\kappa : F(X, Y) \longrightarrow X \wedge^L Y$ as the composite

$$F(X, Y) \xrightarrow{\cong} F(\gamma X, \gamma Y) \xleftarrow{\cong} F(\gamma(X^\flat), \gamma Y) \xrightarrow{\beta_{X^\flat, Y}} \gamma(X^\flat \wedge Y) = X \wedge^L Y$$

and then we have $\psi \circ \kappa(\gamma \times \gamma) = \beta$ [...]. Moreover, $\kappa$ is uniquely determined by this property.

## Exercises

Exercise 6.1. Show that for every injective $\Omega$-spectrum $X$ and all $n, m \geq 0$ the bijection

$$[F_n S^m, X] \longrightarrow \pi_{m-n} X \ , \quad [f] \longmapsto (\pi_{m-n} f)(j) \ ,$$

is additive, hence a group isomorphism. Here $j \in \pi_{m-n}(F_n S^m)$ is the 'fundamental class'.

Exercise 6.2. This exercise generalizes Lemmas 2.7 and 2.8. Let $K$ be a pointed simplicial set whose reduced integral homology is concentrated in one dimension, where it is free abelian of rank 1. The *degree* of a based self map $\tau : K \longrightarrow K$ is the unique integer $\deg(\tau)$ such that $\tau$ induces multiplication by $\deg(\tau)$ on reduced integral homology.

Show that for every injective $\Omega$-spectrum $X$ the induced morphism $\tau^* : X^K \longrightarrow X^K$ equals $\deg(\tau) \cdot \mathrm{Id}$ in the group $[X^K, X^K]$.

Exercise 6.3. [check if this works] Let $R$ be a symmetric ring spectrum. An $R$-module $M$ is *strongly injective* is it has the extension property for all homomorphisms of $R$-modules which are levelwise injective and a weak equivalence of underlying simplicial sets. We define the *derived category* $\mathcal{D}(R)$ of the ring spectrum $R$ as the homotopy category of those strongly injective $R$-modules whose underlying symmetric spectra are $\Omega$-spectra.

(i) Suppose that $R$ is flat as a symmetric spectrum. Show that then the underlying symmetric spectrum of a strongly injective $R$-modules is injective. Give an example showing that the converse is not true.

(ii) Show that the derived category $\mathcal{D}(R)$ has the structure of a triangulated category with shift and distinguished triangles defined after forgetting the $R$-action.

(iii) Show that $\mathcal{D}(R)$ is the target of a universal functor from $R$-modules which takes stable equivalences to isomorphisms.

(iv) Let $f : R \longrightarrow S$ be a homomorphism of symmetric ring spectra which makes $R$ a flat right $S$-module. Show that restriction of scalars from $S$-modules to $R$-modules passes to an exact functor of triangulated categories $f^* : \mathcal{D}(S) \longrightarrow \mathcal{D}(R)$.

(v) Suppose that the underlying symmetric spectrum of $R$ is semistable. Show that then $R$, considered as a module over itself, has a strongly injective $\Omega$-spectrum replacement $\gamma R$ as an $R$-module. Show that the map

$$[\gamma R, \gamma R]_k^{\mathcal{D}(R)} \ \cong \ \pi_k(\gamma R) \cong \pi_k R$$

is an isomorphism of graded rings, where the first map is evaluation at the unit $1 \in \pi_0(\gamma R) \cong \pi_k(\gamma R)[k]$. Show that the map

$$[\gamma R, M]_k^{\mathcal{D}(R)} \ \cong \ \pi_k M$$

is an isomorphism of graded modules over $\pi_* R$ for every strongly injective $\Omega$-$R$-module $M$. Show that $\gamma R$ is a compact weak generator of the triangulated category $\mathcal{D}(R)$.

We shall see later that for $R = HA$ the Eilenberg-Mac Lane ring spectrum associated to a ring $A$ (compare Example I.2.7) the derived category $\mathcal{D}(HA)$ is triangle equivalent to the unbounded derived category of the ring $A$. In fact, the equivalence of triangulated categories will come out as a corollary of a Quillen equivalence of model categories.

### History and credits

The stable homotopy category as we know it today is usually attributed to Boardman, who introduced it in his thesis [**3**] including the triangulated structure and the symmetric monoidal (derived !) smash product. Boardman's stable homotopy category is obtained from a category of *CW-spectra* by passing to homotopy classes of morphisms. Boardman's construction was widely circulated as mimeographed notes [**4**], but he never published these. Accounts of Boardman's construction appear in [**62**], [**65**], and [**2**, Part III]. Strictly speaking the 'correct' stable homotopy category had earlier been introduced by Kan [**30**] based on his notion of *semisimplicial spectra*. Kan and Whitehead [**31**] defined a smash product in the homotopy category of semisimplicial spectra and proved that it is homotopy commutative, but neither they, nor anyone else, ever addressed the associativity of that smash product. Before Kan and Boardman there had been various precursors of the stable homotopy category, and I recommend May's survey article [**43**] for a detailed discussion and an extensive list of references to these.

I am not aware of a complete published account that Boardman's category is really equivalent to the stable homotopy category as defined in Definition 1.5 using injective $\Omega$-spectra. However, here is a short guide through the literature which outlines a comparison. In a first step, Boardman's stable homotopy category can be compared to Kan's homotopy category of semisimplicial spectra, which is done in Chapter IV of Boardman's unpublished notes [**4**]. An alternative source is Tierney's article [**62**] where he promotes the geometric realization functor to a functor from Boardman's category of CW-spectra to Kan's category of semisimplicial spectra. Tierney remarks that the singular complex functor from spaces to simplicial set does not lift to a pointset level functor in the other directions, but Section 3 of [**62**] then ends with the words "(...) it is more or less clear – combining various results of Boardman and Kan – that the singular functor exists at the level of homotopy and provides an inverse to the stable geometric realization, i.e. the two homotopy theories are equivalent. The equivalence of homotopy theories has also been announced by Boardman." I am not aware that the details have been carried out in the published literature.

Kan's semisimplicial spectra predate model categories, but Brown [**13**, Thm. 5] showed later that the $\pi_*$-isomorphisms used by Kan are part of a model structure on semisimplicial spectra. In the paper [**11**] Bousfield and Friedlander introduce a model structure on a category of 'sequential spectra' which are just like symmetric spectra, but without the symmetric group actions. In Section 2.5 of [**11**], Bousfield and Friedlander describe a chain of Quillen equivalences between semisimplicial and sequential spectra, which then in particular have equivalent homotopy categories. Hovey, Shipley and Smith show in [**25**, Thm. 4.2.5] that the forgetful functor is the right adjoint of a Quillen equivalence from symmetric spectra (with the stable absolute projective model structure in the sense of Chapter III) to the Bousfield-Friedlander stable model structure of sequential spectra. Since the weak equivalences used for symmetric spectra are the stable equivalences in the sense of Definition 4.1 we can conclude that altogether that Boardman's stable homotopy category is equivalent to the localization of the category of symmetric spectra at the class of stable equivalences, which coincides with the stable homotopy category in our sense by Theorem 4.12.

A word of warning: the comparison which I just summarized passes through the intermediate homotopy category of sequential spectra for which no intrinsic way to define a derived smash product has been studied. As a consequence, it is not clear to me if the combined equivalence takes Boardman's derived smash product to the derived smash product as discussed in Section 5. However, I would be surprised if the composite equivalence were not strongly symmetric monoidal.

The spectral sequence of Theorem 4.15 was first constructed by Shipley in [**56**], by completely different method; more precisely, Shipley obtains a spectral sequence with isomorphic $E^2$-term and isomorphic abutment, and so it seems very likely that the spectral sequences are isomorphic. Shipley constructs a spectral sequence of the form

$$E^2_{p,q} = \operatorname{colim}^p_I \, (\underline{\pi}^{\mathrm{s}}_q X) \implies \pi_{p+q} DX$$

which also converges to the true homotopy groups of a symmetric spectrum $X$ and whose $E^2$-term depends on the $I$-functor $\underline{\pi}_q^{\mathrm{s}} X$ of stable homotopy groups of the spectrum $X$. Here $D$ is Shipley's *detection functor* [**56**, Def. 3.1.1]. The proof that Shipley's $E^2$-term (the derived functors of colimit) is isomorphic to the Tor groups which arise as the $E^2$-term in Theorem 4.15 was given by the author in [**53**].

The argument used in Lemma 4.22 to reduce the lifting property to a set of morphisms with bounded cardinality is taken from [**25**, Lemma 5.1.4 (6)] and ultimately goes back to Bousfield, who used it in [**8**] to establish a 'local' model structure for simplicial sets with respect to a homology theory.

CHAPTER III

# Model structures

Symmetric spectra support many useful model structures and we will now develop several of these. We will mainly be interested in two kinds, namely *level model structures* (with weak equivalences the level equivalences) and *stable model structures* (with weak equivalences the stable equivalences). The level model structures are really an intermediate steps towards the more interesting stable model structures. We will develop the theory for symmetric spectra of simplicial sets first, and later say how to adapt things to symmetric spectra of topological spaces.

We have already seen pieces of some of the model structures at work. Our definition of the stable homotopy category in Section 1 of Chapter II is implicitly relying on the *absolute injective stable* model structure in which every object is cofibrant (as long as we use simplicial sets, not topological spaces) and the fibrant objects are the injective $\Omega$-spectra. However, this model structure does not interact well with the smash product, so when we constructed the derived smash product in Section 5 of Chapter II we implicitly worked in the flat model structures. So it should already be clear although the homotopy category of a model category only depends on the class of weak equivalences, it can be useful to play different model structures off against each other.

Besides the injective and flat model structures there is another useful kind of cofibration/fibration pair which we will discuss, giving the *projective* model structures. Moreover, we will later need 'positive' versions of the model structures which discard all homotopical information contained in level 0 of a symmetric spectrum.

So each of the model structures which we discuss has four kinds of 'attributes':

- a kind of space (simplicial set or topological space)
- a kind of cofibration/fibration pair (injective, flat or projective)
- a type of equivalence (level or stable)
- which levels are used (absolute or positive)

Since all of these attributes can be combined, this already makes $2\times 3\times 2\times 2 = 24$ different model structures on the two kinds of symmetric spectra. More variations are possible: one can also take $\pi_*$-isomorphisms as weak equivalences, or even isomorphisms in some homology theory (giving model structures which realize Bousfield localizations), or one could study 'more positive' model structures which disregard even more than the level 0 information. And this is certainly not the end of the story...

## 1. Level model structures

**1.1. Types of cofibrations.** The latching space $L_nA$ of a symmetric spectrum $A$ was defined in Definition II.5.9 as the $n$th level of the symmetric spectrum $A\wedge\bar{\mathbb{S}}$, where $\bar{\mathbb{S}}$ is the subspectrum of the sphere spectrum with $\bar{\mathbb{S}}_0 = *$ and $\bar{\mathbb{S}}_n = S^n$ for positive $n$. We also gave a more explicit presentation of $L_nA$ as a quotient of $\Sigma_n^+ \wedge_{\Sigma_{n-1}} A_{n-1} \wedge S^1$. The latching space has a based $\Sigma_n$-action and comes with a natural equivariant map $\nu_n : L_nA \longrightarrow A_n$.

For a morphism $f : A \longrightarrow B$ of symmetric spectra and $n \geq 0$ we have a commutative square of $\Sigma_n$-simplicial sets

$$\begin{tikzcd} L_nA \arrow[r, "L_nf"] \arrow[d, "\nu_n"'] & L_nB \arrow[d, "\nu_n"] \\ A_n \arrow[r, "f_n"'] & B_n \end{tikzcd}$$

We thus get a natural morphism of $\Sigma_n$-simplicial sets

$$\nu_n(f) \ : \ A_n \cup_{L_nA} L_nB \ \longrightarrow \ B_n \ .$$

DEFINITION 1.1. A morphism $f : A \longrightarrow B$ of symmetric spectra of simplicial sets is

- a *projective cofibration* if for every $n \geq 0$ the morphism $\nu_n(f)$ is injective and the symmetric group $\Sigma_n$ acts freely on the complement of its image;
- a *flat cofibration* if for every $n \geq 0$ the morphism $\nu_n(f)$ is injective;
- a *level cofibration* if it is a categorical monomorphism, i.e., if for every $n \geq 0$ the morphism $f_n : A_n \longrightarrow B_n$ is injective.

By the criterion for flatness given in Proposition II.5.10 a symmetric spectrum $A$ is flat in the original sense (i.e., $A \wedge -$ preserves monomorphisms) if and only if the unique morphism $* \longrightarrow A$ is a flat cofibration. We call a symmetric spectrum $A$ *projective* if the unique morphism $* \longrightarrow A$ is a projective cofibration or, equivalently, if for every $n \geq 0$ the morphism $\nu_n : L_nA \longrightarrow A_n$ is injective and the symmetric group $\Sigma_n$ acts freely on the complement of its image. Every symmetric spectrum of simplicial sets is level cofibrant.

Now we define the analogues of the three kinds of cofibrations for symmetric spectra of pointed topological spaces. We refer to the standard model structure on the category of pointed compactly generated weak Hausdorff spaces as described for example in [**24**, Thm. 2.4.25]. In this model structure, the weak equivalences are the weak homotopy equivalences and fibrations are the Serre fibrations. The cofibrations are the retracts of 'generalized CW-complexes', i.e., cell complexes in which cells can be attached in any order and not necessarily to cells of lower dimensions. The term '$\Sigma_n$-cofibration' in the next definition refers to the model structure on pointed $\Sigma_n$-spaces which is created by the forgetful functor to pointed spaces. These cofibrations are the retracts of 'generalized free $\Sigma_n$-CW-complexes', i.e., equivariant cell complexes in which only free $\Sigma_n$-cells are attached.

DEFINITION 1.2. A morphism $f : A \longrightarrow B$ of symmetric spectra of topological spaces is

- a *projective cofibration* if for every $n \geq 0$ the morphism $\nu_n(f)$ is a $\Sigma_n$-cofibration;
- a *flat cofibration* if for every $n \geq 0$ the morphism $\nu_n(f)$ is a cofibration of spaces;
- a *level cofibration* if for every $n \geq 0$ the morphism $f_n : A_n \longrightarrow B_n$ is a cofibration of spaces.

To see the analogy with the earlier definitions for symmetric spectra of simplicial sets one should remember that in the standard model structure for pointed simplicial set the cofibrations are the monomorphisms.

By definition every projective cofibration is also a flat cofibration. Flat cofibrations are level cofibrations by the following lemma.

LEMMA 1.3. *Let $f : A \longrightarrow B$ be a morphism of symmetric spectra of simplicial sets or topological spaces. Then $f$ is a flat cofibration if and only if for every level cofibration $g : X \longrightarrow Y$ the pushout product map*

$$f \wedge g \ : \ B \wedge X \cup_{A \wedge X} A \wedge Y \ \longrightarrow \ B \wedge Y$$

*is a level cofibration. In particular, every flat cofibration is a level cofibration.*

PROOF. The inclusion $\bar{\mathbb{S}} \longrightarrow \mathbb{S}$ is a level cofibration and in level $n$ the pushout product of $f$ with this inclusion is the morphism $\nu_n(f) : A_n \cup_{L_nA} L_nB \longrightarrow B_n$. So the pushout product condition for all level cofibrations implies that $f : A \longrightarrow B$ is a flat cofibration.

For the other direction we first consider the special case where $f$ is of the form $G_m i : G_m K \longrightarrow G_m L$ for a $\Sigma_m$-equivariant map $i : K \longrightarrow L$ of pointed simplicial sets or spaces which is a cofibration in the underlying category. We use the isomorphism $G_m L \wedge X \cong L \rhd_m X$ (compare Proposition I.3.5) to rewrite the pushout product $(G_m i) \wedge g$ as

$$L \rhd_m X \cup_{K \rhd_m X} K \rhd_m Y \ \longrightarrow \ L \rhd_m Y \ .$$

In level $m+n$ this morphism is given by first forming the pushout product of $i \wedge g_n : L \wedge X_n \cup_{K \wedge X_n} K \wedge Y_n \longrightarrow L \wedge Y_n$ as pointed simplicial set (or pointed spaces), and then inducing up from $\Sigma_m \times \Sigma_n$ to $\Sigma_{m+n}$. The map $i \wedge g_n$ is a cofibration by the pushout product property of simplicial sets respectively spaces, and inducing up preserves cofibrations. So in this special case the pushout product map is a level cofibration. [...]

If we apply this to the injective map $* \longrightarrow \mathbb{S}$ the pushout product is isomorphic to the map $f$. So as a special csae we obtain that every flat cofibration is a level cofibration. $\square$

Lemma 1.4. *Let $f : A \longrightarrow B$ be a morphism of symmetric spectra of simplicial sets. Then $f : A \longrightarrow B$ is a projective cofibration if and only if it is a flat cofibration and the cokernel $B/A$ is projective.* [Is a morphism $f : A \longrightarrow B$ is a flat cofibration if and only if it is an injective cofibration (i.e., monomorphism) and the cokernel $B/A$ is flat?]

Proof. This is direct consequence of the definitions since a group acts freely on the complement of the image of an equivariant map $A \longrightarrow B$ if and only if the induced action on the quotient $B/A$ is free away from the basepoint. $\square$

Thus we have the following implications for the various kinds of cofibrations:

$$\text{projective cofibration} \implies \text{flat cofibration} \implies \text{level cofibration}$$

All these containments are strict, as the following examples show. The symmetric spectrum $\bar{\mathbb{S}}$ is not flat since its second latching object $L_2\bar{\mathbb{S}}$ is isomorphic to $S^1 \vee S^1$ and the map $L_2\bar{\mathbb{S}} \longrightarrow \bar{\mathbb{S}}_2 = S^2$ is the fold map, which is not injective. Semifree symmetric spectra $G_m L$ are flat for all pointed $\Sigma_m$-simplicial sets $L$, but they are projective only if $\Sigma_m$ acts freely away from the basepoint (compare the following proposition).

Proposition 1.5. (i) *Let $K$ be a pointed $\Sigma_m$-simplicial set for some $m \geq 0$. Then the $n$th latching spaces of the semifree symmetric spectrum $G_m K$ is trivial for $n \leq m$ and for $n > m$ the map $\nu_n : L_n(G_m K) \longrightarrow (G_m K)_n$ is an isomorphism.*

(ii) *Given $m \geq 0$ and an injective morphism $f : K \longrightarrow L$ of $\Sigma_m$-simplicial sets, the induced map $G_m f : G_m K \longrightarrow G_m L$ on semifree symmetric spectra is a flat cofibration, and it is a projective cofibration if and only if $\Sigma_m$ acts freely on the complement of the image of $f$. In particular, every semifree symmetric spectrum $G_m L$ is flat and it is projective if and only if $\Sigma_m$ acts freely on $L$ away from the basepoint.*

(iii) *If $f : K \longrightarrow L$ is a monomorphism of simplicial sets, then for every $m \geq 0$, the induced map $F_m f : F_m K \longrightarrow F_m L$ is a projective cofibration. In particular, all free symmetric spectra are projectively cofibrant.*

Proof. (i) For $n < m$, the latching space $L_n(G_m K)$ consists only of the basepoint. For $n \geq m$, substitution of the definitions and some rewriting gives

$$\begin{aligned} L_{m+n}(G_m K) \ &= \ (G_m K \wedge \bar{\mathbb{S}})_{m+n} \ \cong \ (K \rhd_m \bar{\mathbb{S}})_{m+n} \ = \ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} K \wedge \bar{\mathbb{S}}_n \\ &= \begin{cases} * & \text{for } n = 0, \text{ and} \\ \Sigma^+_{m+n} \wedge_{\Sigma_m \times \Sigma_n} K \wedge S^n \cong (G_m K)_{m+n} & \text{for } n \geq 1. \end{cases} \end{aligned}$$

where we used the identification of $G_m K \wedge X$ with the twisted smash product $K \rhd_m X$ (see Proposition I.3.5).

(ii) We use part (i) to identify the terms in the commutative square of $\Sigma_n$-simplicial sets

$$\begin{tikzcd} L_n(G_mK) \arrow[r, "L_n(G_mf)"] \arrow[d, "\nu_n"'] & L_n(G_mL) \arrow[d, "\nu_n"] \\ (G_mK)_n \arrow[r, "f_n"'] & (G_mL)_n \end{tikzcd}$$

For $n < m$ all four terms are just points. For $n = m$ the two upper objects are points and the lower vertical map is injective. For $n > m$ both vertical maps are isomorphisms. So the map

$$\nu_n(G_mf) \ : \ (G_mK)_n \cup_{L_n(G_mK)} L_n(G_mL) \ \longrightarrow \ (G_mL)_n$$

is an isomorphism for $n \neq m$ and injective for $n = m$. So $G_mf$ is always a flat cofibration. The only case in which $\nu_n(G_mf)$ is not an isomorphism is $n = m$, and then $\nu_n(G_mf)$ is isomorphic to $f : K \longrightarrow L$. So $G_mf$ is a projective cofibration if and only if $\Sigma_m$ acts freely away from the image of $f$.

The natural isomorphism between $F_mK$ and $G_m(\Sigma_m^+ \wedge K)$ makes (iii) a special case of (ii). □

Proposition 1.6. *Let $A$ be a flat symmetric spectrum and $n \geq 2$. Then the symmetric power spectrum $(A^{\wedge n})/\Sigma_n$ is again flat.*
[is the product of flat spectra flat ? how about $A^K$ and $\mathrm{sh}\, A$ ?]

Theorem 1.7. *Let $f : X \longrightarrow Y$ be an injective morphism of $\Gamma$-spaces of simplicial sets. Then the associated morphism $f(\mathbb{S}) : X(\mathbb{S}) \longrightarrow Y(\mathbb{S})$ is a flat cofibration of symmetric spectra. In particular, for every $\Gamma$-space of simplicial sets $X$, the associated symmetric spectrum $X(\mathbb{S})$ is flat.*

[how do the BF- and Q-cofibrations of $\Gamma$-spaces relate to the various cofibrations ?]

Proof. □

Definition 1.8. A morphism $f : K \longrightarrow L$ of $\Sigma_n$-simplicial sets is a *$\Sigma_n$-fibration* (respectively *$\Sigma_n$-equivalence* ) if the induced map on $H$-fixed points $f^H : K^H \longrightarrow L^H$ is a Kan fibration (respectively weak equivalence) of simplicial sets for all subgroups $H$ of $\Sigma_n$.

Theorem 1.9. *The category of symmetric spectra of simplicial sets admits the following three* level model structures *in which the weak equivalences are those morphisms $f : X \longrightarrow Y$ such that for all $n \geq 0$ the map $f_n : X_n \longrightarrow Y_n$ is a weak equivalence of simplicial sets.*

(i) *In the* projective level model structure *the cofibrations are the projective cofibrations and a morphism $f : X \longrightarrow Y$ is a projective level fibration if and only if for every $n \geq 0$ the map $f_n : Y_n \longrightarrow X_n$ is a Kan fibration of simplicial sets.*
(ii) *In the* flat level model structure *the cofibrations are the flat cofibrations, and a morphism $f : X \longrightarrow Y$ is a flat level fibration if and only if for every $n \geq 0$ the map $f_n : Y_n \longrightarrow X_n$ satisfies the following two equivalent conditions*
   – *the map $f_n$ has the right lifting property for all injective morphisms of $\Sigma_n$-simplicial sets which are weak equivalences on underlying simplicial set;*
   – *map $f_n$ is a $\Sigma_n$-fibration and the commutative square*

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] & \mathrm{map}(E\Sigma_n, X_n) \arrow[d, "\mathrm{map}(E\Sigma_n, f_n)"] \\ Y_n \arrow[r] & \mathrm{map}(E\Sigma_n, Y_n) \end{tikzcd}$$

*is $\Sigma_n$-homotopy cartesian. Here* $\mathrm{map}(E\Sigma_n, X)$ *is the simplicial mapping space of all maps from the contractible free $\Sigma_n$-simplicial set to $X$, with $\Sigma_n$-action by conjugation.*

*A morphism $f : X \longrightarrow Y$ is an acyclic fibration in the flat model structure if and only if for all $n \geq 0$ the map $f_n : X_n \longrightarrow Y_n$ is a $\Sigma_n$-equivariant acyclic fibration.*

(iii) *In the* injective level model structure *the cofibrations are the level cofibrations (i.e., monomorphisms) and the injective fibrations are those morphisms which have the right lifting property with respect to all morphisms which are simultaneously level equivalences and monomorphisms.*

*Moreover we have:*

- *All three level model structures are proper, simplicial and cofibrantly generated.*
- *The flat and projective level model structures are even finitely generated and monoidal with respect to the smash product of symmetric spectra.*

Proof. The category of symmetric spectra of simplicial sets has all set-indexed limits and colimits, the level equivalences satisfy the 2-out-of-3 property and in all three cases the classes of cofibrations, fibrations and weak equivalences are closed under retracts. So it remains to prove the factorization and lifting axioms.

As usual we construct the factorizations using Quillen's small object argument. We first defined the respective classes $I^{lv}_{proj}$, $I^{lv}_{flat}$ and $I^{lv}_{inj}$ of generating cofibrations and $J^{lv}_{proj}$, $J^{lv}_{flat}$ and $J^{lv}_{inj}$ of generating acyclic cofibrations. As generating projective cofibrations we take

$$I^{lv}_{proj} \ = \ \left\{F_n\partial\Delta[m]^+ \longrightarrow \ F_n\Delta[m]^+\right\}_{n,m\geq 0}$$

where $F_n$ is the free symmetric spectrum generated by a pointed simplicial set in level $n$, see Example I.2.12. Since $F_n$ is left adjoint to evaluation at level $n$, a morphism $f : X \longrightarrow Y$ of symmetric spectra has the right lifting property with respect to $I^{lv}_{proj}$ if and only if for every $n \geq 0$ the map $f_n : Y_n \longrightarrow X_n$ has the RLP for the boundary inclusions $\partial\Delta[n] \longrightarrow \Delta[n]$, i.e., if it is a Kan fibration and a weak equivalence. In other words, precisely the morphisms which are both level equivalences and projective level fibrations enjoy the right lifting property with respect to $I^{lv}_{proj}$.

As generating flat cofibrations we take

$$I^{lv}_{flat} \ = \ \left\{G_m\left(\Sigma_m/H \times \partial\Delta[n]\right)^+ \longrightarrow \ G_m\left(\Sigma_m/H \times \Delta[n]\right)^+\right\}_{n,m\geq 0,\ H\leq\Sigma_n}$$

where $G_m$ is the semifree symmetric spectrum generated by a pointed $\Sigma_m$-simplicial set in level $m$, see Example I.2.13. Since $G_m$ is left adjoint to evaluation at level $m$ with values in $\Sigma_m$-simplicial sets, a morphism $f : X \longrightarrow Y$ of symmetric spectra has the right lifting property with respect to $I^{lv}_{flat}$ if and only if for every $m \geq 0$ and every subgroup $H$ of $\Sigma_n$ the map $(f_m)^H : (Y_m)^H \longrightarrow (X_m)^H$ on $H$-fixed points of $f_m$ is a Kan-fibration and weak equivalence of simplicial sets. By Proposition 1.9 of Appendix A this is equivalent to the property that $f_n$ is simultaneously a $\Sigma_n$-fibration, a weak equivalence on underlying simplicial sets and the square above is $\Sigma_n$-homotopy cartesian; in other words, precisely the flat level fibrations defined in (ii) above enjoy the right lifting property with respect to $I^{lv}_{flat}$.

Let $f : X \longrightarrow Y$ be a flat level fibration. If $H$ is a subgroup of $\Sigma_n$ then the semifree symmetric spectrum $G_n(\Sigma_n/H)^+$ is flat. Since the flat model structure is simplicial, the induced map on mapping spaces

$$(X_n)^H \cong \mathrm{map}(G_n(\Sigma_n/H)^+, X) \longrightarrow \mathrm{map}(G_n(\Sigma_n/H)^+, Y) \cong (Y_n)^H$$

is a Kan fibration. Since this holds for all subgroup, the map $f_n : X_n \longrightarrow Y_n$ is a $\Sigma_n$-fibration.

Suppose that $X$ is fibrant in the flat level model structure. Let $L$ be any (unbased) $\Sigma_m$-simplicial set. Then the projection $E\Sigma_n \times L \longrightarrow L$ is $\Sigma_n$-equivariant and a weak equivalence of underlying simplicial sets. So the induced map of semifree symmetric spectra

$$G_n(E\Sigma_n \times L)^+ \longrightarrow G_nL^+$$

is a level equivalence between flat symmetric spectra. Since the flat model structure is simplicial, the induced map on mapping spaces

$$\mathrm{map}(G_nL^+, X) \longrightarrow \mathrm{map}(G_n(E\Sigma_n \times L)^+, X)$$

is a weak equivalence. By adjointness that map is isomorphic to

$$\mathrm{map}^{\Sigma_n}(L, X_n) \longrightarrow \mathrm{map}^{\Sigma_n}(E\Sigma_n \times L, X_n) \cong \mathrm{map}^{\Sigma_n}(L, \mathrm{map}(E\Sigma_n, X_n))$$

where $\mathrm{map}(E\Sigma_n, X_n)$ is the space (i.e., simplicial set) of all morphisms from $E\Sigma_n$ to $X$ with conjugation action by $\Sigma_n$. If we specialize to $L = \Sigma_n/H$ for a subgroup $H$ of $\Sigma_n$ we see that the map

$$X_n \longrightarrow \mathrm{map}(E\Sigma_n, X_n)$$

is a weak equivalence on $H$-fixed points, so it is an equivariant equivalence.

To define the generating injective cofibrations we choose one representative for each isomorphism class of pairs $(B, A)$ consisting of a *countable* symmetric spectrum $B$ and a symmetric subspectrum $A$.

□

We still have to show that the level model structures are simplicial and that the flat and projective level model structures are monoidal with respect to the internal smash product of symmetric spectra. So we have to verify various forms of the *pushout product property*. We recall that the pushout product of a morphism $i : K \longrightarrow L$ of pointed simplicial sets or symmetric spectra and a morphism $j : A \longrightarrow B$ of symmetric spectra is the morphism

$$i \wedge j \ : \ L \wedge A \cup_{K\wedge A} K \wedge B \ \longrightarrow \ L \wedge B \ .$$

The first proposition below is about smash products of simplicial sets with symmetric spectra, and it says that various model structures of symmetric spectra are *simplicial* model structures. The next proposition is about internal smash products of symmetric spectra, and it says that various flat and projective (but not injective) model structures of symmetric spectra are *monoidal* model structures.

PROPOSITION 1.10. *Let $i : K \longrightarrow L$ be a morphism of pointed simplicial sets and $j : A \longrightarrow B$ a morphism of symmetric spectra.*

(i) *If $i$ is injective and $j$ a level cofibration, flat cofibration respectively projective cofibration, then the pushout product $i \wedge j$ is also a level cofibration, flat cofibration respectively projective cofibration.*
(ii) *If $i$ is an injective weak equivalence of simplicial sets, and $j$ is a level cofibration (i.e, monomorphism), then $i \wedge j$ is also a level equivalence of symmetric spectra.*
(iii) *If $i$ is injective and $j$ a level cofibration (i.e, monomorphism) and a level equivalence, $\pi_*$-isomorphism respectively stable equivalence of symmetric spectra, then $i \wedge j$ is also a level equivalence, $\pi_*$-isomorphism respectively stable equivalence.*

*Thus the injective, flat and projective level model structures are* simplicial *model categories.*

PROOF. For every pointed simplicial set $K$ and symmetric spectrum $A$ the smash product $K \wedge L$ is naturally isomorphic to the smash product of the suspension spectrum $\Sigma^\infty K$ with $A$. The suspension spectrum functor takes injective maps of simplicial sets to projective cofibrations (see Proposition 1.5 (iii) for $m = 0$) and it takes weak equivalences to level equivalences. So this proposition is a special case of Proposition 1.11 below. □

PROPOSITION 1.11. *Let $i : K \longrightarrow L$ and $j : A \longrightarrow B$ be morphisms of symmetric spectra.*

(i) *If $i$ is a level cofibration and $j$ is a flat cofibration, then $i \wedge j$ is a level cofibration.*
(ii) *If both $i$ and $j$ are flat cofibrations, then so is $i \wedge j$.*
(iii) *If both $i$ and $j$ are projective cofibrations, then so is $i \wedge j$.*
(iv) *If $i$ is a level cofibration, $j$ a flat cofibration and one of $i$ or $j$ a level equivalence, $\pi_*$-isomorphism respectively stable equivalence, then $i \wedge j$ is also a level equivalence, $\pi_*$-isomorphism respectively stable equivalence.*

*Thus the flat and projective level model structures are* monoidal *model categories with respect to the smash product of symmetric spectra.*

PROOF. Check on generators. □

[State all adjoint forms of the simplicial and monoidal axiom]

DEFINITION 1.12. A morphism $f : K \longrightarrow L$ of $\Sigma_n$-spaces is a $\Sigma_n$*-fibration* (respectively $\Sigma_n$*-equivalence*) if the induced map on $H$-fixed points $f^H : K^H \longrightarrow L^H$ is a Serre fibration (respectively weak equivalence) for all subgroups $H$ of $\Sigma_n$.

THEOREM 1.13. *The category of symmetric spectra of topological spaces admits the following two* level model structures *in which the weak equivalences are those morphisms $f : X \longrightarrow Y$ such that for all $n \geq 0$ the map $f_n : X_n \longrightarrow Y_n$ is a weak equivalence of spaces.*

(i) *In the* projective level model structure *the cofibrations are the projective cofibrations and a morphism $f : X \longrightarrow Y$ is a projective level fibration if and only if for every $n \geq 0$ the map $f_n : Y_n \longrightarrow X_n$ is a Serre fibration.*

(ii) *In the* flat level model structure *the cofibrations are the flat cofibrations, and a morphism $f : X \longrightarrow Y$ is a flat level fibration if and only if for every $n \geq 0$ the map $f_n : Y_n \longrightarrow X_n$ satisfies the following two equivalent conditions*
  - *the map $f_n$ has the right lifting property for all cofibrations of pointed $\Sigma_n$-spaces which are weak equivalences on underlying spaces;*
  - *map $f_n$ is a $\Sigma_n$-fibration and the commutative square*

$$\begin{tikzcd} X_n \arrow[r] \arrow[d, "f_n"'] & \mathrm{map}(E\Sigma_n, X_n) \arrow[d, "\mathrm{map}(E\Sigma_n, f_n)"] \\ Y_n \arrow[r] & \mathrm{map}(E\Sigma_n, Y_n) \end{tikzcd}$$

  *is $\Sigma_n$-homotopy cartesian. Here $\mathrm{map}(E\Sigma_n, X)$ is the space of all maps from the contractible free $\Sigma_n$-space to $X$, with $\Sigma_n$-action by conjugation.*

*A morphism $f : X \longrightarrow Y$ is an acyclic fibration in the flat model structure if and only if for all $n \geq 0$ the map $f_n : X_n \longrightarrow Y_n$ is a $\Sigma_n$-equivariant acyclic fibration.*

*Moreover, both level model structures are proper, topological and finitely generated, and monoidal with respect to the smash product of symmetric spectra.*

[Is there are an *injective level model structure* for symmetric spectra of spaces ?]
[positive model structures]

## 2. Stable model structures

Recall from Definition II.4.1 that a morphism $f : A \longrightarrow B$ if symmetric spectra of simplicial sets is a stable equivalence if for every injective $\Omega$-spectrum $X$ the induced map

$$[f, X] \ : \ [B, X] \ \longrightarrow \ [A, X]$$

on homotopy classes of spectrum morphisms is a bijection.

For every morphism $f : X \longrightarrow Y$ the natural morphism $\lambda_X^* : X \longrightarrow \Omega(\mathrm{sh}\, X)$ adjoint to $\lambda_X : S^1 \wedge X \longrightarrow \mathrm{sh}\, X$ gives rise to a commutative square of symmetric spectra

$$\begin{tikzcd} X \arrow[r, "\lambda_X^*"] \arrow[d, "f"'] & \Omega(\mathrm{sh}\, X) \arrow[d, "\Omega(\mathrm{sh}\, f)"] \\ Y \arrow[r, "\lambda_Y^*"'] & \Omega(\mathrm{sh}\, Y) \end{tikzcd} \tag{2.1}$$

THEOREM 2.2. *The category of symmetric spectra of simplicial sets admits the following three* stable model structures *in which the weak equivalences are the stable equivalences.*

(i) *In the* projective stable model structure *the cofibrations are the projective cofibrations and the fibrations are those projective level fibrations $f : X \longrightarrow Y$ for which the commutative square* (2.1) *is levelwise homotopy cartesian.*

(ii) *In the* flat stable model structure *the cofibrations are the flat cofibrations, and the fibrations are those flat level fibrations $f : X \longrightarrow Y$ for which the commutative square* (2.1) *is levelwise homotopy cartesian.*

(iii) *In the* injective stable model structure *the cofibrations are the level cofibrations (i.e., monomorphisms) and the fibrations are those those injective fibrations $f : X \longrightarrow Y$ for which the commutative square* (2.1) *is levelwise homotopy cartesian.*

*Moreover we have:*

- *All three stable model structures are proper, simplicial and cofibrantly generated.*
- *The flat and projective stable model structures are even finitely generated and monoidal with respect to the smash product of symmetric spectra.*
- *In all three cases a morphism is a stable acyclic fibration if and only if it is a level acyclic fibration.*

Proof. We reduce the proof of the stable model structures to the level model structures by applying a general localization theorem of Bousfield, see Theorem 1.8 of Appendix A. In Proposition II.4.21 we constructed a functor $Q : \mathcal{S}p^\Sigma \longrightarrow \mathcal{S}p^\Sigma$ with values in $\Omega$-spectra and a natural stable equivalence $\alpha : A \longrightarrow QA$. We note that a morphism $f : A \longrightarrow B$ of symmetric spectra is a stable equivalence if and only if $Qf : QA \longrightarrow QB$ is a level equivalence. Indeed, since $\alpha_A : A \longrightarrow QA$ and $\alpha_B : B \longrightarrow QB$ are stable equivalences, $f$ is a stable equivalence if and only if $Qf$ is. But $Qf$ is a morphism between $\Omega$-spectra, so it is a stable equivalence if and only if it is a level equivalence.

We now apply Bousfield's Theorem A.1.8 to the injective, flat and projective level model structures. All three level model structures are proper by Theorem 1.9. Axiom (A1) holds since we have a commutative square

(2.3)
$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & QX \arrow[d, "Qf"] \\ Y \arrow[r, "\alpha"'] & QY \end{tikzcd}$$

If $f$ is a level equivalences, then $Qf$ is a stable equivalence between $\Omega$-spectra, hence a level equivalence. Axiom (A2) holds: $\alpha_{QX}$ is a stable equivalence between $\Omega$-spectra, hence a level equivalence. Then $Q\alpha_X : QX \longrightarrow QQX$ is a level equivalence since $Q$ takes all stable equivalences, in particular $\alpha_X$, to level equivalences.

We prove (A3) in the projective level model structure. Since the projective fibrations include the flat and injective fibrations, it then also holds in the flat and injective level model structures. So we are given a pullback square

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ W \arrow[r, "j"'] & Y \end{tikzcd}$$

of symmetric spectra in which $X$ and $Y$ are $\Omega$-spectra (possibly not levelwise Kan), $f$ is levelwise a Kan fibration and $j$ is a stable equivalence. We showed in part (iv) of Proposition II.4.5 that then $i$ is also a stable equivalence. This proves (A3), and thus Bousfield's theorem provides three model structures with stable equivalences as weak equivalence and with cofibrations the projective, flat or level cofibrations respectively.

Bousfield's theorem characterizes the fibrations as those level fibrations $f : X \longrightarrow Y$ for which the commutative square (2.3) is homotopy cartesian. So it remains to shows that for a morphism $f : X \longrightarrow Y$ which is levelwise a Kan fibration the square (2.1) is levelwise homotopy cartesian if and only if the square (2.3) is levelwise homotopy cartesian. □

Corollary 2.4. *The following categories are equivalent*

- *the stable homotopy category, i.e., the homotopy category of injective $\Omega$-spectra of simplicial sets;*
- *the homotopy category of those flat $\Omega$-spectra of simplicial sets for which all $X_n$ are $\Sigma_n$-fibrant and the maps $X_n \longrightarrow \mathrm{map}(E\Sigma_n, X_n)$ are $\Sigma_n$-equivalences;*
- *the homotopy category of projective $\Omega$-spectra which are levelwise Kan complexes.*

## 3. Model structures for modules

With the symmetric monoidal smash product and a compatible model structure in place, we are ready to explore ring and module spectra. In this section we construct model structures on the category of modules over a symmetric ring spectrum. We restrict our attention to stable model structures and show that the forgetful functor to symmetric spectra 'creates' various such model structure. The forgetful functor also creates various level model structures, but we have no use for that and so will not discuss level model structures for $R$-modules.

The various stable model structures are also 'stable' in the technical sense that the suspension functor on the homotopy category is an equivalence of categories. As consequence of this is that stable homotopy category of modules over a ring spectrum is a triangulated category. The free module of rank one is a small generator.

We originally defined a symmetric ring spectrum in Definition I.1.3 in the 'explicit' form, i.e., as a family $\{R_n\}_{n\geq 0}$ of pointed simplicial sets with a pointed $\Sigma_n$-action on $R_n$ and $\Sigma_p \times \Sigma_q$-equivariant multiplication maps $\mu_{p,q} : R_p \wedge R_q \longrightarrow R_{p+q}$ and two unit maps subject to an associativity, unit and centrality condition. Using the internal smash product of symmetric spectra we saw in Theorem I.3.8 that a symmetric ring spectrum can equivalently be defined as a symmetric spectrum $R$ together with morphisms $\mu : R \wedge R \longrightarrow R$ and $\iota : \mathbb{S} \longrightarrow R$, called the multiplication and unit map, which satisfy certain associativity and unit conditions. In this 'implicit' picture a morphism of symmetric ring spectra is a morphism $f : R \longrightarrow S$ of symmetric spectra commuting with the multiplication and unit maps, i.e., such that $f \circ \mu = \mu \circ (f \wedge f)$ and $f \circ \iota = \iota$.

Similarly, if $R$ is a symmetric ring spectrum, a *right $R$-module* was originally defined explicitly, but it can also be given in an implicit form as a symmetric spectrum $M$ together with an action map $M \wedge R \longrightarrow M$ satisfying associativity and unit conditions. A morphism of right $R$-modules is a morphism of symmetric spectra commuting with the action of $R$. We denote the category of right $R$-modules by mod-$R$.

The unit $\mathbb{S}$ of the smash product is a ring spectrum in a unique way, and $\mathbb{S}$-modules are the same as symmetric spectra. The smash product of two ring spectra is naturally a ring spectrum. For a ring spectrum $R$ the opposite ring spectrum $R^{\mathrm{op}}$ is defined by composing the multiplication with the twist map $R \wedge R \longrightarrow R \wedge R$ (so in terms of the bilinear maps $\mu_{p,q} : R_p \wedge R_q \longrightarrow R_{p+q}$, a block permutation appears). The definitions of left modules and bimodules is hopefully clear; left $R$-modules and $R$-$T$-bimodule can also be defined as right modules over the opposite ring spectrum $R^{op}$, respectively right modules over the ring spectrum $R^{\mathrm{op}} \wedge T$.

A formal consequence of having a closed symmetric monoidal smash product of symmetric spectra is that the category of $R$-modules inherits a smash product and function objects. The smash product $M \wedge_R N$ of a right $R$-module $M$ and a left $R$-module $N$ can be defined as the coequalizer, in the category of symmetric spectra, of the two maps

$$M \wedge R \wedge N \rightrightarrows M \wedge N$$

given by the action of $R$ on $M$ and $N$ respectively. Alternatively, one can characterize $M \wedge_R N$ as the universal example of a symmetric spectrum which receives a bilinear map from $M$ and $N$ which is $R$-*balanced*, i.e., all the diagrams

$$\begin{tikzcd} M_p \wedge R_q \wedge N_r \arrow[r, "\mathrm{Id} \wedge \alpha_{q,r}"] \arrow[d, "\alpha_{p,q} \wedge \mathrm{Id}"'] & M_p \wedge N_{q+r} \arrow[d, "\iota_{p,q+r}"] \\ M_{p+q} \wedge N_r \arrow[r, "\iota_{p+q,r}"'] & (M \wedge N)_{p+q+r} \end{tikzcd} \tag{3.1}$$

commute. If $M$ happens to be a $T$-$R$-bimodule and $N$ an $R$-$S$-bimodule, then $M \wedge_R N$ is naturally a $T$-$S$-bimodule. If $R$ is a commutative ring spectrum, the notions of left and right module coincide and agree with the notion of a symmetric bimodule. In this case $\wedge_R$ is an internal symmetric monoidal smash product for $R$-modules. There are also symmetric function spectra $\mathrm{Hom}_R(M, N)$ defined as the equalizer of two maps

$$\mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(R\wedge M,N) \ .$$

The first map is induced by the action of $R$ on $M$, the second map is the composition of $R\wedge - : \mathrm{Hom}(M,N) \longrightarrow \mathrm{Hom}(R\wedge M, R\wedge N)$ followed by the map induced by the action of $R$ on $N$. The internal function spectra and function modules enjoy the 'usual' adjointness properties with respect to the various smash products. [spell out]

THEOREM 3.2. *Let $R$ be a symmetric ring spectrum of topological spaces or simplicial sets. The category of right $R$-modules admits the following four* stable model structures *in which the weak equivalences are those morphisms of $R$-modules which are stable equivalences on underlying symmetric spectra.*

(i) *In the* absolute projective stable model structure *the fibrations are those morphisms of $R$-modules which are absolute projective stable fibrations on underlying symmetric spectra.*
(ii) *In the* positive projective stable model structure *the fibrations are those morphisms of $R$-modules which are positive projective stable fibrations on underlying symmetric spectra.*
(iii) *In the* absolute flat stable model structure *the fibrations are those morphisms of $R$-modules which are absolute flat stable fibrations on underlying symmetric spectra.*
(iv) *In the* positive flat stable model structure *the fibrations are those morphisms of $R$-modules which are positive flat stable fibrations on underlying symmetric spectra.*

*Moreover we have:*

- *All four stable model structures are proper, simplicial and cofibrantly generated.*
- *If $R$ is commutative then all four stable model structures are monoidal with respect to the smash product over $R$.*

*If underlying symmetric spectrum of $R$ is flat, then the category of right $R$-modules admits the following two* injective stable model structures *in which the weak equivalences are those morphisms of $R$-modules which are stable equivalences on underlying symmetric spectra.*

(v) *In the* absolute injective stable model structure *the fibrations are those morphisms of $R$-modules which are absolute injective stable fibrations on underlying symmetric spectra.*
(vi) *In the* positive injective stable model structure *the fibrations are those morphisms of $R$-modules which are positive injective stable fibrations on underlying symmetric spectra.*

*Moreover, both injective stable model structures are proper, simplicial and cofibrantly generated.*

*In all six model structures, a cofibration of $R$-modules is a monomorphism of underlying symmetric spectra.*

PROOF. In the language of Definition 1.3 of Appendix A we claim that in all of the six cases the forgetful functor from $R$-modules to symmetric spectra creates a model structure on $R$-modules. In Theorem A.1.4 we can find sufficient conditions for this, which we will now verify.

The category of $R$-modules is complete, cocomplete and simplicial; in fact all limits, colimits, tensors and cotensors with simplicial sets are created on underlying symmetric spectra. In particular the forgetful functor preserves filtered colimits. The forgetful functor has a left adjoint free functor, given by smashing with $R$. [Smallness]

It remains to check the condition which in practice is often the most difficult one, namely that every $(J\wedge R)$-cell complex is a weak equivalence. We claim that in all six cases the free functor $X \mapsto X\wedge R$ takes stable acyclic cofibrations of symmetric spectra of the respective kind to stable equivalences of $R$-modules which are monomorphisms. In the first four cases (where we have no assumption on $R$) this uses that every generating acyclic cofibration $i : A \longrightarrow B$ is in particular a flat cofibration, so $i\wedge \mathrm{Id} : A\wedge R \longrightarrow B\wedge R$ is injective and a stable equivalence by parts (i) and (iv) of Proposition 1.11. In the 'injective' cases (v) and (vi) the argument is slightly different; then the assumption that $R$ is flat assures that for every injective stable equivalence $i : A \longrightarrow B$ the morphism $i\wedge \mathrm{Id} : A\wedge R \longrightarrow B\wedge R$ is again injective (by the definition of flatness) and a stable equivalence (by Proposition II.5.14).

So in all the six cases, the free functor $-\wedge R$ takes the generating stable acyclic cofibrations to injective stable equivalences of $R$-modules. Since colimits of $R$-modules are created on underlying symmetric spectra, the class of injective stable equivalences is closed under wedges, cobase change and transfinite composition. So every $(J \wedge R)$-cell complex is a stable equivalence. So we have verified the hypothesis of Theorem 1.4, which thus shows that the forgetful functor creates the six model structure. It also shows that the model structures are simplicial and right proper.

[left proper] [monoidal if $R$ commutative.] [preservation of cofibrations] $\square$

[Is there an 'strongly injective' stable model structure in which cofibrations are the monomorphisms of $R$-modules ? make exercise?]

Proposition 3.3. *A morphism $f : M \longrightarrow N$ of right $R$-modules is a flat cofibration if and only if for every morphism $g : V \longrightarrow W$ of left $R$-modules the pushout product map*

$$f \wedge_R g \;:\; M \wedge_R W \cup_{M \wedge_R V} N \wedge_R W \;\longrightarrow\; N \wedge_R W$$

*is an injective morphism of symmetric spectra.*

There are also characterizations of flat and projective cofibrations in terms of '$R$-module latching objects', see Exercise 4.1.

As we just proved, cofibrations of $R$-modules are always monomorphisms of underlying symmetric spectra, but sometimes more is true. As the special case $S = \mathbb{S}$ of Theorem 3.4 (iii) below we will see that if $R$ is flat as a symmetric spectrum, then every flat cofibration of $R$-modules is also a flat cofibration on underlying symmetric spectra. Similarly, if $R$ is projective as a symmetric spectrum, then every projective cofibration of $R$-modules is also a projective cofibration on underlying symmetric spectra.

For a morphism $f : S \longrightarrow R$ of symmetric ring spectra, there is are two adjoint functor pairs relating the modules over $S$ and $R$. The functors are analogous to restriction and extension respectively coextension of scalars. Every $R$-module becomes an $S$-module if we let $S$ act through the homomorphism $f$; more precisely, given an $R$-module $M$ we define an $S$-module $f^*M$ as the same underlying symmetric spectrum as $M$ and with $S$-action given by the composite

$$(f^*M) \wedge S \;=\; M \wedge S \xrightarrow{\mathrm{Id} \wedge f} M \wedge R \xrightarrow{\alpha} M\ .$$

We call the resulting functor $f^* : \text{mod-}R \longrightarrow \text{mod-}S$ *restriction of scalars* along $f$ and note that it has both a left and right adjoint. We call the left adjoint *extension of scalars* and denote it by $f_* : \text{mod-}S \longrightarrow \text{mod-}R$. The left adjoint takes an $S$-module $N$ to the $R$-module $f_*N = N \wedge_S R$, where $S$ is a left $R$-module via $f$, and with right $R$-action through the right multiplication action of $R$ on itself. We call the right adjoint of $f^*$ the *coextension of scalars* and denote it by $f_! : \text{mod-}S \longrightarrow \text{mod-}R$. The right adjoint takes an $S$-module $N$ to the $R$-module $f_!N = \mathrm{Hom}_{\text{mod-}S}(R, N)$, where $S$ is a right $R$-module via $f$, and with right $R$-action through the left multiplication action of $R$ on itself.

Theorem 3.4. *Let $f : S \longrightarrow R$ be a homomorphism of symmetric ring spectra.*

(i) *The functor pair*

$$\text{mod-}S \;\underset{f^*}{\overset{f_*}{\rightleftarrows}}\; \text{mod-}R$$

*is a Quillen adjoint functor pair with respect to the absolute projective, the positive projective, the absolute flat and the positive flat stable model structures on both sides.*

(ii) *If $S$ and $R$ are flat as symmetric spectra then $(f_*, f^*)$ is a Quillen adjoint functor pair with respect to the absolute injective and the positive injective stable model structures on both sides.*

(iii) *Suppose that the morphism $f : S \longrightarrow R$ makes $R$ into a flat (respectively projective) right $S$-module. Then the functor pair*

$$\text{mod-}R \;\underset{f_!}{\overset{f^*}{\rightleftarrows}}\; \text{mod-}S$$

*is a Quillen adjoint functor pair with respect to the absolute and positive flat stable (respectively absolute and positive projective stable) model structures on both sides. In particular, the restriction* *of scalars $f^*$ then takes flat (respectively projective) cofibrations of $R$-modules to flat (respectively projective) cofibrations of $S$-modules.*

(iv) *If the homomorphism $f : S \longrightarrow R$ is a stable equivalence, then the adjoint functor pairs $(f_*, f^*)$ and $(f^*, f_!)$ are a Quillen equivalences in all the cases when they are Quillen adjoint functors.*

PROOF. (i) In each case, the weak (i.e., stable) equivalences and the various kinds of fibrations are defined on underlying symmetric spectra, hence the restriction functor preserves fibrations and acyclic fibrations. By adjointness, the extension functor preserves cofibrations and trivial cofibrations.

(iv) If $f : S \longrightarrow R$ is a stable equivalence, then for every flat right $S$-module $N$ the morphism

$$N \;\cong\; N \wedge_S S \;\longrightarrow\; N \wedge_S R \;=\; f_* N$$

is a stable equivalence. Thus if $Y$ is a fibrant left $R$-module, an $S$-module map $N \longrightarrow Y$ is a weak equivalence if and only if the adjoint $R$-module map $f_* N \longrightarrow Y$ is a weak equivalence. □

EXAMPLE 3.5 (Modules over Eilenberg-Mac Lane spectra). For every ring $A$ we have an associated Eilenberg-Mac Lane ring spectrum, see Example I.2.7. This symmetric spectrum arises from a $\Gamma$-space by evaluation on spheres, so it is flat as a symmetric spectrum (Proposition II.5.18). Hence all six model structure of Theorem 3.2 are defined on the category of $HA$-modules, and they are Quillen-equivalent to each other.

The homotopy category of $HA$-modules can be described purely algebraically in terms of $A$-modules. More precisely, the stable model structures of $HA$-modules are Quillen equivalent to the category of chain complexes of $A$-modules in any of the model structures which have the quasi-isomorphisms as weak equivalences. In particular, we get an equivalence of triangulated categories

$$\mathrm{Ho}(\text{mod-}HA) \;\cong\; \mathcal{D}(A)$$

to the unbounded derived category of the ring $A$.

## Exercises

EXERCISE 4.1. Let $R$ by a symmetric ring spectrum. We define an $R$-bimodule $\bar{R}$ by

$$\bar{R}_n \;=\; \begin{cases} * & \text{for } n = 0 \\ R_n & \text{for } n \geq 1. \end{cases}$$

We define the *$n$-latching object* $L_n^R M$ of a right $R$-module $M$ by $L_n^R M = (M \wedge_R \bar{R})_n$. The latching object has a left action of the symmetric group $\Sigma_n$ and a right action of the pointed monoid $R_0$. The inclusion $\bar{R} \longrightarrow R$ is a morphism of $R$-bimodules and thus induces a morphism of $\Sigma_n$-$R_0$ simplicial bisets

$$\nu_n \;:\; L_n^R M = (M \wedge_R \bar{R})_n \;\longrightarrow\; (M \wedge_R R)_n \cong M_n \;.$$

Show:

(i) A morphism $f : M \longrightarrow N$ is a flat cofibration of $R$-modules if and only if the maps $\nu_n(f) : L_n^R N \cup_{L_n^R M} M_n \longrightarrow N_n$ are cofibrations of right $R_0$-simplicial sets.

(ii) A morphism $f : M \longrightarrow N$ is a projective cofibration of $R$-modules if and only if the maps $\nu_n(f) : L_n^R N \cup_{L_n^R M} M_n \longrightarrow N_n$ are cofibrations of $\Sigma_n$-$R_0$-simplicial bisets.

(Hint: define a suitable $R$-module analog of the filtration $F^m A$ of a symmetric spectrum $A$ so that the proof of Proposition II.5.10 can be adapted.)

## History and credits

The projective and injective level and stable model structures for symmetric spectra are constructed in the original paper [**25**] of Hovey, Shipley and Smith. The flat model structures show up in the literature under the name of *$S$-model structure.* (the '$S$' refers to the sphere spectrum). The cofibrant objects in this model structure (which we call 'flat' and Hovey, Shipley and Smith call '$S$-cofibrant') and parts of the model structures show up in [**25**] and in [**52**], but the first verification of the full model axioms appears in Shipley's paper [**58**]. I prefer the term 'flat' model structure because the cofibrant objects are very analogous to flat modules in algebra and because we can then also use the term 'flat model structure' for modules over a symmetric ring spectrum. Shipley [**58**] calls the flat model structure for modules over a symmetric ring spectrum $R$ the '$R$-model structure'.

APPENDIX A

## 1. Tools from model category theory

**1.1. Cofibrantly generated model categories and a lifting theorem.** In this section we review cofibrantly generated model categories and a general method for creating model category structures. If a model category is cofibrantly generated, its model category structure is completely determined by a set of cofibrations and a set of acyclic cofibrations. The transfinite version of Quillen's small object argument allows functorial factorization of maps as cofibrations followed by acyclic fibrations and as acyclic cofibrations followed by fibrations. Most of the model categories in the literature are cofibrantly generated, e.g. topological spaces and simplicial sets, as are all model structures involving symmetric spectra which we discuss in this book.

The only complicated part of the definition of a cofibrantly generated model category is formulating the definition of relative smallness. For this we need to consider the following set theoretic concepts. The reader might keep in mind the example of a compact topological space which is $\aleph_0$-small relative to closed inclusions.

*Ordinals and cardinals.* An *ordinal* $\gamma$ is an ordered isomorphism class of well ordered sets; it can be identified with the well ordered set of all preceding ordinals. For an ordinal $\gamma$, the same symbol will denote the associated poset category. The latter has an initial object $\emptyset$, the empty ordinal. An ordinal $\kappa$ is a *cardinal* if its cardinality is larger than that of any preceding ordinal. A cardinal $\kappa$ is called *regular* if for every set of sets $\{X_j\}_{j\in J}$ indexed by a set $J$ of cardinality less than $\kappa$ such that the cardinality of each $X_j$ is less than that of $\kappa$, then the cardinality of the union $\bigcup_J X_j$ is also less than that of $\kappa$. The successor cardinal (the smallest cardinal of larger cardinality) of every cardinal is regular.

*Transfinite composition.* Let $\mathcal{C}$ be a cocomplete category and $\gamma$ a well ordered set which we identify with its poset category. A functor $V : \gamma \longrightarrow \mathcal{C}$ is called a $\gamma$-sequence if for every limit ordinal $\beta < \gamma$ the natural map $\mathrm{colim} V|_\beta \longrightarrow V(\beta)$ is an isomorphism. The map $V(\emptyset) \longrightarrow \mathrm{colim}_\gamma V$ is called the transfinite composition of the maps of $V$. A subcategory $\mathcal{C}_1 \subset \mathcal{C}$ is said to be closed under transfinite composition if for every ordinal $\gamma$ and every $\gamma$-sequence $V : \gamma \longrightarrow \mathcal{C}$ with the map $V(\alpha) \longrightarrow V(\alpha+1)$ in $\mathcal{C}_1$ for every ordinal $\alpha < \gamma$, the induced map $V(\emptyset) \longrightarrow \mathrm{colim}_\gamma V$ is also in $\mathcal{C}_1$. Examples of such subcategories are the cofibrations or the acyclic cofibrations in a closed model category.

*Relatively small objects.* Consider a cocomplete category $\mathcal{C}$ and a subcategory $\mathcal{C}_1 \subset \mathcal{C}$ closed under transfinite composition. If $\kappa$ is a regular cardinal, an object $C \in \mathcal{C}$ is called *$\kappa$-small relative to* $\mathcal{C}_1$ if for every regular cardinal $\lambda \geq \kappa$ and every functor $V : \lambda \longrightarrow \mathcal{C}_1$ which is a $\lambda$-sequence in $\mathcal{C}$, the map

$$\mathrm{colim}_\lambda \operatorname{Hom}_{\mathcal{C}}(C, V) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(C, \mathrm{colim}_\lambda V)$$

is an isomorphism. An object $C \in \mathcal{C}$ is called *small relative to* $\mathcal{C}_1$ if there exists a regular cardinal $\kappa$ such that $C$ is $\kappa$-small relative to $\mathcal{C}_1$.

*$I$-injectives, $I$-cofibrations and $I$-cell complexes.* Given a cocomplete category $\mathcal{C}$ and a class $I$ of maps, we denote

- by $I$-inj the class of maps which have the right lifting property with respect to the maps in $I$. Maps in $I$-inj are referred to as *$I$-injectives.*
- by $I$-cof the class of maps which have the left lifting property with respect to the maps in $I$-inj. Maps in $I$-cof are referred to as *$I$-cofibrations.*

- by $I$-cell $\subset$ $I$-cof the class of the (possibly transfinite) compositions of pushouts (cobase changes) of maps in $I$. Maps in $I$-cell are referred to as *$I$-cell complexes*.

In [**46**, p. II 3.4] Quillen formulates his *small object argument*, which immediately became a standard tool in model category theory. In our context we will need a transfinite version of the small object argument, so we work with the 'cofibrantly generated model category', which we now recall. Note that here $I$ has to be a *set*, not just a class of maps. The obvious analogue of Quillen's small object argument would seem to require that coproducts are included in the $I$-cell complexes. In fact, any coproduct of an $I$-cell complex is already an $I$-cell complex, see [**24**, 2.1.6].

Lemma 1.1. *Let $\mathcal{C}$ be a cocomplete category and $I$ a set of maps in $\mathcal{C}$ whose domains are small relative to $I$-*cell*. Then*

- *there is a functorial factorization of any map $f$ in $\mathcal{C}$ as $f = qi$ with $q \in I$-*inj *and $i \in I$-*cell *and thus*
- *every $I$-cofibration is a retract of an $I$-cell complex.*

Definition 1.2. A model category $\mathcal{C}$ is called *cofibrantly generated* if it is complete and cocomplete and there exists a set of cofibrations $I$ and a set of acyclic cofibrations $J$ such that

- the fibrations are precisely the $J$-injectives;
- the acyclic fibrations are precisely the $I$-injectives;
- the domain of each map in $I$ (resp. in $J$) is small relative to $I$-cell (resp. $J$-cell).

Moreover, here the (acyclic) cofibrations are the $I$ ($J$)-cofibrations.

For a specific choice of $I$ and $J$ as in the definition of a cofibrantly generated model category, the maps in $I$ (resp. $J$) will be referred to as generating cofibrations (resp. generating acyclic cofibrations). In cofibrantly generated model categories, a map may be functorially factored as an acyclic cofibration followed by a fibration and as a cofibration followed by an acyclic fibration.

Definition 1.3. Let $\mathcal{C}$ be a model category

$$R \,:\, \mathcal{D} \longrightarrow \mathcal{C}$$

a functor. We say that $R$ *creates a model structure* on the category $\mathcal{D}$ if the following definitions make $\mathcal{D}$ into a model category: a morphism $f$ in $\mathcal{D}$ is a

- weak equivalence if the morphism $R(f)$ is a weak equivalence in $\mathcal{C}$,
- fibration if the morphism $R(f)$ is a fibration in $\mathcal{C}$,
- cofibration if it has the left lifting property with respect to all morphisms in $\mathcal{D}$ which are both fibrations and weak equivalences.

Theorem 1.4. *Let $\mathcal{C}$ be a model category, $\mathcal{D}$ a category which is complete and cocomplete and let*

$$R \,:\, \mathcal{D} \;\longrightarrow\; \mathcal{C} \,:\, L$$

*be a pair of adjoint functors such that $R$ commutes with filtered colimits. Let $I$ ($J$) be a set of generating cofibrations (resp. acyclic cofibrations) for the cofibrantly generated model category $\mathcal{C}$. Let $LI$ (resp. $LJ$) be the image of these sets under the left adjoint $L$. Assume that the domains of $LI$ ($LJ$) are small relative to $LI$-*cell *($LJ$-*cell*). Finally, suppose every $LJ$-cell complex is a weak equivalence. Then $R : \mathcal{D} \longrightarrow \mathcal{C}$ creates a model structure on $\mathcal{D}$ which is cofibrantly generated with $LI$ ($LJ$) a generating set of (acyclic) cofibrations.*

*If the model category $\mathcal{C}$ is right proper, then so is the model structure on $\mathcal{D}$.*

*If $\mathcal{C}$ and $\mathcal{D}$ are simplicially enriched, the adjunction $(L, R)$ is simplicial, and the model structure of $\mathcal{C}$ is simplicial, then the model structure on $\mathcal{D}$ is again simplicial.*

*If $\mathcal{C}$ and $\mathcal{D}$ are topologically enriched, the adjunction $(L, R)$ is continuous, and the model structure of $\mathcal{C}$ is topological, then the model structure on $\mathcal{D}$ is again topological.*

Proof. Model category axiom MC1 (limits and colimits) holds by hypothesis. Model category axioms MC2 (saturation) and MC3 (closure properties under retracts) are clear. One half of MC4 (lifting properties) holds by the definition of cofibrations in $\mathcal{D}$.

The proof of the remaining axioms uses the transfinite small object argument (Lemma 1.1), which applies because of the hypothesis about the smallness of the domains. We begin with the factorization axiom, MC5. Every map in $LI$ and $LJ$ is a cofibration in $\mathcal{D}$ by adjointness. Hence every $LI$-cofibration or $LJ$-cofibration is a cofibration in $\mathcal{D}$. By adjointness and the fact that $I$ is a generating set of cofibrations for $\mathcal{C}$, a map is $LI$-injective precisely when the map becomes an acyclic fibration in $\mathcal{C}$ after application of $R$, i.e., an acyclic fibration in $\mathcal{D}$. Hence the small object argument applied to the set $LI$ gives a (functorial) factorization of any map in $\mathcal{D}$ as a cofibration followed by an acyclic fibration.

The other half of the factorization axiom, MC5, needs the hypothesis. Applying the small object argument to the set of maps $LJ$ gives a functorial factorization of a map in $\mathcal{D}$ as an $LJ$-cell complex followed by a $LJ$-injective. Since $J$ is a generating set for the acyclic cofibrations in $\mathcal{C}$, the $LJ$-injectives are precisely the fibrations among the $\mathcal{D}$-morphisms, once more by adjointness. We assume that every $LJ$-cell complex is a weak equivalence. We noted above that every $LJ$-cofibration is a cofibration in $\mathcal{D}$. So we see that the factorization above is an acyclic cofibration followed by a fibration.

It remains to prove the other half of MC4, i.e., that any acyclic cofibration $A \longrightarrow B$ in $\mathcal{D}$ has the left lifting property with respect to fibrations. In other words, we need to show that the acyclic cofibrations are contained in the $LJ$-cofibrations. The small object argument provides a factorization

$$A \longrightarrow W \longrightarrow B$$

with $A \longrightarrow W$ a $LJ$-cofibration and $W \longrightarrow B$ a fibration. In addition, $W \longrightarrow B$ is a weak equivalence since $A \longrightarrow B$ is. Since $A \longrightarrow B$ is a cofibration, a lifting in

$$\begin{tikzcd} A \arrow[r] \arrow[d] & W \arrow[d, "\sim"] \\ B \arrow[r, equal] \arrow[ur, dotted] & B \end{tikzcd}$$

exists. Thus $A \longrightarrow B$ is a retract of a $LJ$-cofibration, hence it is a $LJ$-cofibration. □

In cofibrantly generated model categories fibrations can be detected by checking the right lifting property against a *set* of maps, the generating acyclic cofibrations, and similarly for acyclic fibrations. This is in contrast to general model categories where the lifting property has to be checked against the whole class of acyclic cofibrations. Similarly, in cofibrantly generated model categories, the pushout product axiom and the monoid axiom only have to be checked for a set of generating (acyclic) cofibrations:

Lemma 1.5. *Let $\mathcal{C}$ be a cofibrantly generated model category endowed with a closed symmetric monoidal structure. If the pushout product axiom holds for a set of generating cofibrations and a set of generating acyclic cofibrations, then it holds in general.*

Proof. For the first statement consider a map $i : A \longrightarrow B$ in $\mathcal{C}$. Denote by $G(i)$ the class of maps $j : K \longrightarrow L$ such that the pushout product

$$A \wedge L \cup_{A\wedge K} B \wedge K \;\longrightarrow\; B \wedge L$$

is a cofibration. This pushout product has the left lifting property with respect to a map $f : X \longrightarrow Y$ if and only if $j$ has the left lifting property with respect to the map

$$p : [B, X] \;\longrightarrow\; [B, Y] \times_{[A,Y]} [A, X].$$

Hence, a map is in $G(i)$ if and only if it has the left lifting property with respect to the map $p$ for all $f : X \longrightarrow Y$ which are acyclic fibrations in $\mathcal{C}$.

$G(i)$ is thus closed under cobase change, transfinite composition and retracts. If $i : A \longrightarrow B$ is a generating cofibration, $G(i)$ contains all generating cofibrations by assumption; because of the closure properties it thus contains all cofibrations, see Lemma 1.1. Reversing the roles of $i$ and an arbitrary cofibration $j : K \longrightarrow L$ we thus know that $G(j)$ contains all generating cofibrations. Again by the closure properties, $G(j)$ contains all cofibrations, which proves the pushout product axiom for two cofibrations. The proof of the pushout product being an acyclic cofibration when one of the constituents is, follows in the same manner. $\square$

We now spell out the small object argument for symmetric spectra.

THEOREM 1.6 (Small object argument). *Let $I$ be a set of morphisms of symmetric spectra based on simplicial sets. Then there exists a functorial factorization of morphisms as $I$-cell complexes followed by $I$-injective morphisms.*

PROOF. In the first step we construct a functor $F$ from the category of morphisms of symmetric spectra to symmetric spectra as follows. Given a morphism $f : X \longrightarrow Y$ and a morphism $i : S_i \longrightarrow T_i$ in the set $I$ we let $D_i$ denote the set of all pairs $(a : S_i \longrightarrow X, b : T_i \longrightarrow Y)$ of morphisms satisfying $fa = bi$, i.e., which make the square

$$\begin{tikzcd} S_i \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T_i \arrow[r, "b"'] & Y \end{tikzcd}$$

commute. We define $F(f)$ as the pushout in the diagram

$$\begin{tikzcd} \bigvee_{i\in I}\bigvee_{D_i} S_i \arrow[r, "\vee a"] \arrow[d, "\vee i"'] & X \arrow[d, "j"] \\ \bigvee_{i\in I}\bigvee_{D_i} T_i \arrow[r] & F(f) \end{tikzcd}$$

The morphisms $b : T_i \longrightarrow Y$ and $f : X \longrightarrow Y$ glue to a morphism $p : F(f) \longrightarrow Y$ such that $pj = f$. The factorization we are looking for is now obtained by iterating this construction infinitely often, possibly transfinitely many times.

We define functors $F^n : Ar(\mathcal{S}p^\Sigma) \longrightarrow \mathrm{Spec}^\Sigma$ and natural transformations $X \xrightarrow{j_n} F^n(f) \xrightarrow{p_n} Y$ for every ordinal $n$ by transfinite induction. We start with $F^0(f) = X$, $j_0 = \mathrm{Id}$ and $p_0 = f$. For successor ordinal we set $F^{n+1}(f) = F(p_n : F^n(f) \longrightarrow Y)$ with the morphisms $j_{n+1} = j \circ j_n$ respectively $p_{n+1} = p(p_n)$. For limit ordinals $\lambda$ we set $F^\lambda(f) = \mathrm{colim}_{\mu<\lambda} F^\mu(f)$ with morphisms induced by the $j_\mu$ and $p_\mu$. By construction, all morphisms $j_n : X \longrightarrow F^n(f)$ are $I$-cell complexes.

We claim that there exists a limit ordinal $\kappa$, depending on the set $I$, such that for every morphism $f$ the map $p_\kappa : F^\kappa(f) \longrightarrow Y$ is $I$-injective. Then $f = p_\kappa j_\kappa$ is the required factorization.

We prove the claim under the simplifying hypothesis that for each morphism $i \in I$ the source $S_i$ is *finitely presented* as a symmetric spectrum, i.e., for every sequence $Z_0 \longrightarrow Z_1 \longrightarrow Z_2 \longrightarrow \dots$ the natural map

$$\mathrm{colim}_n\, \mathcal{S}p^\Sigma(S_i, Z_n) \;\longrightarrow\; \mathcal{S}p^\Sigma(S_i, \mathrm{colim}_n\, Z_n)$$

is bijective. In that case, the first infinite ordinal $\omega$ will do the job. Indeed, $F^\omega(f)$ is the colimit over the sequence

$$X = F^0(f) \xrightarrow{j_1} F^1(f) \xrightarrow{j_2} F^2(f) \cdots .$$

Given a morphism $i \in I$ and a lifting problem

$$\begin{tikzcd} S_i \arrow[r, "a"] \arrow[d, "i"'] & F^\omega(f) \arrow[d, "p_\omega"] \\ T_i \arrow[r, "b"'] & Y \end{tikzcd} \tag{1.7}$$

there exists a factorization $a = ca_n$ for some $n \geq 0$ and some morphism $a_n : S_i \longrightarrow F^n(f)$ since $S_i$ is finitely presented (where $c : F^n(f) \longrightarrow F^\omega(f)$ is the canonical morphism to the colimit). The commutative square

$$\begin{tikzcd} S_i \arrow[r, "a_n"] \arrow[d, "i"'] & F^n(f) \arrow[d, "p_\omega c = p_n"] \\ T_i \arrow[r, "b"'] & Y \end{tikzcd}$$

is an element in the set $D_i$ which is used to define $F^{n+1}(f) = F(p_n)$. Thus the canonical morphism $C : T_i \longrightarrow F^{n+1}(f)$ makes the diagram

$$\begin{tikzcd} S_i \arrow[r, "a_n"] \arrow[d, "i"'] & F^{n+1}(f) \arrow[d, "p_{n+1}"] \\ T_i \arrow[r, "b"'] \arrow[ur, "C"'] & Y \end{tikzcd}$$

commute. Then the composite of $C$ with the canonical morphism $F^{n+1}(f) \longrightarrow F^\omega(f)$ solves the lifting problem (1.7). $\square$

### 1.2. Bousfield's localization theorem.

THEOREM 1.8 (Bousfield). *Let $\mathcal{C}$ be a proper model category with a functor $Q : \mathcal{C} \longrightarrow \mathcal{C}$ and a natural transformation $\alpha : 1 \longrightarrow Q$ such that the following three axioms hold:*

(A1) *if $f : X \longrightarrow Y$ is a weak equivalence, then so is $Qf : QX \longrightarrow QY$;*

(A2) *for each object $X$ of $\mathcal{C}$, the maps $\alpha_{QX}, Q\alpha_X : QX \longrightarrow QQX$ are weak equivalences;*

(A3) *for a pullback square*

$$\begin{tikzcd} V \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ W \arrow[r, "h"'] & Y \end{tikzcd}$$

*in $\mathcal{C}$, if $f$ is a fibration between fibrant objects such that $\alpha : X \longrightarrow QX$, $\alpha : Y \longrightarrow QY$ and $Qh : QW \longrightarrow QY$ are weak equivalences, then $Qk : QV \longrightarrow QX$ is a weak equivalence.*

*Then the following notions define a proper model structure on $\mathcal{C}$: a morphisms $f : X \longrightarrow Y$ is a $Q$-cofibration if and only if it is a cofibration, a $Q$-equivalence if and only if $Qf : QX \longrightarrow QY$ is a weak equivalence, and $Q$-fibration if and only if $f$ is a fibration and the commutative square*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & QX \arrow[d, "Qf"] \\ Y \arrow[r, "\alpha"'] & QY \end{tikzcd}$$

*is homotopy cartesian.*

The reference is [**9**, Thm. 9.3].

### 1.3. Some equivariant homotopy theory.

PROPOSITION 1.9. *Let $G$ be a finite groups and $f : X \longrightarrow Y$ a morphism of $G$-simplicial sets. Then the following are equivalent.*

(i) *The morphism $f$ has the right lifting property for all injective morphism of $G$-simplicial sets which are weak equivalences of underlying simplicial sets.*

(ii) *The morphism $f$ is a $G$-fibration and the commutative square*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & \mathrm{map}(EG, X) \arrow[d, "\mathrm{map}(EG,f)"] \\ Y \arrow[r] & \mathrm{map}(EG, Y) \end{tikzcd}$$

*is $G$-homotopy cartesian. Here* $\mathrm{map}(EG, X)$ *is the simplicial mapping space of all maps from the contractible free $G$-simplicial set to $X$, with $G$-action by conjugation.*

*Moreover, the following are equivalent:*

(a) *$f$ is a weak equivalence of underlying simplicial set and has the equivalent properties* (i) *and* (ii) *above.*

(b) *$f$ is a $G$-acyclic fibration, i.e., induced a weak equivalence and Kan fibration on fixed points for all subgroups of $G$.*

Proof. (i)⇒(ii) If $H$ is a subgroup of $G$ then for all $n \geq 0$ and $0 \leq i \leq n$ the map $G/H \times \Lambda^i[n] \longrightarrow G/H \times \Delta[n]$ is an injective $G$-morphism and weak equivalence on underlying simplicial sets. So $f$ has the right lifting property with respect to it which means by adjunction that the induced morphism on $H$-fixed points $f^H : X^H \longrightarrow Y^H$ has the right lifting property for all horn inclusions, i.e., is a Kan fibration. So $f$ is $G$-fibration.

Now we claim that if $f$ has the right lifting property of (i), then for every injective $G$-morphism $K \longrightarrow L$ which is an underlying weak equivalence, the induced map

$$\mathrm{map}(L, X) \longrightarrow \mathrm{map}(K, X) \times_{\mathrm{map}(K,Y)} \mathrm{map}(L, Y) \tag{1.10}$$

is a $G$-fibration and $G$-equivalence. To prove this, we note that the pushout product morphism

$$G/H \times (L \times \partial\Delta[n] \cup_{K \times \partial\Delta^i[n]} K \times \Delta[n]) \longrightarrow G/H \times L \times \Delta[n]$$

is injective, equivariant and an underlying weak equivalence for all subgroups $H$ of $G$ and all boundary inclusions. Since $f : X \longrightarrow Y$ has the RLP for such maps, by adjointness the $H$-fixed points of the map (1.10) have the right lifting property for all boundary inclusions, so they are weak equivalences and Kan fibrations. Since this holds for all subgroups $H$ of $G$, the map (1.10) is a $G$-acyclic fibration.

Now we show that the square of property (ii) is $G$-homotopy cartesian. Since $f$ is $G$-fibration, so is the morphism $\mathrm{map}(EG, f)$, and hence it suffices to show that the morphism

$$X \longrightarrow Y \times_{\mathrm{map}(EG,Y)} \mathrm{map}(EG, X)$$

is a $G$-weak equivalence. The inclusion $EG \longrightarrow C(EG)$ of $EG$ into its cone is $G$-equivariant and an injective weak equivalence of underlying simplicial sets (but not a $G$-weak equivalence !). So by the previous paragraph the induced morphism

$$\mathrm{map}(C(EG), X) \longrightarrow \mathrm{map}(EG, X) \times_{\mathrm{map}(EG,Y)} \mathrm{map}(C(EG), Y)$$

is a $G$-acyclic fibration. In the commutative square

$$\begin{tikzcd} X = \mathrm{map}(*, X) \arrow[r] \arrow[d] & \mathrm{map}(EG, X) \times_{\mathrm{map}(EG,Y)} \mathrm{map}(*, Y) \arrow[d] \\ \mathrm{map}(C(EG), X) \arrow[r] & \mathrm{map}(EG, X) \times_{\mathrm{map}(EG,Y)} \mathrm{map}(C(EG), Y) \end{tikzcd}$$

the vertical maps are induced by the unique morphism $C(EG) \longrightarrow *$ which is a $G$-equivariant homotopy equivalence, so induces a homotopy equivalence on mapping spaces. So the top horizontal map is a $G$-weak equivalence since the other three maps are.

(ii)⇒(i) Let $i : K \longrightarrow L$ be a monomorphism of pointed $G$-simplicial sets which is a weak equivalence of underlying simplicial sets. Then $i$ is a cofibration in the strong $G$-model structure. Since that model structure is monoidal, the induced map

$$\mathrm{map}(L, X) \ \longrightarrow \ \mathrm{map}(K, X) \times_{\mathrm{map}(K,Y)} \mathrm{map}(L, Y)$$

is a $G$-fibration. We show that it is also a $G$-weak equivalence, thus a $G$-acyclic fibration. Since the square is $G$-homotopy cartesian, we can replace the $G$-fibration $f : X \longrightarrow Y$ be the $G$-fibration $\mathrm{map}(EG, f)$ and show that the $G$-fibration.

$$\mathrm{map}(L, \mathrm{map}(EG, X)) \ \longrightarrow \ \mathrm{map}(K, \mathrm{map}(EG, X)) \times_{\mathrm{map}(K,\mathrm{map}(EG,Y))} \mathrm{map}(L, \mathrm{map}(EG, Y))$$

is a $G$-weak equivalence. This map is isomorphic to

$$\mathrm{map}(L \times EG, X) \ \longrightarrow \ \mathrm{map}(K \times EG, X) \times_{\mathrm{map}(K\times EG,Y)} \mathrm{map}(L \times EG, Y) \ .$$

What we have gained now is that the morphism $i \times \mathrm{Id} : K \times EG \longrightarrow L \times EG$ is a $G$-equivariant weak equivalence between *free* $G$-simplicial sets, thus a $G$-weak equivalence. So the latter morphism is a $G$-acyclic fibration by the adjoint of the pushout product property.

By taking $G$-fixed points we then get an acyclic fibration of simplicial sets

$$\mathrm{map}_G(L, X) \ \longrightarrow \ \mathrm{map}_G(K, X) \times_{\mathrm{map}_G(K,Y)} \mathrm{map}_G(L, Y)$$

which is in particular surjective on vertices. This exactly means that $f : X \longrightarrow Y$ has the right lifting property with respect to $i : K \longrightarrow L$.

(a)⇒(b) The map $f$ is a $G$-fibration by assumption (ii), so we need to show that it is also a $G$-weak equivalence. We show that for every *free* $G$-simplicial set $L$ the induced map $\mathrm{map}(L, f) : \mathrm{map}(L, X) \longrightarrow \mathrm{map}(L, Y)$ is a $G$-acyclic fibration. We first show this for finite-dimensional $L$ by induction over the dimension of $L$. We start the induction with $L = \emptyset$ being empty, when there is nothing to show. If $L$ has dimension $n \geq 0$ then there exists a pushout square of $G$-simplicial sets

$$\begin{tikzcd} X \times \partial\Delta[n] \arrow[r] \arrow[d] & X \times \Delta[n] \arrow[d] \\ L' \arrow[r] & L \end{tikzcd}$$

where $L'$ has strictly smaller dimension than $L$ and $X$ is a free $G$-set. [...] In the general case of an arbitrary free $G$-simplicial set $L$ we use that $L$ is the union of its simplicial skeleta, for which we have proved the claim. So the morphism $\mathrm{map}(L, f) : \mathrm{map}(L, X) \longrightarrow \mathrm{map}(L, Y)$ is the inverse limit of a tower of $G$-acyclic fibrations, thus itself a $G$-acyclic fibration.

Now we specialize the above to the free $G$-simplicial set $L = EG$. We deduce that $\mathrm{map}(EG, f) : \mathrm{map}(EG, X) \longrightarrow \mathrm{map}(EG, Y)$ is a $G$-acyclic fibration. Since the square of condition (ii) is $G$-homotopy cartesian, the map $f : X \longrightarrow Y$ is then also a $G$-weak equivalence.

(b)⇒(a) If $f$ is a $G$-acyclic fibration, then so is $\mathrm{map}(EG, X)$. Thus the square of condition (ii) is $G$-homotopy cartesian. □

[remark about Shipley's mixed model structure]

## 2. Triangulated categories

Let $\mathcal{T}$ be a category equipped with an endofunctor $[1] : \mathcal{T} \longrightarrow \mathcal{T}$. A *triangle* in $\mathcal{T}$ (with respect to the functor [1]) is a triple $(f, g, h)$ of composable morphisms in $\mathcal{T}$ such that the target of $h$ is equal to [1] applied to the source of $f$. We will often display a triangle in the form

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1] \ .$$

A *morphism* from a triangle $(f, g, h)$ to a triangle $(f', g', h')$ is a triple of morphisms $\alpha : A \longrightarrow A'$, $\beta : B \longrightarrow B'$ and $\gamma : C \longrightarrow C'$ in $\mathcal{T}$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & A'[1] \end{tikzcd}$$

commutes. A morphism of triangles is an isomorphism (i.e., has an inverse morphism) if and only all three components are isomorphisms in $\mathcal{T}$.

Definition 2.1. A *triangulated category* is an additive category $\mathcal{T}$ together with a self-equivalence $[1] : \mathcal{T} \longrightarrow \mathcal{T}$ and a collection of triangles, called *distinguished triangles* which satisfy the following axioms (T1), (T2) and (T3).

**(T1)**

(a) For every object $X$ and every zero object 0 the triangle $X \xrightarrow{\mathrm{Id}} X \longrightarrow 0 \longrightarrow X[1]$ is distinguished.
(b) Every morphism $f$ is part of a distinguished triangle $(f, g, h)$.
(c) Any triangle which is isomorphic to a distinguished triangle is itself distinguished.

**(T2)** Distinguished triangles can be rotated: if a triangle $(f, g, h)$ is distinguished, then so is the triangle $(g, h, -f[1])$.

**(TR3)** [Octahedral axiom] Consider distinguished triangles $(f_1, g_1, h_1)$, $(f_2, g_2, h_2)$ and $(f_3, g_3, h_3)$ such that $f_1$ and $f_2$ are composable and $f_3 = f_2 f_1$. Then there exist morphisms $x$ and $y$ such that $(x, y, g_1[1] \circ h_2)$ is a distinguished triangle and the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f_1"] \arrow[d, equal] & B \arrow[r, "g_1"] \arrow[d, "f_2"'] & C \arrow[r, "h_1"] \arrow[d, dotted, "x"] & A[1] \arrow[d, equal] \\ A \arrow[r, "f_3"'] & D \arrow[r, "g_3"'] \arrow[d, "g_2"'] & E \arrow[r, "h_3"'] \arrow[d, dotted, "y"] & A[1] \arrow[d, "f_1[1]"] \\ & F \arrow[r, equal] \arrow[d, "h_2"'] & F \arrow[r, "h_2"'] \arrow[d, "g_1[1]\circ h_2"] & B[1] \\ & B[1] \arrow[r, "g_1[1]"'] & C[1] & \end{tikzcd}$$

This formulation of the axioms is due to May [**44**] who noticed a redundancy in Verdier's original axioms [**63**]. So the axioms are seemingly weaker, but in fact equivalent to the ones of Verdier [**63**]. There are two differences: Verdier's formulation (TR2) asks that distinguished triangles can be rotated in both directions, where (T2) above only asks for one direction; the extra implication is part (v) of the following proposition. Verdier has another axiom (TR3) which is the content of part (i) of the following proposition.

For the convenience of the reader we recall various useful standard facts about distinguished triangles, including Verdier's stronger forms of the axioms.

Proposition 2.2. *Let $\mathcal{T}$ be a triangulated category in the sense of Definition 2.1. Then the following two properties hold.*

(i) *Consider two distinguished triangles $(f, g, h)$ and $(f', g', h')$. Any pair $(\alpha, \beta)$ of morphisms satisfying $\beta f = f'\alpha$ can be extended to a morphism of triangles, i.e., there exists a morphism $\gamma$ making the following diagram commute*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, dotted, "\gamma"] & A[1] \arrow[d, "\alpha[1]"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & A'[1] \end{tikzcd}$$

(ii) *For every distinguished triangle $(f, g, h)$ and every object $X$ of $\mathcal{T}$, the two sequences of abelian groups*

$$\mathcal{T}(X, A) \xrightarrow{\mathcal{T}(X,f)} \mathcal{T}(X, B) \xrightarrow{\mathcal{T}(X,g)} \mathcal{T}(X, C) \xrightarrow{\mathcal{T}(X,h)} \mathcal{T}(X, A[1])$$

*and*

$$\mathcal{T}(A[1], X) \xrightarrow{\mathcal{T}(h,X)} \mathcal{T}(C, X) \xrightarrow{\mathcal{T}(g,X)} \mathcal{T}(B, X) \xrightarrow{\mathcal{T}(f,X)} \mathcal{T}(A, X)$$

*are exact.*

(iii) *Let $(\alpha, \beta, \gamma)$ be a morphism of distinguished triangles. If two out of the three morphisms are isomorphisms, then so is the third.*

(iv) *A triangle $(f, g, h)$ is distinguished if and only if the triangle $(-f[1], -g[1], -h[1])$ is distinguished.*

(v) *Distinguished triangles can be rotated to the left: given any triangle $(f, g, h)$ such that $(g, h, -f[1])$ is distinguished, then so is the original triangle $(f, g, h)$.*

PROOF. (i) [...]

(ii) We start by showing exactness of the first sequence at $\mathcal{T}(X, B)$. By part (i) applied to the pair $(\mathrm{Id}, f)$ there is a (necessarily unique) morphism from any zero object to $C$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\mathrm{Id}"] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d, dotted] & A[1] \arrow[d, equal] \\ A \arrow[r, "f"'] & B \arrow[r, "g"'] & C \arrow[r, "h"'] & A[1] \end{tikzcd}$$

commutes (the top row is distinguished by (T1 a)). So $gf = 0$ and thus the image of $\mathcal{T}(X, f)$ is contained in the kernel of $\mathcal{T}(X, g)$ for every object $X$.

Conversely, let $\psi : X \longrightarrow B$ be a morphism in the kernel of $\mathcal{T}(X, g)$, i.e., such that $g\psi = 0$. Applying part (i) to the pair $(\psi, 0)$ gives a morphism $\bar{\varphi} : X[1] \longrightarrow A[1]$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\psi"'] & 0 \arrow[r] \arrow[d] & X[1] \arrow[r, "-\mathrm{Id}"] \arrow[d, dotted, "\bar{\varphi}"] & X[1] \arrow[d, "\psi[1]"] \\ B \arrow[r, "g"'] & C \arrow[r, "h"'] & A[1] \arrow[r, "-f[1]"'] & B[1] \end{tikzcd}$$

commutes (both rows are distinguished by (T1 a) and (T2)). Since shifting is full, there exists a morphism $\varphi : X \longrightarrow A$ such that $\bar{\varphi} = \varphi[1]$, and since shifting is faithful we have $f\varphi = \psi$, so $\psi$ is in the image of $\mathcal{T}(X, f)$. Altogether, the first sequence is exact at $\mathcal{T}(X, B)$. If we apply this to the triangle $(g, h, -f[1])$ (which is distinguished by (T2)), we deduce that the first sequence is also exact at $\mathcal{T}(X, C)$. Exactness of the second sequence is similar, using part (i) for suitable maps out of the distinguished triangle $(f, g, h)$ and its rotations.

(iii) We first treat the case where $\alpha$ and $\beta$ are isomorphisms. If $X$ is any object of $\mathcal{T}$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{T}(X,A) \arrow[r, "f_*"] \arrow[d, "\alpha_*"'] & \mathcal{T}(X,B) \arrow[r, "g_*"] \arrow[d, "\beta_*"'] & \mathcal{T}(X,C) \arrow[r, "h_*"] \arrow[d, "\gamma_*"'] & \mathcal{T}(X,A[1]) \arrow[r, "(-f[1])_*"] \arrow[d, "\alpha[1]_*"] & \mathcal{T}(X,B[1]) \arrow[d, "\beta[1]_*"] \\
\mathcal{T}(X,A') \arrow[r, "f'_*"'] & \mathcal{T}(X,B') \arrow[r, "g'_*"'] & \mathcal{T}(X,C') \arrow[r, "h'_*"'] & \mathcal{T}(X,A'[1]) \arrow[r, "(-f'[1])_*"'] & \mathcal{T}(X,B'[1])
\end{tikzcd}$$

where we write $f_*$ for $\mathcal{T}(X,f)$, etc. The top row is exact by part (ii) applied to the triangles $(f,g,h)$ and $(g,h,-f[1])$, which are distinguished by hypothesis, respectively axiom (T2). Similarly, the bottom row is exact. Since $\alpha$ and $\beta$ (and hence $\alpha[1]$ and $\beta[1]$) are isomorphisms, all vertical maps except possibly the middle one are isomorphisms of abelian groups. So the five lemma says that $\gamma_*$ is an isomorphisms. Since this holds for all objects $X$, the morphism $\gamma : C \longrightarrow C'$ is an isomorphism.

If $\beta$ and $\gamma$ are isomorphisms, we apply the same argument to the triple $(\beta, \gamma, \alpha[1])$. This is a morphism from the distinguished (by (T2)) triangle $(g,h,-f[1])$ to the distinguished triangle $(g',h',-f'[1])$. By the above, $\alpha[1]$ is an isomorphism, hence so is $\alpha$ since shifting is an equivalence of categories. The third case is similar.

(iv) One direction is a direct consequence of the axioms: if $(f,g,h)$ is distinguished, then so is $(-f[1],-g[1],-h[1])$ by three applications of (T2). Now suppose that $(-f[1],-g[1],-h[1])$ is distinguished. Axiom (T1 b) let's us choose a distinguished triangle

$$A \xrightarrow{f} B \xrightarrow{\bar{g}} \bar{C} \xrightarrow{\bar{h}} A[1]$$

and by the first sentence, the triangle $(-f[1],-\bar{g}[1],-\bar{h}[1])$ is distinguished. By (i) there is a morphism $\bar{\gamma} : C[1] \longrightarrow \bar{C}[1]$ such that the diagram

$$\begin{tikzcd}
A[1] \arrow[r, "-f[1]"] \arrow[d, equal] & B[1] \arrow[r, "-g[1]"] \arrow[d, equal] & C[1] \arrow[r, "-h[1]"] \arrow[d, "\bar{\gamma}"] & A[2] \arrow[d, equal] \\
A[1] \arrow[r, "-f[1]"'] & B[1] \arrow[r, "-\bar{g}[1]"'] & \bar{C}[1] \arrow[r, "-\bar{h}[1]"'] & A[2]
\end{tikzcd}$$

commutes. By part (iii), $\gamma$ is an isomorphism. Since shifting is an equivalence of categories, we have $\bar{\gamma} = \gamma[1]$ for a unique isomorphism $\gamma : C \longrightarrow \bar{C}$. Thus $(f,g,h)$ is isomorphic to the distinguished triangle $(f,\bar{g},\bar{h})$, so it is itself distinguished by axiom (T1 c).

(v) If $(g,h,-f[1])$ is distinguished, then so is $(-f[1],-g[1],-h[1])$ by two applications of (T2). So $(f,g,h)$ is distinguished by part (v). $\square$

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