Exercise Session 2

Remark: Let E be a proper integral smooth group sch$/k$ of dim 1 then

$E \times_k E$ is still integral.

- If $k = \bar{k}$ then if A, B integral domains of fin type$/k$ then $A \otimes_k B$ is an integral domain.
- If $k \neq \bar{k}$ then this fails, e.g. $\mathbb{Q}(i) \otimes_{\mathbb{Q}} \mathbb{Q}(i) \cong \mathbb{Q}(i) \times \mathbb{Q}(i)$.
- For E: · Note that $E \times_k E$ is still smooth, i.p. reduced.
  - $E \times_k E$ is a group scheme, so connected $\Rightarrow$ irreducible.

  Why connected? E has k-rational point (neutral element)

  $\xRightarrow{+\text{connected}}$ E geometrically connected

  $\Rightarrow E \times_k E$ connected (use that all fibers of $E \times_k E \to E$ are connected)

Generic Smoothness

Lemma: Given $A \to B \to C$, there is an exact sequence

$$C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

Proof: · $\{\Omega^1_{C/A} \to \Omega^1_{C/B}\} \hat{=} \{C \xrightarrow{d} \Omega^1_{C/B}\}$. Surj. by explicit construction of Ω^1.

· Kernel gen. by db, $b \in \operatorname{im}(B \to C)$. $\square$

Different approach: Apply $\operatorname{Hom}_C(-, M)$ to the sequence and prove exactness then.

Lemma: K/k algebraic field ext. K/k separable $\Leftrightarrow \Omega^1_{K/k}=0$.

Proof: "$\Rightarrow$": Let $a\in K$, $f\in k[x]$ min poly. Then $0=d(f(a))=f'(a)\,da$.

a separable $\Rightarrow f'(a)\neq 0 \Rightarrow da=0$.

"$\Leftarrow$": Suppose K/k inseparable. Let $k'\subset K$ be max sep. ext of k. By above Lemma, $\Omega^1_{K/k}\twoheadrightarrow \Omega^1_{K/k'}$, so enough to show $\Omega^1_{K/k'}\neq 0$.

$\rightsquigarrow$ w.l.o.g. K/k purely inseparable.

Same reasoning $\rightsquigarrow$ w.l.o.g. $K=k(\alpha)$. Let $f=$ min poly (α). Then $K=k[x]/f$.

$$\rightsquigarrow \Omega^1_{K/k}=Kdx/\underbrace{f'(\alpha)}_{=0}\cong K\neq 0. \qquad \square$$

Lemma: Let char $k=0$, K/k any extension. Then

$$\dim_K \Omega^1_{K/k}=\operatorname{trdeg} K/k.$$

Proof: W.l.o.g. K/k fin. gen. Write $K\supseteq \underbrace{k(x_1,\dots,x_n)}_{k'}\supseteq k$ s.t. K/k' algebraic and k'/k purely transcendental. Then $\operatorname{trdeg} K/k=n$.

$$\overset{(a)}{\Rightarrow}\quad \underbrace{K\otimes_{k'}\Omega^1_{k'/k}}_{\cong K^n}\twoheadrightarrow \Omega^1_{K/k}\to \underset{\overset{\shortparallel\,(b)}{0}}{\Omega^1_{K/k'}}\to 0$$

$\Omega^1_{S^{-1}A/B}=S^{-1}\Omega^1_{A/B}$ and $k'=S^{-1}k[x_1,\dots,x_n]$

To show $K\otimes_{k'}\Omega^1_{k'/k}\to\Omega^1_{K/k}$ is isom, enough to show this after $\operatorname{Hom}_K(-,M)$ $\forall K$-v.s. M.

$\rightsquigarrow$ Need that $\mathrm{Der}_k(K, M) \longrightarrow \mathrm{Der}_k(k', M)$ is bijective.

(inj: By above; left to show surj.)

W.l.o.g. $K = k'(\alpha)$. Given $\delta : k' \to M$, need to extend to $k'(\alpha)$. ~~Can (even must) set $\delta(\alpha) = 0$.~~ $\square$

Let $f(x) = x^n + \ldots + b_1 x + b_0$ be min poly of α / k'. Then $0 = df(\alpha) = f'(\alpha)\, d\alpha + db_0 + \alpha\, db_1 + \ldots + \alpha^n \rightsquigarrow d\alpha$ uniquely determined

<u>Prop</u>: Let char $k = 0$, X/k reduced scheme locally of fin type. Then $\exists$ open dense $U \subset X$ s.t. U/k smooth.

<u>Proof</u>: W.l.o.g. X quasicompact (even affine). Then $X = \bigcup_{i=1}^{n} X_i$ for X_i irreducible. Let $Z = \bigcup_{i \neq j} X_i \cap X_j$. Then Z is nowhere dense in $X \rightsquigarrow$ w.l.o.g. replace X by $X \setminus Z$. $\rightsquigarrow$ W.l.o.g. X irreducible

$\Rightarrow X$ is integral, let $\eta(X)$ be the generic point.

By (c),

$$\dim_{\kappa(\eta(X))} \Omega^1_{X/k, \eta(X)} = \operatorname{trdeg} \kappa(\eta(X))/k = \dim X$$

Thus, locally around $\eta(X)$, $\Omega^1_{X/k}$ is free of rank $\dim X$. $\square$

(In general: A noeth ring, M fin. A-module, $\mathfrak{p} \subseteq A$ prime ideal s.t. $M_\mathfrak{p} \cong A_\mathfrak{p}^n$. Then $\exists g \in A \setminus \mathfrak{p}$ s.t. $M_g \cong A_g^n$.)

<u>Cor</u>: Let char $k = 0$, G reduced grp sch$/k$ of loc. fin. type then G is smooth.

<u>Proof</u>: W.l.o.g. $k = \bar{k}$ (reducedness is stable under passing to separable field ext). By Prop, $\exists$ dense open smooth $U \subseteq G$.

Suppose $\exists$ closed pt $x \in X \setminus U$. Choose closed pt $x' \in U$. Then

{closed pts $\in G$} = $G(k)$.

$$\cdot\left(\tfrac{x}{x'}\right) : G \to G$$

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is isom and maps x' to x. $\Rightarrow X$ is smooth. $\square$

Lifting Properties

Prop: Let X be a k-scheme loc. of fin. type. Then X is smooth iff $\forall$ k-alg R, ideals $I \subseteq R$ s.t. $I^2 = 0$, $X(R) \to X(R/I)$ is surj.

(2) (6) X, R as above, with $X = \operatorname{Spec} A/J$, $A = k[x_1, \dots, x_n]$, s.t.

$$0 \to J/J^2 \to \Omega^1_{A/k} \otimes_A A/J \to \Omega^1_{X/k} \to 0$$

is split exact. Then $X(R) \twoheadrightarrow X(R/I)$ surj.

Let $\bar\varphi : A/J \to R/I$ given. Can lift this to a map $\varphi : A \to R$. Then $\varphi(J) \subseteq I \rightsquigarrow$ get A/J-linear map $\varphi' : J/J^2 \to I$. By above splitting, can extend φ' to

$$\varphi'' : \Omega^1_{A/k} \otimes_A A/J \to I,$$

$\operatorname{Hom}_{A/J}(M \otimes_A A/J, N)$

$= \operatorname{Hom}_A(M, N)$

equiv. a derivation $\delta : A \to I$.

Then $\delta|_J = \varphi|_J$. Hence $(\varphi - \delta) : A \to R$ factors through B. $\square$