Solutions for exercises, Algebra I (Commutative Algebra) – Week 3

Exercise 9. (Adjunction)
Let us define $\chi : \mathrm{Hom}_A(M, {}_AN) \to \mathrm{Hom}_B(M \otimes_A B, N)$ by $\varphi \mapsto \chi(\varphi) = [m \otimes b \mapsto b\varphi(m)]$.
For $\varphi \in \mathrm{Hom}_A(M, {}_AN)$, $\chi(\varphi) \in \mathrm{Hom}_B(M \otimes_A B, N)$: indeed for $m, m' \in M$ and $b, b', b'' \in B$,

$$\begin{aligned}\chi(\varphi)(b''(m \otimes b + m' \otimes b')) = \chi(\varphi)(m \otimes b''b + m' \otimes b''b') &= \chi(\varphi)(m \otimes b''b) + \chi(\varphi)(m' \otimes b''b') \\ &= b''b\varphi(m) + b''b'\varphi(m') \\ &= b''\chi(\varphi)(m \otimes b) + b''\chi(\varphi)(m' \otimes b').\end{aligned}$$

χ is a group homomorphism: for $\varphi, \psi \in \mathrm{Hom}_A(M, {}_AN)$,

$$\begin{aligned}\chi(\varphi - \psi) = [m \otimes b \mapsto b(\varphi(m) - \psi(m))] &= [m \otimes b \mapsto b\varphi(m) - b\psi(m))] \\ &= [m \otimes b \mapsto \chi(\varphi(m \otimes b)) - \chi(\psi(m \otimes b))] \\ &= \chi(\varphi) - \chi(\psi).\end{aligned}$$

χ is injective: if $\chi(\varphi) = 0$, then for $m \in M$, we have $0 = \chi(\varphi)(m \otimes 1) = 1 \cdot \varphi(m) = \varphi(m)$ so $\varphi = 0$.

χ is surjective: given $\psi \in \mathrm{Hom}_B(M \otimes_A B, N)$, let us define $\varphi : M \to N$ by $m \mapsto \psi(m \otimes 1)$. Then φ is clearly a group homomorphism and for $a \in A$, and $m \in M$, $\varphi(am) = \psi(am \otimes 1) = \psi(m \otimes f(a)) = f(a)\psi(m \otimes 1) = f(a)\varphi(m)$ so $\varphi \in \mathrm{Hom}_A(M, {}_AN)$. Now, we have $\chi(\varphi) = [m \otimes b \mapsto b\varphi(m)] = [m \otimes b \mapsto b\psi(m \otimes 1)] = [m \otimes b \mapsto \psi(m \otimes b)] = \psi$.
So χ is a group isomorphism.

The B-module structure on $\mathrm{Hom}_B(M \otimes_A B, N)$ is the structure seen in Exercise 7. For any $\varphi \in \mathrm{Hom}_A(M, {}_AN)$ and $b \in B$ let us define, using the structure of B-module on N, $b\varphi : m \mapsto b\varphi(m)$; then $b\varphi \in \mathrm{Hom}_A(M, {}_AN)$: for $a \in A$ and $m, m' \in M$,

$$\begin{aligned}b\varphi(a(m + m')) = b\varphi(am) + b\varphi(am') &= bf(a)\varphi(m) + bf(a)\varphi(m') \\ &= f(a)b\varphi(m) + f(a)b\varphi(m') \\ &= a \cdot b\varphi(m) + a \cdot b\varphi(m')\end{aligned}$$

Because of the structure of B-module on N, (it is easy to check that) the operation just defined $B \times \mathrm{Hom}_A(M, {}_AN) \to \mathrm{Hom}_A(M, {}_AN)$ satisfies all the axioms required to give a B-module structure on $\mathrm{Hom}_A(M, {}_AN)$.
Moreover, $\chi(b\varphi) = [m \otimes b' \mapsto b'b\varphi(m)] = [m \otimes b' \mapsto bb'\varphi(m)] = b[m \otimes b' \mapsto b'\varphi(m)] = b\chi(\varphi)$ so χ is a homomorphism of B-modules (thus an isomorphism of B-modules) when $\mathrm{Hom}_A(M, {}_AN)$ is given B-module structure just defined.

The A-module structure on $\mathrm{Hom}_A(M, {}_AN)$ is the structure seen in Exercise 7. We give $\mathrm{Hom}_B(M \otimes_A B, N)$ the A-module structure ${}_A\mathrm{Hom}_B(M \otimes_A B, N)$. Then for $a \in A$ and $\varphi \in \mathrm{Hom}_A(M, {}_AN)$, $\chi(a \cdot \varphi) = [m \otimes b \mapsto b(a \cdot \varphi)(m)] = [m \otimes b \mapsto bf(a)\varphi(m)] = [m \otimes b \mapsto f(a)b(m)] = f(a)[m \otimes b \mapsto b\varphi(m)] = a \cdot \chi(\varphi)$ so χ is a homomorphism of A-modules.

Solutions to be handed in before Monday April 27, 4pm.

Exercise 10. (Deducing exactness)
Let us start by proving that $f \circ g = 0$ i.e. that $\mathrm{im}(g) \subset \ker(f)$: apply the assumption to $N = M_3$, we get that

$$0 \to \mathrm{Hom}(M_3, M_3) \overset{\circ f}{\to} \mathrm{Hom}(M_2, M_3) \overset{\circ g}{\to} \mathrm{Hom}(M_1, M_3)$$

is exact. In particular, $\mathrm{id}_{M_3} \circ f \circ g = 0 \in \mathrm{Hom}(M_1, M_3)$ i.e. $f \circ g = 0$.

To prove the reverse inclusion i.e. $\ker(f) \subset \mathrm{im}(g)$, apply the assumption to $N = M_2/\mathrm{im}(g)$:

$$0 \to \mathrm{Hom}(M_3, M_2/\mathrm{im}(g)) \overset{\circ f}{\to} \mathrm{Hom}(M_2, M_2/\mathrm{im}(g)) \overset{\circ g}{\to} \mathrm{Hom}(M_1, M_2/\mathrm{im}(g))$$

is exact. The exactness in the middle can be written $\ker(- \circ g) = \mathrm{im}(- \circ f)$. Now, consider the projection homomorphism $\pi : M_2 \to M_2/\mathrm{im}(g)$. We have $\pi \in \ker(- \circ g)$ so there is a $\varphi \in \mathrm{Hom}(M_3, M_2/\mathrm{im}(g))$ such that $\pi = \varphi \circ f$. Let $m_2 \in \ker(f)$, we have $\pi(m_2) = \varphi \circ f(m_2) = \varphi(f(m_2)) = \varphi(0) = 0$ i.e. $m_2 \in \mathrm{im}(g)$. So we get $\ker(f) \subset \mathrm{im}(g)$. Hence $\ker(f) = \mathrm{im}(g)$.

To prove that f is surjective, apply the assumption to $N = M_3/\mathrm{im}(f)$:

$$0 \to \mathrm{Hom}(M_3, M_3/\mathrm{im}(f)) \overset{\circ f}{\to} \mathrm{Hom}(M_2, M_3/\mathrm{im}(f)) \overset{\circ g}{\to} \mathrm{Hom}(M_1, M_3/\mathrm{im}(f))$$

is exact. Consider the projection $\pi : M_3 \to M_3/\mathrm{im}(f) \in \mathrm{Hom}(M_3, M_3/\mathrm{im}(f))$; we have of course $\pi \circ f = 0 \in \mathrm{Hom}(M_2, M_3/\mathrm{im}(f))$ but since $- \circ f$ is injective, we get $\pi = 0$ i.e. $M_3/\mathrm{im}(f) = 0$ i.e. $M_3 = \mathrm{im}(f)$.

Exercise 11. (Examples of exact sequences)

1. The map β is surjective: $(m_1, -m_2) \in M_1 \oplus M_2$ is a preimage of $m_1 + m_2 \in M_1 + M_2$.
   The map α is injective: if $\alpha(m) = (0, 0)$, then $m = 0$.
   For $m \in M_1 \cap M_2$, we have $\beta \circ \alpha(m) = \beta((m, m)) = m - m = 0$ i.e. $\mathrm{im}(\alpha) \subset \ker(\beta)$.
   Now, let $(m_1, m_2) \in \ker(\beta)$, then $m_1 - m_2 = 0$ i.e. $M_1 \ni m_1 = m_2 \in M_2$ hence $m_1 = m_2 \in M_1 \cap M_2$. Thus $(m_1, m_2) = \alpha(m_1)$. So we get $\mathrm{im}(\alpha) = \ker(\beta)$.

   We start by proving some properties of the sequence of the two last items that are independent of $f \in k[x, y, z]$.
   φ_1 is surjective: by assumption, an element $a \in \mathfrak{a}$ can be written $a = (x+z)p + qy + rf$ for some $p, q, r \in A$ so we have $\varphi_1(p, q, r) = a$.
   We have $\varphi_1 \circ \varphi_2 = 0$ i.e. $\mathrm{im}(\varphi_2) \subset \ker(\varphi_1)$: for $(p, q, r) \in A^3$,

$$\begin{aligned}\varphi_1 \circ \varphi_2(pe_1 \wedge e_2 + qe_1 \wedge e_3 + re_2 \wedge e_3) &= \varphi_1((x+z)pe_2 - ype_1 + q(x+z)e_3 - qfe_1 + yre_3 - rfe_2) \\ &= (x+z)py - yp(x+z) + (x+z)qf - qf(x+z) + yrf - rfy \\ &= 0\end{aligned}$$

   We have $\varphi_2 \circ \varphi_3 = 0$ i.e. $\mathrm{im}(\varphi_3) \subset \ker(\varphi_2)$: for $p \in A$,

$$\begin{aligned}\varphi_2 \circ \varphi_3(pe_1 \wedge e_2 \wedge e_3) &= \varphi_2((x+z)pe_2 \wedge e_3 - ype_1 \wedge e_3 + pfe_1 \wedge e_2) \\ &= (x+z)pye_3 - (x+z)pfe_2 - yp(x+z)e_3 + ypfe_1 + pf(x+z)e_2 - pfye_1 \\ &= 0\end{aligned}$$

   φ_3 is injective: we have $\wedge^3 A^3 \simeq Ae_1 \wedge e_2 \wedge e_3$ and the image of the generator is not 0 and A is an integral domain.

2. Let us show that $\ker(\varphi_1) \subset \operatorname{im}(\varphi_2)$: By a direct calculation $(0,-z,y)$, $(-z,0,x+z)$ and $(-y,x+z,0)$ belong to $\ker(\varphi_1)$. By a direct calculation, we also see that (taking the basis $(e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3)$ for $\wedge^2 A^3$) $\varphi_2(1,0,0) = (-y,x+z,0)$, $\varphi_2(0,1,0) = (-z,0,x+z)$ and $\varphi_2(0,0,1) = (0,-z,y)$. So to prove the claim, it is sufficient to prove that $(0,-z,y)$, $(-z,0,x+z)$ and $(-y,x+z,0)$ generate $\ker(\varphi_1)$.
So let $(p,q,r) \in \ker(\varphi_1)$ then

$$p(x+z) + qy + rz = 0. \tag{*}$$

(Partially) evaluating (*) at $(x,y,0)$, we get $p(x,y,0)x + q(x,y,0)y = 0$ in $k[x,y]$. In particular $y|p(x,y,0)$ and $x|q(x,y,0)$. So we can write $p = yp_1 + zp_2$ and $q = xq_1 + zq_2$ for some polynomials $p_1, q_1 \in k[x,y]$ and $p_2, q_2 \in A$. Looking back to the evaluation at $(x,y,0)$, we have $xy(p_1+q_1) = 0$ so $p_1 = -q_1$ in $k[x,y]$.

Now, evaluating (*) at $(x,0,z)$, we get in $k[x,z]$,

$$0 = p(x,0,z)(x+z) + r(x,0,z)z = z((x+z)p_2(x,0,z) + r(x,0,z))$$

So $(x+z)|r(x,0,z)$ i.e. we can write $r = (x+z)r_1 + yr_2$ for some polynomials $r_1 \in k[x,z]$ and $r_2 \in A$. Looking back to the evaluation at $(x,0,z)$, we get $0 = z(x+z)(p_2(x,0,z) + r_1)$ in $k[x,z]$. Thus $p_2 = -r_1 + yp_3$ for some $p_3 \in A$.
Evaluating (*) at $(x,y,-x)$, we get in $k[x,y]$,

$$0 = q(x,y,-x)y - xr(x,y,-x) = xy(q_1(x,y) - q_2(x,y,-x) - r_2(x,y,-x))$$

so we can write $q_2 = q_3 + (x+z)q_4$, $r_2 = q_1 - q_3 + (x+z)r_3$ for some $q_3 \in k[x,y]$ and $q_4, r_3 \in A$. At this point, we have:

$$p = p_1y - zr_1 + yzp_3$$

$$q = -p_1x + zq_3 + (x+z)zq_4$$
$$r = (x+z)r_1 - (p_1+q_3)y + y(x+z)r_3$$

Now plugging it into (*), we get $p_3 + q_4 + r_3 = 0$.
Now check that $\begin{pmatrix} p \\ q \\ r \end{pmatrix} = -p_1 \begin{pmatrix} -y \\ x+z \\ 0 \end{pmatrix} - (yp_3 - r_1) \begin{pmatrix} -z \\ 0 \\ x+z \end{pmatrix} - (p_1 + q_3 + (x+z)q_4) \begin{pmatrix} 0 \\ -z \\ y \end{pmatrix}$
proving that $\ker(\varphi_1) \subset \operatorname{im}(\varphi_2)$.

Let us show that $\ker(\varphi_2) \subset \operatorname{im}(\varphi_3)$: let $(p,q,r) \in \ker(\varphi_2)$ then we have

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \varphi_2\left(\begin{pmatrix} p \\ q \\ r \end{pmatrix}\right) = \begin{pmatrix} -py - qz \\ (x+z)p - rz \\ q(x+z) + ry \end{pmatrix}$$

Looking at the first line: we get $z|p$ and $y|q$; so let us write $p = zp_1$ and $q = yq_1$. Looking again at the first line, we get $p_1 = -q_1$.
Looking at the second line, we get $(x+z)|r$ so we can write $r = (x+z)r_1$. The second line again, gives $p_1 = r_1$. So $\varphi_3(p_1) = \begin{pmatrix} p_1 z \\ -yp_1 \\ p_1(x+z) \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$ proving $\ker(\varphi_2) \subset \operatorname{im}(\varphi_3)$.

3. It is immediate to check that $(1,-y,-1) \in A^3$ (i.e. $e_1 - ye_2 - e_3$) is in the kernel of φ_1 since $(x+z) + (-y)y + (-1)(x - y^2 + z) = 0$. Suppose that the sequence is exact. Then we have a $(p,q,r) \in \wedge^2 A^3$ (i.e. $pe_1\wedge e_2 + qe_1\wedge e_3 + re_2\wedge e_3$), such that $\varphi_2(p,q,r) = (1,-y,-1)$. On the first component, we get $1 = py - q(x - y^2 + z)$. But evaluating the equality at $(0,0,0) \in k^3$, we have $1 = p(0,0,0)\cdot 0 - q(0,0,0)\cdot 0$ which is absurd so the inclusion $\operatorname{im}(\varphi_2) \subset \ker(\varphi_1)$ is strict i.e. the sequence is not exact.

Exercise 12. (Flat, free, projective)

1. Since A is an integral domain, the principal ideal (a) is a free module $A \overset{\varphi}{\simeq} Aa = M$ as A module (if $ax = \varphi(x) = 0$ then $x = 0$ and by definition an element $x \in M$ can be written $x = ay$, with $y \in A$, so $x = \varphi(y)$).

2. Let us prove that $k(x)$ is a flat $k[x]$-module. Let $\alpha : N \hookrightarrow N'$ be an injective homomorphism of $k[x]$-modules; we want to see that $\alpha \otimes \mathrm{id}_{k(x)} : N \otimes_{k[x]} k(x) \to N' \otimes_{k[x]} k(x)$ is injective [[[be careful; the proof in the previous version contained a mistake]]]. Let $\sum_i n_i \otimes \frac{p_i}{q_i} \in N \otimes_{k[x]} k(x)$ such that $\alpha \otimes \mathrm{id}_{k(x)}(\sum_i n_i \otimes \frac{p_i}{q_i}) = 0$, then:

$$\begin{aligned} 0 = \alpha \otimes \mathrm{id}_{k(x)}(\sum_i n_i \otimes \frac{p_i}{q_i}) &= \alpha \otimes \mathrm{id}_{k(x)}(\sum_i n_i \otimes_{k[x]} \frac{p_i}{q_i}) \\ &= \alpha \otimes \mathrm{id}_{k(x)}(\sum_i n_i \otimes_{k[x]} \frac{p_i}{\Pi_k q_k} \Pi_{k \neq i} q_k) \\ &= \alpha \otimes \mathrm{id}_{k(x)}(\sum_i p_i(\Pi_{k\neq i} q_k) n_i \otimes_{k[x]} \frac{1}{\Pi_k q_k}) \\ &= \alpha(\sum_i p_i(\Pi_{k\neq i} q_k) n_i) \otimes_{k[x]} \frac{1}{\Pi_k q_k} \end{aligned}$$

Now look at the homomorphism of $k[x]$-modules $\mu : k[x] \to N$ given by $f \mapsto f \sum_i p_i(\Pi_{k\neq i} q_k) n_i$. If $\alpha \circ \mu$ is injective, it gives an isomorphism of $k[x]$-modules $k[x] \simeq \mathrm{im}(\alpha \circ \mu) = \langle \alpha(\sum_i p_i(\Pi_{k\neq i} q_k) n_i) \rangle$ (i.e. $\mathrm{im}(\alpha \circ \mu)$ is a free submodule of N'). Then $\mathrm{im}(\alpha \circ \mu) \otimes_{k[x]} k(x) \simeq k[x] \otimes_{k[x]} k(x) \simeq k(x)$. In particular $\alpha(\sum_i p_i(\Pi_{k\neq i} q_k) n_i) \otimes_{k[x]} \frac{1}{\Pi_k q_k} =\neq 0$; contradiction. So $\alpha \circ \mu$ is not injective and since α is injective, we get that μ is not injective. Its kernel is a $k[x]$-submodule of $k[x]$ i.e. an ideal (the annihilator of $\sum_i p_i(\Pi_{k\neq i} q_k) n_i$) and since $k[x]$ is a principal ideal domain, $\ker(\mu) = (g)$ for some $g \in k[x] \backslash \{0\}$. Then we have in $N \otimes_{k[x]} k(x)$:

$$\begin{aligned} \sum_i n_i \otimes_{k[x]} \frac{p_i}{q_i} &= \sum_i n_i \otimes_{k[x]} \frac{g p_i}{g q_i} \\ &= \sum_i n_i \otimes_{k[x]} \frac{g p_i}{g \Pi_k q_k} \Pi_{k\neq i} q_k \\ &= \sum_i g p_i (\Pi_{k\neq i} q_k) n_i \otimes_{k[x]} \frac{1}{g \Pi_k q_k} \\ &= g(\sum_i p_i(\Pi_{k\neq i} q_k) n_i) \otimes_{k[x]} \frac{1}{g \Pi_k q_k} \\ &= 0 \otimes_{k[x]} \frac{1}{g \Pi_k q_k} = 0 \end{aligned}$$

so $\alpha \otimes \mathrm{id}_{k(x)}$ is injective.

The $k[x]$-module $k(x)$ is not projective. An easy way to see that is to use the following fact:

$$\text{Let } P \text{ be a } A\text{-module. Then } P \text{ is projective if and only if } \exists M \simeq \oplus_{i \in I} A \text{ and an } A\text{-module } N \text{ such that } M \simeq P \oplus N \qquad (*)$$

(i.e. P is a direct summand of a free module). To prove this, look at the surjective morphism $\oplus_{p \in P} A \overset{\alpha}{\to} P$ given, on the component associated to $p \in P$, by $a \mapsto ap$ and use the fact that P is projective to lift id_P. Conversely, if P is a direct summand of a free module $\oplus_i A \simeq P \oplus Q$, then the projection $p_P : \oplus_i A \to P$ and the inclusion $i_P : P \to \oplus_i A$ satisfy $p_P \circ i_P = \mathrm{id}_P$. Now let $g : M \to N$ be a surjective homomorphism of A-modules, and $f : P \to N$ a homomorphism. Then $f \circ p_P : \oplus_i A \to N$ gives us a homomorphism and since free modules are flat, there is a $f' : \oplus_i A \to M$ such that $g \circ f' = f \circ p_P$. Now $f' \circ i_P : P \to M$ satisfies $g \circ f' \circ i_P = f \circ p_P \circ i_P = f$.

So if $k(x)$ is projective, we should have, in particular, an injective homomorphism of $k[x]$-module $\alpha : k(x) \to \oplus_{i\in I} k[x]$ for some set I. Looking at one of its components (compose α with the projection $\oplus_{i\in I} k[x] \to k[x]$), we get a homomorphism of $k[x]$-modules $\alpha_i : k(x) \to k[x]$. Let us denote $f = \alpha_i(1) \in k[x]$. If $f \neq 0$, it as finitely many irreducible divisors so take $g \in k[x]$ irreducible not dividing f. We have $g \underbrace{\alpha_i(\frac{1}{g})}_{\in k[x]} = \alpha_i(g\frac{1}{g}) = \alpha_i(1) = f$ so $g|f$. Contradiction. So $\alpha_i(1) = 0$. Thus (i was arbitrary), $\alpha = 0$. In particular there is no injection of $k[x]$-module from $k(x)$ to a free $k[x]$-module. So $k(x)$ is not projective (in particular not free).

3. The injection $M \hookrightarrow A$ is a homomorphism of A modules. So (by definition of A), M is a direct summand of the free A-module A and as such, it is projective (in particular it is flat).
   But M is not free: indeed M is a finitely generated non-zero A-module so if M is free, there is an isomorphism of A-modules $M \simeq A^d$ for a $d > 0$. But have $\dim_k(A^d) = d\dim_k(A) = d(\deg(f) + 1) > 1 = \dim_k M$. Contradiction.

Exercise 13. (Long exact cohomology sequences)
Let us first prove that for any i, the sequence

$$0 \to \ker(a_i) \overset{f_{i|\ker(a_i)}}{\longrightarrow} \ker(b_i) \overset{g_{i|\ker(b_i)}}{\longrightarrow} \ker(c_i)$$

is exact. First, the sequence is well-defined:
For $x \in \ker(a_i)$, $b_i(f_i(x)) = b_i \circ f_i(x) = f_{i+1} \circ a_i(x) = f_{i+1}(0) = 0$. Thus $\mathrm{im}(f_{i|\ker(a_i)}) \subset \ker(b_i)$. Similarly, using $c_i \circ g_i = g_{i+1} \circ b_i$, one sees that $\mathrm{im}(g_{i|\ker(b_i)}) \subset \ker(c_i)$. So the sequence is well-defined.
The restriction of an injective morphism to a subset is clearly injective (as composition of two injective maps) so $f_{i|\ker(a_i)}$ is injective.
As $g_i \circ f_i = 0$, by restriction $g_{i|\ker(b_i)} \circ f_{i|\ker(a_i)} = 0$ i.e. $\mathrm{im}(f_{i|\ker(a_i)}) \subset \ker(g_{i|\ker(b_i)})$. For $y \in \ker(g_{i|\ker(b_i)})$ let $x \in M^i$ such that $f_i(x) = y$ (by exactness $0 \to M^i \to N^i \to P^i \to 0$); then $f_{i+1} \circ a_i(x) = b_i(f_i(x)) = b_i(y) = 0$ ($y \in \ker(b_i)$) so $a_i(x) \in \ker(f_{i+1})$; but f_{i+1} is assumed to be injective so $a_i(x) = 0$ i.e. $x \in \ker(a_i)$ i.e. $\mathrm{im}(f_{i|\ker(a_i)}) = \ker(g_{i|\ker(b_i)})$.

Similarly, for any i, the sequence:

$$M^{i+1}/\mathrm{im}(a_i) \overset{\overline{f_{i+1}}}{\to} N^{i+1}/\mathrm{im}(b_i) \overset{\overline{g_{i+1}}}{\to} P^{i+1}/\mathrm{im}(c_i) \to 0$$

is exact. It is a well-defined since for $x \in M^{i+1}$ and $x' \in M^i$, $f_{i+i}(x + a_i(x')) = f_{i+1}(x) + f_{i+1} \circ a_i(x') = f_{i+1}(x) + \underbrace{b_i(f_i(x'))}_{\in \mathrm{im}(b_i)}$. A similar calculation shows that $\overline{g_{i+1}}$ is a well defined homomorphism of A-modules.
The surjectivity of $\overline{g_{i+1}}$ follows directly from the surjectivity of g_{i+1} so does the equality $\overline{g_{i+1}} \circ \overline{f_{i+1}} = 0$ from $g_{i+1} \circ f_{i+1} = 0$. The equality $\mathrm{im}(\overline{f_{i+1}}) = \ker(\overline{g_{i+1}})$ follows also from the corresponding the corresponding equality before passing to the quotients.

For any i, by assumption, we have: $\mathrm{im}(a_i) \subset \ker(a_{i+1})$, $\mathrm{im}(b_i) \subset \ker(b_{i+1})$ and $\mathrm{im}(c_i) \subset \ker(c_{i+1})$. So have the following commutative (follows from the commutativity $b_i \circ f_i = f_{i+1} \circ a_i$, $c_i \circ g_i = g_{i+1} \circ b_i$) diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^0 & \xrightarrow{f_0} & N^0 & \xrightarrow{g_0} & P^0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_0} & & \downarrow{\scriptstyle b_0} & & \downarrow{\scriptstyle c_0} & & \\
0 & \longrightarrow & \ker(a_1) & \xrightarrow{f_{1|\ker(a_1)}} & \ker(b_1) & \xrightarrow{g_{1|\ker(b_1)}} & \ker(c_1) & &
\end{array}$$

Now go through the proof of the snake lemma and check that neither the surjectivity of (what corresponds here to) $g_{1|\ker(b_1)}$ nor the injectivity of (what corresponds here to) f_0 were used to construction of the boundary homomorphism $\delta : \ker(c_0) = H^0(M^\bullet) \to \mathrm{Coker}(a_0) = \ker(a_1)/\mathrm{im}(a_0) = H^1(M^\bullet)$ and neither were they used to prove the exactness of the induced sequence; so the following sequence of A-modules is exact:

$$H^0(M^\bullet) \to H^0(N^\bullet) \to H^0(P^\bullet) \to H^1(M^\bullet) \to H^1(N^\bullet) \to H^1(P^\bullet).$$

Moreover, we have also seen that $H^0(M^\bullet) = \ker(a_0) \hookrightarrow \ker(b_0) = H^0(N^\bullet)$.

Using the preliminary discussion, and again that $\mathrm{im}(a_i) \subset \ker(a_{i+1})$, $\mathrm{im}(b_i) \subset \ker(b_{i+1})$ and $\mathrm{im}(c_i) \subset \ker(c_{i+1})$, we have, for $i \geq 1$, the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} & & M^i/\mathrm{im}(a_{i-1}) & \xrightarrow{\overline{f_i}} & N^i/\mathrm{im}(b_{i-1}) & \xrightarrow{\overline{g_i}} & P^i/\mathrm{im}(c_{i-1}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{a_i}} & & \downarrow{\scriptstyle \overline{b_i}} & & \downarrow{\scriptstyle \overline{c_i}} & & \\ 0 & \longrightarrow & \ker(a_{i+1}) & \xrightarrow{f_{i+1|\ker(a_{i+1})}} & \ker(b_{i+1}) & \xrightarrow{g_{i+1|\ker(b_{i+1})}} & \ker(c_{i+1}) & & \end{array}$$

By the previous remark (namely that the proof of the snake lemma presented in the lecture requires less hypothesis than assumed in the statement) we get the following exact sequence:

$$H^i(M^\bullet) \to H^i(N^\bullet) \to H^i(P^\bullet) \to H^{i+1}(M^\bullet) \to H^{i+1}(N^\bullet) \to H^{i+1}(P^\bullet).$$

Exercise 14. (Direct limit)
Let us denote $\pi_M : \oplus M_i \to \varinjlim M_i$ the canonical projection.

1. For $x \in \varinjlim M_i$, take $m \in \oplus_i M_i$ such that $x = \pi_M(m)$. We can write $m = \sum_{k=1}^n m_{i_k}$ with $m_{i_k} \in M_{i_k}$. By hypothesis, we can find a $i_1, i_2 \leq \ell'$ and next a $i_3, \ell' \leq \ell''$. Then $i_1, i_2, i_3 \leq \ell''$. So we see that by an elementary induction, we can find a $\ell \in I$ such that $i_1, \ldots, i_k \leq \ell$. Set $m' = \sum_{k=1}^n f_{i_k \ell}(m_{i_k}) \in M_\ell$. We have $m - m' = \sum_{k=1}^n m_{i_k} - f_{i_k \ell}(m_{i_k})$; in particular $m - m' \in \ker(\pi_M)$ so $\pi_M(m') = x$.

2. Let us begin by proving the following fact:

$$\text{Let } m \in M_i \cap \ker(\pi_M), \text{ then } \exists j \geq i \text{ such that } f_{ij}(m) = 0 \in M_j \qquad (*)$$

   For such $m \in M_i \cap \ker(\pi_M)$, we can write $m = \sum_{k=1}^n n_{i_k} - f_{i_k j_k}(n_{i_k})$ for some elements $i_k \leq j_k$ $(k = 1, \ldots n)$ of I and $n_{i_k} \in M_{i_k}$. Since we have a direct sum $(\oplus_i M_i)$, and $m \in M_i$, in the previous sum, all the terms that are lying on a M_l with $l \neq i$ have to vanish. So let us reorganise the sum: $m = \sum_{k=1}^n n_{i_k} - f_{i_k j_k}(n_{i_k}) = \sum_k w_{p_k}$ where $w_{p_k} \in M_{p_k}$, the p_k's are chosen among $\cup_{\ell=1}^n \{i_\ell, j_\ell\}$ and $w_{p_k} = 0$ for $p_k \neq i$ (so in the sum there is just $m = w_i$). Let us choose $r \in I$ such that $r \geq j_k \geq i_k$ for any $k \in \{1, \ldots, n\}$. Then

$$f_{i,r}(m) = f_{i,r}(w_i) = f_{i,r}(w_i) + \sum_{p_k \neq i} \underbrace{f_{p_k r}(w_{p_k})}_{f_{p_k r}(0)=0}.$$

   Now, each w_{p_k} is of the form $\sum n_a - \sum f_{q p_k}(n_q)$ for some $n_a \in M_{p_k}$ and $q \leq p_k$ and $n_q \in M_q$; so $f_{p_k r}(w_{p_k}) = \sum f_{p_k r}(n_a) - \sum f_{p_k r} \circ f_{q p_k}(n_q)$ so we can reorganize terms as follow:

$$\begin{aligned} f_{i,r}(m) = f_{i,r}(w_i) + \sum_{p_k \neq i} f_{p_k r}(w_{p_k}) &= \sum_{p_k} f_{p_k r}(w_{p_k}) \\ &= \sum_{k=1}^n f_{i_k r}(n_{i_k}) - f_{j_k r} \circ f_{i_k j_k}(n_{i_k}) \\ &= \sum_{k=1}^n f_{i_k r}(n_{i_k}) - f_{i_k r}(n_{i_k}) \\ &= 0 \text{ proving the fact.} \end{aligned}$$

Now, let $(g_i : M_i \to N)_{i\in I}$ be a system of homomorphisms of A-modules, such that $g_i = g_j \circ f_{ij}$ for any $i \le j$. Define a map $g : \varinjlim M_i \to N$ by $x \mapsto g_i(m)$ where $m \in M_i$ is such that $\pi_M(m) = x$ (which exists by the first question).
Let us first prove that it is well-defined. For $x \in \varinjlim M_i$, let $m \in M_i$ and $m' \in M_j$ such that $\pi_M(m) = x = \pi_M(m')$. Pick a $i, j \le k$ then by definition $m - f_{ik}(m) \in \ker(\pi_M)$, $m' - f_{ik}(m') \in \ker(\pi_M)$ and by assumption $m - m' \in \ker(\pi_M)$ so $f_{ik}(m) - f_{jk}(m') \in \ker(\pi_M) \cap M_k$. By (*), there is a $\ell \ge k$, such that $f_{k\ell}(f_{ik}(m) - f_{jk}(m')) = 0 \in M_\ell$ which can be written $f_{i\ell}(m) = f_{j\ell}(m')$. So we get

$$g_i(m) = g_\ell \circ f_{i\ell}(m) = g_\ell(f_{i\ell}(m)) = g_\ell(f_{j\ell}(m')) = g_j(m)$$

so the map g is well-defined. Now for $x, y \in \varinjlim M_i$ and $a \in A$, pick $m \in M_i$ and $n \in M_j$ such that $\pi_M(m) = x$ and $\pi_M(n) = y$. Choose $k \ge i, j$. We have $a(f_{ik}(m)+f_{jk}(n)) \in M_k$ and

$$\pi_M(a(f_{ik}(m)+f_{jk}(n))) = \pi_M(a(m+n))+\pi_M(a(f_{ik}(m)-m+f_{jk}(n)-n)) = \pi_M(a(m+n)) = a(x+y)$$

so $g(a(x+y)) = g_k(a(f_{ik}(m)+f_{jk}(n))) = ag_k \circ f_{ik}(m) + ag_k \circ f_{jk}(n)$ since g_k and f_{ik}, f_{jk} are homomorphism of A-modules and since $\pi_M(f_{ik}(m)) = x$, $\pi_M(f_{jk}(n)) = y$, the previous equality can be written $g(a(x+y)) = ag(x) + ag(y)$. So g is a homomorphism of A-modules.
Let $h : \varinjlim M_i \to N$ be another homomorphism of A-modules through which the system (g_i) factorizes. For $x \in \varinjlim M_i$, take $m \in M_i$ lifting x i.e. $f_i(m) = \pi_M(m) = x$; we have $h(x) = h(f_i(m)) = g_i(m)$ since h factorizes (g_i); but by definition of g, $g_i(m) = g(x)$ thus $h = g$ hence the uniqueness of the homomorphism factorizing (g_i).

Now, let $(g_i : M_i \to N)_{i\in I}$ be a system of homomorphisms of A-modules, for which there are $i_0 \le j_0$ such that $g_{i_0} \ne g_{j_0} \circ f_{i_0j_0}$. Assume there a homomorphism $g : \varinjlim M_i \to N$ factorizing (g_i). By assumption, there is a $m \in M_{i_0}$ such that $g_{i_0}(m) \ne g_{j_0} \circ f_{i_0j_0}(m)$. Then for $x = \pi_M(m) = f_{i_0}(m)$, we have on one hand $g(x) = g(f_{i_0}(m)) = g_{i_0}(m)$ and on the other, $x = \pi_M(f_{i_0j_0}(m) + m - f_{i_0j_0}(m)) = \pi_M(f_{i_0j_0}(m)) = f_{j_0}(f_{i_0j_0}(m))$ so $g(x) = g(f_{j_0}(f_{i_0j_0}(m))) = g_{j_0}(f_{i_0j_0}(m))$. Thus $g(x) = g_{i_0}(m) \ne g_{j_0}(f_{i_0j_0}(m)) = g(x)$. So there is no such map g.

3. The sequence exists because the homomorphisms in each exact sequence commute with the homomorphisms in the directed systems. For example, denoting α_i the homomorphism $M_i \to N_i$ for each i, and $\overline{\alpha_i} : M_i \to \varinjlim N_k$ the composition $\pi_N \circ \alpha_i = f_i^N \circ \alpha_i$, we have for any $i \le j$,

$$\begin{aligned} \overline{\alpha_j} \circ f_{ij}^M = f_j^N \circ \alpha_j \circ f_{ij}^M = f_j^N \circ f_{ij}^N \circ \alpha_i &= \pi_{N|N_j} \circ f_{ij}^N \circ \alpha_i \\ &= \pi_N \circ (\underbrace{f_{ij}^N - \mathrm{id}_{M_i}}_{\mathrm{im}(-)\subset\ker(\pi_N)} + \mathrm{id}_{M_i}) \circ \alpha_i \\ &= \pi_{N|M_i} \circ \alpha_i \\ &= f_i^N \circ \alpha_i = \overline{\alpha_i} \end{aligned}$$

So by the universal property there is a unique homomorphism of A-modules $\alpha : \varinjlim M_i \to \varinjlim N_i$.
Let us denote β_i the homomorphism $N_i \to P_i$ for each i, and $\beta : \varinjlim N_i \to \varinjlim P_i$ the homomorphism given by the universal property.
α is injective: let $x \in \varinjlim M_i$, such that $\alpha(x) = 0$. Take $m \in M_i$ (by item 1) lifting x. Then $0 = \alpha(x) = \alpha(f_i^M(m)) = \pi_N \circ \alpha_i(m)$. By (*), there is a $j \ge i$ such that $f_{ij}^N(\alpha_i(m)) = 0$ but $\alpha_j \circ f_{ij}^M = f_{ij}^N \circ \alpha_i$ by hypothesis; so $\alpha_j \circ f_{ij}^M(m) = 0$. But since α_j is injective (exactness of the j^{th}-sequence), we get $f_{ij}^M(m) = 0$. So projecting to $\varinjlim M_i$,

we get $x = 0$.
$\mathrm{im}(\alpha) \subset \ker(\beta)$: let $x \in \varinjlim M_i$ and $m \in M_i$ lifting x. Then $\alpha_i(m) \in N_i$ lifts $\alpha(x)$ and $\beta_i \circ \alpha_i(m) = 0$ by assumption (exactness of the i^{th}-sequence). So we get $\beta(\alpha(x)) = 0$.
$\ker(\beta) \subset \mathrm{im}(\alpha)$: let $x \in \ker(\beta)$ and $n \in N_i$ lifting x. We have $\pi_P(\beta_i(n)) = 0$. By (*), there is a $j \geq i$ such that $f^P_{ij}(\beta_i(n)) = 0 \in P_j$; using the commutativity we get $\beta_j(f^N_{ij}(n)) = f^P_{ij}(\beta_i(n)) = 0$. By exactness of the j^{th}-sequence, there is a $m \in M_j$, such that $\alpha_j(m) = f^N_{ij}(n)$. Since $\pi_N(n) = \pi_N(f^N_{ij}(n))$, we get $\alpha(y) = x$ for $y = \pi_M(m)$.
β is surjective: let $y \in \varinjlim P_i$ and $p \in P_i$ lifting y. By exactness of the i^{th}-sequence, there is a $n \in N_i$ such that $\beta(n) = y$. Then $\beta(x) = y$ for $x = \pi_N(n)$.