Computing a Model of Set Theory

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New Computational Paradigms
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Natural Numbers

Finite numbers, natural numbers, number theory, computability theory:

$$0, 1, 2, \ldots, n, n+1, \ldots$$

Induction and recursion, defining and computing:

$$f(n) = G(f(0), \ldots, f(n-1)).$$

A Standard Turing Computation

			S P A C E								
		0	1	2	3	4	5	6	7	...	...
	0	**1**	0	0	1	1	1	0	0	0	0
	1	0	**0**	0	1	1	1	0	0		
T	2	0	0	**0**	1	1	1	0	0		
I	3	0	**0**	1	1	1	1	0	0		
M	4	0	1	**1**	1	1	1	0	0		
E	⋮										
	n	1	1	1	1	**0**	1	1	1		
	$n+1$	1	1	1	1	1	**1**	1	1		
	⋮										

A standard Turing computation. Head positions are indicated by shading.

Ordinal Numbers (1)

Ordinal numbers, infinite numbers, set theory, higher recursion theory (?)

$$
\begin{aligned}
0 &= \emptyset, \text{ the empty set;} \\
1 &= \{0\}, \text{ a singleton set;} \\
2 &= \{0,1\}, \text{ a pair set;} \\
3 &= \{0,1,2\}; \\
&\vdots \\
n &= \{0,1,2,\ldots,n-1\} = \{m \mid m < n\}; \\
&\vdots \\
\omega &= \{0,1,2,\ldots,n,\ldots\}, \text{ the set of natural numbers;} \\
\omega+1 &= \{0,1,2,\ldots,\omega\} = \omega \cup \{\omega\}, \text{ the successor of } \omega, \ldots
\end{aligned}
$$

Ordinal Numbers (2)

$$
\begin{aligned}
&\vdots \\
\alpha &= \{\beta \mid \beta < \alpha\} \\
\alpha + 1 &= \{\beta \mid \beta < \alpha + 1\} = \alpha \cup \{\alpha\} \\
&\vdots \\
\aleph_1 &= \text{the first uncountable ordinal / cardinal} \\
&\vdots \\
\aleph_2 &= \text{the second uncountable cardinal} \\
&\vdots \\
\aleph_\omega &= \bigcup_{n<\omega} \aleph_n \\
\aleph_\omega + 1 &= \aleph_\omega \cup \{\aleph_\omega\} \\
&\vdots
\end{aligned}
$$

Ordinal Numbers (3)

Transfinite induction: there is no infinite sequence

$$\alpha_0 > \alpha_1 > \alpha_2 > \ldots$$

of ordinals.

Transfinite recursion:

$$f(\alpha) = G(f \restriction \{\beta \mid \beta < \alpha\}) = G(f \restriction \alpha).$$

Limit ordinals: $\omega, \omega+\omega, \ldots, \aleph_1, \ldots$ are not of the form $\alpha+1$.

Ordinal Turing Computation

		O	r	d	i	n	a	l		S	p	a	c	e	…	…	…
		0	1	2	3	4	5	6	7	…	…	ω	…	α	…	κ	…
O	0	**1**	1	0	1	0	0	1	1	…	…	1	…	1	0	0	0
r	1	0	**1**	0	1	0	0	1	1			1					
d	2	0	0	**0**	1	0	0	1	1			1					
i	3	0	0	0	**1**	0	0	1	1			1					
n	4	0	0	0	0	**0**	0	1	1			1					
a	⋮																
l	n	1	1	1	1	**0**	1	0	1			1					
	n+1	1	1	1	**1**	1	1	0	1			1					
T	⋮	⋮	⋮	⋮	⋮	⋮											
i	ω	0	0	1	0	0	0	1	1	…	…	**1**					
m	$\omega+1$	**0**	0	1	0	0	0	1	1			0					
e	⋮																
⋮	θ	1	0	0	1	1	1	1	0	…	…	…	…	**0**	…	…	…
⋮	⋮			⋮			⋮			⋮	⋮						
⋮	⋮																

Computation Rules

Use standard Turing programs

Computation rules:

for time $t = 0$ or successor time $t = \alpha + 1$: use standard computation rules;

If $t < \theta$ is a limit time (ordinal), the machine constellation at t is determined by taking inferior limits:

$$\begin{aligned} \forall \xi \in \mathrm{Ord}\ T(t)_\xi &= \liminf_{r \to t} T(r)_\xi \text{ (tape contents)}; \\ S(t) &= \liminf_{r \to t} S(r) \text{ (program states)}; \\ H(t) &= \liminf_{s \to t, S(s) = S(t)} H(s) \text{ (head position)}. \end{aligned}$$

The machine may stop or run forever.

Ordinal Computability

If the machine stops, the result is a transfinite 0-1-sequence

$$(T(\theta)_\xi)_{\xi \in \mathrm{Ord}} ,$$

i.e. a characteristic function χ_A of a set or class of ordinals.

A subset $x \subseteq \mathrm{Ord}$ is ordinal computable (from finitely many ordinal parameters) if there a finite subset $z \subseteq \mathrm{Ord}$ and a program P which takes the characteristic function χ_z of z as initial tape content and stops with the tape content χ_x :

$$P: \chi_z \mapsto \chi_x$$

Which sets are ordinal computable?

The Class $\mathcal{S}$

Let

$$\mathcal{S} = \{x \subset \mathrm{Ord} \mid x \text{ is ordinal computable}\}.$$

- $\mathcal{S}$ is closed under unions, intersections, relative complements.
- $\mathcal{S}$ is closed with respect to definable subsets:
  if $x, y \in \mathcal{S}$ and $\varphi(u, v)$ is an $\in$-formula then

  $$\{u \in x \mid (\mathcal{S}, \in) \models \varphi(u, y)\} \in \mathcal{S}.$$

- The proof requires to code all ordinal computations into one universal program so that the quantifiers in φ can be evaluated (an ''ordinal Kleene predicate'').
- $\mathcal{S} = M \cap \{x \mid x \subset \mathrm{Ord}\}$ for some model $(M, \in)$ of set theory.

The Theory SO

$(\mathcal{S}, \in)$ satisfies an axiom system SO (''Sets of Ordinals'') which axiomatizes classes of the form $M \cap \{x | x \subset \mathrm{Ord}\}$ for some model M of set theory.

Gödel's Constructible Universe

Gödel defined a model L of set theory satisfying

$$L = \bigcap \{M \mid M \text{ is a transitive model of set theory and } \mathrm{Ord} \subseteq M\}.$$

- $L_0 = \emptyset$
- $L_{\alpha+1} =$ the set of all definable subsets of L_α
- $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for all limit ordinals λ
- $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$

The Main Theorem

Theorem $\mathcal{S} = L \cap \{x \mid x \subset \mathrm{Ord}\}$.

Proof. If $x \in \mathcal{S}$ then it can be computed by a program P and some ordinal parameters $\vec{\alpha}$. The same computation can then be carried out inside the model L with the same result. Hence $x \in L$.

For the converse note that $\mathcal{S} = M \cap \{x \mid x \subset \mathrm{Ord}\}$ for some transitive model M of set theory. Since $L \subseteq M$ we have

$$\mathcal{S} \subseteq L \cap \{x \mid x \subset \mathrm{Ord}\}.$$

Finally:

- theory of inspired by infinite time Turing machines of Hamkins, Kidder, Lewis
- generalizations of other notions of computability into the transfinite yield same result, e.g. ordinal register machines with Ryan Siders, Helsinki.
- introduction of the computability paradigm into the theory of constructible models of sets theory.
- transfinite analogues of Turing machine notions
- applications: can prove Cantor's continuum hypothesis in L with ordinal Turing machines.

Reference

P. Koepke, Turing Computations on Ordinals, to appear in the Bulletin of Symbolic Logic, 2005; also available at the arxiv.

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