The Center of the Affine NilTemperley-Lieb Algebra

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The affine nilTemperley-Lieb algebra

Definition

The affine nilTemperley-Lieb algebra of rank N is given by

- the unital associative $\mathbb{C}$-algebra $\widehat{\mathrm{nTL}}_N$
- with generators $a_0, \ldots, a_{N-1}$
- and relations (take all indices modulo N)

$$\begin{aligned} a_i^2 &= 0 && \text{for all } 0 \le i \le N-1, \\ a_i a_j &= a_j a_i && \text{for all } |i-j| > 1, \\ a_i a_{i+1} a_i &= a_{i+1} a_i a_{i+1} = 0 && \text{for all } 0 \le i \le N-1. \end{aligned}$$

The nonzero monomials in $\widehat{\mathrm{nTL}}_N$ form a basis (because the relations are given by monomials).

Write $a(\underline{j}) = a_{j_1} \ldots a_{j_m}$ for a sequence $\underline{j} = (j_1, \ldots, j_m)$ with $j_k \in \{0, \ldots, N-1\}$.

Gradings

- $\mathbb{Z}$-grading: By the length of a monomial.
  The degree of a generator a_i is $1 \in \mathbb{Z}$.
- $\mathbb{Z}^N$-grading: By the number of generators appearing in a monomial.
  The degree of a generator a_i is e_i, the ith standard basis vector in $\mathbb{Z}^N$.

The $\mathbb{Z}^N$-grading is finer than the $\mathbb{Z}$-grading!

Examples

In $\widehat{\mathrm{nTL}}_4$: $a_0 a_1 a_3 a_0 \neq 0$.
In $\widehat{\mathrm{nTL}}_5$: $a_0 a_1 a_3 a_0 = 0$ (here $a_3 a_0 = a_0 a_3$).

Notice that in $\widehat{\mathrm{nTL}}_N$ ($N \ge 3$), the element $(a_0 a_1 \ldots a_{N-1})^s$ is nonzero for all $s \in \mathbb{Z}_{\ge 0}$.

Compare the gradings:

$$\begin{aligned} a_0 a_1 a_3 a_0 &\in \widehat{\mathrm{nTL}}_4 : && \mathbb{Z}\text{-degree} = 4, && \mathbb{Z}^4\text{-degree} = (2,1,0,1) \\ a_0 a_1 a_2 a_3 &\in \widehat{\mathrm{nTL}}_4 : && \mathbb{Z}\text{-degree} = 4, && \mathbb{Z}^4\text{-degree} = (1,1,1,1). \end{aligned}$$

A faithful representation

Exterior algebra

Take $V = \bigoplus_{n=0}^{N} \left(\bigwedge^n \mathbb{C}^N\right) \otimes \mathbb{C}[z]$ with standard basis $v_{\underline{k}} := v_{k_1} \wedge \ldots \wedge v_{k_n}$ for $\underline{k} = (0 \le k_1 < \ldots < k_n \le N-1)$.

Action of $\widehat{\mathrm{nTL}}_N$ on V (take all indices modulo N):

$$a_i v_{\underline{k}} = \begin{cases} v_{k_1} \wedge \ldots \wedge v_{k_{r-1}} \wedge v_{i+1} \wedge v_{k_{r+1}} \wedge \ldots \wedge v_{k_n}, & \text{if } k_r = i \text{ for some } r, \\ 0, & \text{otherwise,} \end{cases}$$

$$a_0 v_{\underline{k}} = \begin{cases} z \cdot v_{k_1} \wedge \ldots \wedge v_{k_{r-1}} \wedge v_1 \wedge v_{k_{r+1}} \wedge \ldots \wedge v_{k_n}, & \text{if } k_r = 0 \text{ for some } r, \\ 0, & \text{otherwise.} \end{cases}$$

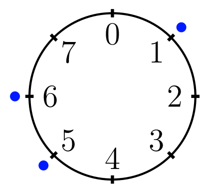
So a_i replaces v_i by v_{i+1}, and one keeps track of 'passing the 0' by multiplication by z.

Theorem 1 ([BFZ]) *For $N \ge 3$, V is a faithful $\widehat{\mathrm{nTL}}_N$-module.*

Graphical description

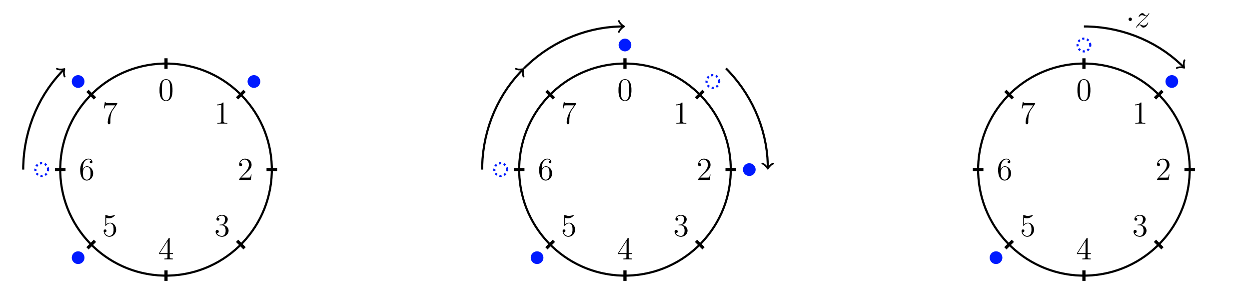
V is the $\mathbb{C}[z]$-span of particle configurations with

- up to N particles on a circle with N positions
- at most one particle on each position



$= v_1 \wedge v_5 \wedge v_6$

In the graphical description, a_i moves a particle clockwise from position i to position $i+1$.



(a) $a_6(v_1 \wedge v_5 \wedge v_6) = v_1 \wedge v_5 \wedge v_7$ (b) $a_7 a_1 a_6(v_1 \wedge v_5 \wedge v_6) = v_2 \wedge v_5 \wedge v_0$ (c) $a_0(v_0 \wedge v_5) = z \cdot v_1 \wedge v_5$.

Facts about the center

The center and the grading

Lemma 1 *The center of a graded algebra is homogeneous.*

$\rightsquigarrow$ We can determine the center by looking at graded components!

Trivially, $\mathbb{C} \cdot 1$ is central in $\widehat{\mathrm{nTL}}_N$ $\rightsquigarrow$ Only search for central elements in $\mathbb{Z}$-degree > 0.

Example 1 *One can determine the center of $\widehat{nTL}_2$ by hand:*

- $\widehat{nTL}_2 = \mathbb{C}\langle 1,\ a_0,\ a_1,\ a_0 a_1,\ a_1 a_0 \rangle$.
- $\mathit{Center}(\widehat{nTL}_2) = \mathbb{C}\langle 1,\ a_0 a_1,\ a_1 a_0 \rangle$.

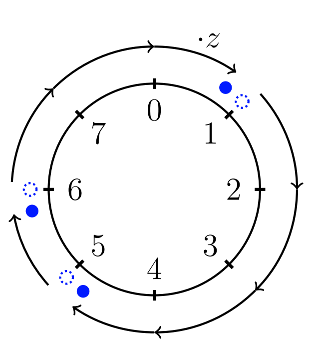
Lemma 2 *Any central element in $\widehat{nTL}_N$ is a linear combination of monomials in which every generator a_i appears at least once (besides 1).*

$\rightsquigarrow$ Homogeneous central elements (besides 1) have $\mathbb{Z}$-degree at least N,
$\mathbb{Z}^N$-degree at least $(1, \ldots, 1)$.

Special monomials in $\widehat{\mathrm{nTL}}_N$

For a basis element $v_{\underline{k}} \in V$ define a monomial $a(\widehat{\underline{k}})$
that moves every particle in $v_{\underline{k}}$ to the position of the precessing particle:

$$a(\widehat{\underline{k}}) v_{\underline{k'}} = \begin{cases} z \cdot v_{\underline{k}} & \text{if } \underline{k'} = \underline{k} \\ 0 & \text{for all } \underline{k'} \neq \underline{k} \text{ (of any length).} \end{cases}$$



$= z \cdot v_1 \wedge v_5 \wedge v_6 = a_4 a_3 a_2 \cdot a_0 a_7 \cdot a_1 a_5 a_6 \cdot v_1 \wedge v_5 \wedge v_6$

$\rightsquigarrow$ We can pick single basis elements in V.

Lemma 3 *Any central element in $\widehat{nTL}_N$ with constant term 0 acts on a wedge $v_{\underline{k}} \in V$ by multiplication with an element of $z\mathbb{C}[z]$.*
This factor only depends on the length $r = |k|$ of $v_{\underline{k}}$ resp. on the number of particles (not their positions).

Description of the center

The main result

Define for $1 \le r \le N-1$

$$t(r) := \sum_{|k|=r} a(\widehat{\underline{k}}),$$

the sum over all monomials that move r particles once around the circle.

Theorem 2 *The $t(r)$ are central, for all $1 \le r \le N-1$. Moreover,*

- *The center of $\widehat{nTL}_N$ is generated by 1 and the $t(r)$.*
- *$t(r) \cdot t(m) = 0$ for all $r \neq m$.*

$$\begin{aligned} \mathrm{Center}(\widehat{\mathrm{nTL}}_N) &= \mathbb{C} \oplus t(1) \cdot \mathbb{C}[t(1)] \oplus \ldots \oplus t(N-1) \cdot \mathbb{C}[t(N-1)] \\ &\cong \frac{\mathbb{C}[t(1), \ldots, t(N-1)]}{(t(i)t(j) \mid i \neq j)}. \end{aligned}$$

Examples

$$\begin{aligned} \widehat{\mathrm{nTL}}_3 : \quad t(1) &= a_2 a_1 a_0 + a_0 a_2 a_1 + a_1 a_0 a_2, \\ t(2) &= a_0 a_1 a_2 + a_1 a_2 a_0 + a_2 a_0 a_1. \end{aligned}$$

$$\begin{aligned} \widehat{\mathrm{nTL}}_4 : \quad t(1) &= a_3 a_2 a_1 a_0 + a_0 a_3 a_2 a_1 + a_1 a_0 a_3 a_2 + a_2 a_1 a_0 a_3, \\ t(2) &= a_0 a_2 a_1 a_3 + a_1 a_3 a_0 a_2 + a_0 a_1 a_3 a_2 + a_1 a_2 a_0 a_3 + a_2 a_3 a_1 a_0 + a_3 a_0 a_2 a_1, \\ t(3) &= a_0 a_1 a_2 a_3 + a_1 a_2 a_3 a_0 + a_2 a_3 a_0 a_1 + a_3 a_0 a_1 a_2. \end{aligned}$$

References

[BFZ] **A. Berenstein, S. Fomin, A. Zelevinsky**, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math., **122**, (1996), p. 49–149.

[BMS] **G. Benkart, J. Meinel, C. Stroppel**, *The center of the affine nilTemperley-Lieb algebra*, in preparation.

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