Seminar on ‘Motivic Donaldson–Thomas invariants’[1]

Content. In the summer term 2009 we will set out to read the recent paper [KS] by Kontsevich and Soibelman. The seminar will be a natural sequel to last term’s seminar ‘Stability conditions and Stokes factors’ (based on the paper of Bridgeland and Toledano Laredo), but yet completely independent of it.

The paper [KS] is long (148 pages), rich, sometimes sketchy and often requires a strong background from various areas, so a real challenge. The aim of the seminar will be modest. Lets try to understand a few of the many interesting concepts introduced by Kontsevich and Soibelman and try to get a glimpse of the overall picture with the many relations to the work of Joyce, Bridgeland, and many others.

So the main reference will be:

[KS] M. Kontsevich, Y. Soibelman *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations.* arXiv:0811.2435.

Related activities. There will be a number of activities that will be closely related to our seminar:

– The Summer School *BPS state counting, stability structures and derived algebraic geometry* August 31- September 4, Hamburg. The main speaker will be Yan Soibelman himself. For more details see:
http://www.math.uni-hamburg.de/home/mohrdieck/summerschool.html

– The Felix Klein lectures *Cluster algebras, cluster categories and periodicity* by *Bernhard Keller*, June 15-July 18 (Bonn). For more details see:
http://www.hausdorff-research-institute.uni-bonn.de/

– Workshop *Mirror Symmetry*, June 2 - June 5, Bonn. For more details see: coming soon.

Organization. The seminar is organized together with people from Mainz and will take place roughly every second Tuesday, 2-6 pm starting April 28. (With some of the talks taking place in Mainz depending on the number of people from Mainz. Travel expenses will be covered by the SFB/TR.) Dates for the following meetings: May 12 and 26, June 9 and 23, and July 7.

For further information or if you are interested in giving a talk in the seminar please contact Daniel Huybrechts (huybrech at math.uni-bonn.de) or Jacopo Stoppa (jacopostoppa at gmail.com)

Below we sketch a plan (also provisional) for the seminar which we should feel free to change. We will not be able to cover the whole paper anyway and should pick the parts that we think accessible, most inspiring. For the moment we stick to [KS], but we could also have interludes on more geometric aspects, e.g. the actual definition of DT, PT or GV invariants. As usual, some of the talks will take longer than anticipated.

[1]Sponsored by the SFB/TR 45. Bonn–Essen–Mainz.

Program

I. Introduction and stability conditions

I.1. Somebody brave should try to give an overview and maybe link the topic to the one of last term. Explain roughly what Donaldson–Thomas invariants are and what wall-crossing phenomena are about. (60min)

I.2. Based on Section 2.1.-2.6 of [KS]. Introduce stability conditions for graded Lie algebras. The definition in [KS] differs slightly from the one of Bridgeland. The difference should be explained. Discuss wall crossing. (120min)

I.3. Based on Section 2.8.-2.9. We could explain the relation to the formulae we have seen last term. We have to decide whether we want to go into any kind of analytical detail or just recall certain things and sketch what is done here. This talks is independent of the rest of the seminar and in this sense optional. We could also do it later. (60min)

II. Motivic functions and motivic Hall algebra.

II.1. We should start with a gentle introduction in the yoga of motives and various realizations. Of course, we will have to be sketchy and accept basic results. Then discuss Section 4.1-4.2.

Next introduce the notion of ind-constructible sets see Section 3.1., page 45. We suggest to work, as a first approximation, over an algebraically closed field or even over $\mathbb{C}$. Later we will need to work over other fields however. Introduce the notion of an ind-constructible triangulated category, in which the objects form an ind-constructible set. (We will need the A_∞ version only later.) This is explained in Section 3.1.-3.2. (90min)

II.2. Based on Section 3.4. and Bridgeland's papers on stability conditions. Explain the notion of constructible stability conditions on ind-constructible triangulated categories. We have seen examples of stability conditions on abelian categories last term and some of us know about stability conditions on triangulated categories. Sketch the main features. (60min)

II.3. Based on Section 6.1. Discuss the notion of the motivic Hall algebra. Last term the Hall algebra was introduced as an algebra of constructible functions, but its motivic nature was already visible. This is worked out here. The main result of this section is Prop. 11, page 88, which states that a stability condition on the category induces a stability condition on its motivic Hall algebra via the functions A^{Hall}. The much deeper result concerning stability conditions on the motivic quantum torus will be proved later.

If we find a speaker who is familiar with Toën's derived Hall algebra we could compare the notion of the motivic Hall algebra with it (Prop. 12). The last part explaining how to avoid stability conditions could be useful to get further acquainted with stability structures and t-structures, but does not seem strictly necessary for what follows.

III. Ind-constructible A_∞-categories

III.1. Recall the notion of A_∞-algebras and modules over them. Recall/explain the notion of constructible and ind-constructible sets. Explain the ind-constructible structure on the moduli space of A_∞-modules. The representation theory is only sketched on page 6 in [KS], but may be standard for the experts. A speaker with a background in representation theory would be good for this part. (60min)

III.2. Recall the notion of A_∞-categories and dg-categories. State Keller's theorem saying that triangulated categories with a classical generator are equivalent to perfect complexes over a dg-algebra. See the paper of Bondal and van den Bergh (or the lectures of Orlov) for the result and references. This part is needed to be sure that most of the categories of interest are equivalent to an A_∞-module category and thus carry an ind-constructible structure. (60min)

III.3. Section 3.3. We need a discussion of the potential of an A_∞-Calabi-Yau category. The splitting of it, asserted in Prop. 7, will be important later. Maybe this can be done first or more explicitly for an A_∞-algebra? We have not checked the literature for this part.

Mention that the potential in an ind-constructible A_∞-category is ind-constructible. We should be careful not get lost in the technical details of A_∞-categories or algebraic stacks.

IV. Motivic Milnor fibre and motivic Donaldson–Thomas

IV.1. Based on Section 4.3. Recall the definition of the Milnor fibre. Introduce its motivic version. The speaker should be familiar with the basic concepts of motivic integration in the sense of Denef and Loeser (following Kontsevich). Thm. 6, page 68, is used in the next talk in the proof of Thm. 8, page 96 .

IV.2. Based on Section 6.2. This is maybe the main result of the paper: A stability condition on the A_∞-CY category induces a stability condition on the motivic quantum torus, the latter is by definition what is called *motivic DT invariants* (see Def. 18, page 101).

Introduce the motivic weight function (Def. 17, page 93) which uses the potential given by the A_∞-structure. Recall the definition of the motivic quantum torus R_Γ, page 94, and define A^{mot}. The latter are related to A^{Hall} via the morphism $\Phi : H(\mathcal{C}) \to R_\Gamma$. The proof of Thm. 8, page 98, will take up most of the talk. The proof of Thm. 7, page 95, then follows directly. The proof of Thm. 8 uses results of Section 5, which we would suggest to use as a black-box.

IV.3. At this point we probably all will want to see examples. Categories generated by one or two spherical objects as generators are treated in Section 6.4.

V. Quasi-classical limit, numerical DT invariants

In this part one finds a series of conjectures, e.g. the integrality conjecture (Conjecture 6). Maybe we could also come back to Conjecture 1 in the introduction and discuss Reineke's recent work on the subject. More details later.

VI. Explicit examples.

Maybe something based on Section 8, but there may be no time for it. More details later.

Selection of further (related) references

Tom Bridgeland *Hall algebras and curve-counting invariants* http://www.tombridgeland.staff.shef.ac.uk/papers/dtpt.pdf

Dominic Joyce, Yinan Song *A theory of generalized Donaldson-Thomas invariants. I. An invariant counting stable pairs* arXiv:0810.5645

Kentaro Nagao *Derived categories of small toric Calabi-Yau 3-folds and counting invariants* arXiv:0809.2994

Rahul Pandharipande, Richard Thomas *Curve counting via stable pairs in the derived category* arXiv:0707.2348

Markus Reineke *Cohomology of quiver moduli, functional equations, and integrality of Donaldson-Thomas type invariants* arXiv:0903.0261.

Jacopo Stoppa, Richard Thomas *Hilbert schemes and stable pairs: GIT and derived category wall crossings* arXiv:0903.1444

Yukinobu Toda *Curve counting theories via stable objects I. DT/PT correspondence* arXiv:0902.4371