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Rigid ring spectra

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Motivation

When working with topological spaces, simplicial sets or other categories, we sometimes would like to consider certain morphisms as invertible, even though they are not isomorphisms. For example, in algebraic topology we are, among other things, interested in the classification of topological spaces up to homotopy equivalence. The usual approach, namely localizing the category with respect to these morphisms in order to obtain a new category where these morphisms are invertible, does not always work: The new 'category' does not need to be a category i.e. the morphisms between two objects might not form a set.

One way to avoid this problem is to work with model categories, which are categories with additional structure and have been introduced by Quillen [Qui]: A *model category* $\mathcal{M}$ is a bicomplete category with three classes of maps, *weak equivalences*, *cofibrations* and *fibrations*, satisfying certain axioms (see [Hov]). This structure ensures that we can localize a model category with respect to the class of weak equivalences and obtain a new category, which is called the homotopy category of $\mathcal{M}$ and is denoted by $\mathrm{Ho}(\mathcal{M})$. There are many — not only topological — examples of model categories: (pointed) simplicial sets, (pointed) topological spaces, chain complexes of modules over a ring R, symmetric spectra, etc.

Two model categories $\mathcal{M}$, $\tilde{\mathcal{M}}$ are called *Quillen equivalent* if there exists an adjunction (F, U, ϕ) from $\mathcal{M}$ to $\tilde{\mathcal{M}}$ which is compatible with the model structure and induces an equivalence on their homotopy categories $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\tilde{\mathcal{M}})$. A pointed model category $\mathcal{M}$ is *stable* if the suspension functor $\Sigma : \mathrm{Ho}(\mathcal{M}) \to \mathrm{Ho}(\mathcal{M})$, $\Sigma(X) = \mathrm{Hocolim}(* \leftarrow X \to *)$ is an equivalence of categories.
An important example of a stable model category is the category of chain complexes $Ch(R)$ with the weak equivalences being the class of quasi-isomorphisms. Here the suspension functor is given by the shift functor. The associated homotopy category is the derived category $\mathcal{D}(R)$. Another example of a stable model category is the category of symmetric spectra Sp^{Σ} together with the class of stable equivalences [HSS]. Its homotopy category $\mathrm{Ho}(Sp^{\Sigma})$ is the classical stable homotopy category $\mathcal{SHC}$.
A stable homotopy category $\mathrm{Ho}(\mathcal{M})$ can be canonically equipped with the structure of a triangulated category.

In general, it is not possible to recover the model category $\mathcal{M}$ from its homotopy category $\mathrm{Ho}(\mathcal{M})$ — even if one also demands that the model category $\mathcal{M}$ is stable, and hence its homotopy category triangulated. Therefore, one could raise the following question: *Is there a stable model category $\mathcal{M}$ such that every stable model category $\tilde{\mathcal{M}}$, whose homotopy category $\mathrm{Ho}(\tilde{\mathcal{M}})$ is triangulated equivalent to $\mathrm{Ho}(\mathcal{M})$, is already Quillen equivalent to $\mathcal{M}$?*
Model categories with this property are called *rigid*. A ring spectrum R, whose model category of R-modules is rigid, is also said to be *rigid*. For example, the sphere spectrum $\mathbb{S}$ is rigid, since the stable model category $Sp^{\Sigma} = \mathbb{S}$-Mod is rigid [Sch07]. Thus, every stable model category, whose homotopy category is triangulated equivalent to $\mathcal{SHC}$, is Quillen equivalent to Sp^{Σ}.

The aim of this project is to prove that the Postnikov sections of the sphere spectrum $P_n(\mathbb{S})$, for $n > 0$, and the 2-local real topological K-theory ring spectra $ko_{(2)}$ and $KO_{(2)}$ are rigid. Furthermore, I would like to find some algebraic criteria for detecting rigidity of ring spectra.

Some known results and one possible approach

If one requires additional technical properties of the model categories, then it is sufficient to consider only model categories of modules in order to answer the question raised above [SS, Theorem 3.1.1.]:

Theorem 1 (Schwede-Shipley) *Let $\mathcal{M}$ be a simplicial, cofibrantly generated, proper, stable model category with a compact generator P. Then there exists a chain of simplicial Quillen equivalences between $\mathcal{M}$ and the model category of $End_{\mathcal{M}}(P)$-modules, where $End_{\mathcal{M}}(P)$ denotes the endomorphism ring spectrum of P.*

Theorem 2 (Schwede-Shipley) *Let R and S be two symmetric ring spectra. Then the model categories R-Mod and S-Mod are Quillen equivalent if the category S-Mod has a compact, cofibrant and fibrant generator P such that R is stably equivalent to the endomorphism ring spectrum $End_{S\text{-Mod}}(P)$.*

In oder to prove that a ring spectrum R is rigid, it thus suffices to show that R is stably equivalent to the ring spectra $\tilde{R} := End_{S\text{-Mod}}(P)$ given in Theorem 2 [SS]. The ring of homotopy groups $\pi_*(\tilde{R})$ of such a ring spectrum $\tilde{R}$ is isomorphic to $\pi_*(R)$ for the following reason:
Let R and S be ring spectra and let $\Phi : \mathrm{Ho}(R\text{-Mod}) \to \mathrm{Ho}(S\text{-Mod})$ be a triangulated equivalence between the homotopy categories of modules over these ring spectra. Then the following holds:

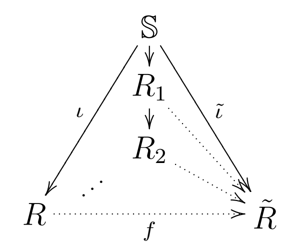
$$\pi_*(End_{S\text{-Mod}}(P)) \cong \mathrm{Ho}(S\text{-Mod})(P, P)_* \cong \mathrm{Ho}(R\text{-Mod})(R, R)_* \cong \pi_*(R),$$

where $P := \Phi(R)^{cf}$ is a cofibrant and fibrant replacement of $\Phi(R)$ in the model category S-Mod.
Furthermore, this isomorphism preserves the Toda brackets of R and $\tilde{R}$, since triangulated equivalences preserve Toda brackets.

Therefore, a ring spectrum R is rigid if every ring spectrum $\tilde{R}$, having the same ring of homotopy groups and the same Toda brackets as R, is stably equivalent to R.

In case that the ring spectrum R is connective, that is $\pi_k(R) = 0$ for all $k < 0$, there are two methods for proving rigidity of R:

(i) The first approach is to find a map of ring spectra $f : R \to \tilde{R}$ such that $f \circ \iota = \tilde{\iota}$, where the maps ι and $\tilde{\iota}$ are the unit maps of the ring spectra R and $\tilde{R}$.
In order to obtain the map f, one can construct the ring spectrum R by gluing ring spectra cells to the sphere spectrum $\mathbb{S}$ and trying to factorize the map $\tilde{\iota}$ inductively through the resulting ring spectra R_i. Then it remains to show that the resulting map of ring spectra $f : R \to \tilde{R}$ is a stable equivalence.



Another approach

(ii) The second approach is to consider the two Postnikov towers of the ring spectra R and $\tilde{R}$

$$\cdots \longrightarrow P_{n+1}(R) \longrightarrow P_n(R) \longrightarrow \cdots \longrightarrow P_2(R) \longrightarrow P_1(R) \longrightarrow P_0(R) \qquad \text{and}$$
$$\cdots \longrightarrow P_{n+1}(\tilde{R}) \longrightarrow P_n(\tilde{R}) \longrightarrow \cdots \longrightarrow P_2(\tilde{R}) \longrightarrow P_1(\tilde{R}) \longrightarrow P_0(\tilde{R})$$

and to prove inductively that the ring spectra $P_n(R)$ and $P_n(\tilde{R})$ are stably equivalent for every $n \geq 0$: For each natural number n, one has to show that there exists exactly one extension of ring spectra

$$e : Y \longrightarrow P_n(R)$$

such that $\pi_k(e)$ is an isomorphism for $k \leq n$ and such that the ring spectra Y and $P_{n+1}(R)$ have the same ring of homotopy groups and the same Toda brackets.

One can do this by using a result of Dugger and Shipley: In [DS], they classify the possible extensions $e : Y \to C$ of a cofibrant ring spectrum $C := P_n(R)^c$ to a ring spectrum Y by a $\pi_0(C)$-module $M := \pi_{n+1}(R)$ such that

- $\pi_i(Y) \to \pi_i(C)$ is an isomorphism for $i \leq n$,
- $\pi_{n+1}(Y)$ and M are isomorphic as $\pi_0(Y)$-bimodules and
- $\pi_i(Y) = 0$ for all $i > n + 1$.

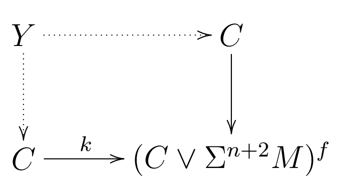
Dugger and Shipley prove that, up to a zig-zag of weak equivalences, all these extensions are classified by

$$\mathrm{Ho}(\mathbb{S} - Alg_{/C})(C, C \vee \Sigma^{(n+1)+1}M)/Aut(M) \cong \mathrm{THH}^{(n+1)+2}(C, M)/Aut(M),$$

where $\mathrm{Ho}(\mathbb{S} - Alg_{/C})(A, B)$ denotes the homotopy mapping space from A to B in the category of $\mathbb{S}$-algebras over C and $\mathrm{THH}^m(C, M)$ is the m-th topological Hochschild cohomology group of C with coefficients in M [DS, Theorem 8.1. and 8.8.].

To give a vague idea of this classification, we will briefly sketch how Dugger and Shipley obtain the extension Y corresponding to a homotopy class $[k] \in \mathrm{Ho}(\mathbb{S} - Alg_{/C})(C, C \vee \Sigma^{n+2}M)$:

We choose a representing homomorphism $k : C \to (C \vee \Sigma^{n+2}M)^f$ of this k-invariant, where $(C \vee \Sigma^{n+2}M)^f$ denotes a fibrant replacement of $C \vee \Sigma^{n+2}M$ in $\mathbb{S} - Alg_{/C}$. The homotopy fiber of k, i.e. the homotopy pullback in the category of $\mathbb{S}$-algebras, is an extension Y of C by M.



Unfortunately, this theorem of Dugger and Shipley provides not much information about the product of two elements $x \cdot y \in \pi_{n+1}(Y)$ or about Toda brackets which are subsets of $\pi_{n+1}(Y)$. In general, we only know the product $x \cdot y \in \pi_{n+1}(Y)$ in 'trivial' cases, for example if x or y lies in $\pi_0(Y)$ or if $Y \simeq M \vee C$.

Remark 1 *Using these two methods, one can prove that the ring spectra $P_4(ko_{(2)})$, $P_8(ko_{(2)})$ and $P_9(ko_{(2)})$ are rigid.*

Examples

Below, we list some examples of rigid and non-rigid ring spectra:

- The sphere spectrum $\mathbb{S}$ is a rigid ring spectrum [Sch07].
- Further examples for rigid ring spectra are the Eilenberg-MacLane ring spectra $H(R)$ for any ring R [SS, Theorem 5.1.1.]. This holds due to Theorem 2 and since Eilenberg-MacLane ring spectra are uniquely determined by their homotopy rings $\pi_*(\mathrm{HR})$.
- The ring spectra $P_n(\mathbb{S}) = P_n(ko)$ are rigid for $n = 0, 1, 2$.
- In contrast, the Morava K-theory ring spectra are not rigid:
  Let $K(n)$ be the n^{th} Morava K-theory ring spectrum for a fixed prime p and a natural number $n > 0$. Then the homotopy category $\mathrm{Ho}(K(n)\text{-Mod})$ is triangulated equivalent to the derived category of the graded field $\pi_*(K(n)) = \mathbb{F}_p[v_n, v_n^{-1}]$, where $|v_n| = 2p^n - 2$. However, the model categories $K(n)$-Mod and $\mathbb{F}_p[v_n, v_n^{-1}]$-Mod are not Quillen equivalent (see [Sch01]).

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