An introduction to Elliptic Cohomology and Conformal Field Theories

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From Field Theories to Elliptic Objects

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Genera

A measure for spaces

1. $\mu(*) = 1$
2. $\mu(X) = \mu(X_1) + \mu(X_2) - \mu(X_1 \cap X_2)$ if $X = X_1 \cup X_2$
3. $\mu(X) = \mu(Y)$ if X is diffeomorphic to Y

μ exists, essentially unique, takes values in the integers

μ is the Euler-Poincaré characteristic

Further properties:

4. $\mu(X \times Y) = \mu(X)\mu(Y)$
5. $\mu(X \sqcup Y) = \mu(X) + \mu(Y)$

Methods of calculation

- covering
- $\mu(X) = \sum_i (-1)^i \dim H^i(X;\mathbb{R})$
- F oriented closed surface in $\mathbb{R}^3$

  Gauss curvature: $\kappa = \kappa_{min}\kappa_{max}$

$$\mu(F) = \int_F \frac{\kappa}{2\pi} d\sigma = \int_F e(TF)$$

  with $e(TF) \in H^2(F;\mathbb{R})$ the Euler class
- X oriented, closed Riemannian , $\dim(X) = 2n$

$$\mu(X) = \int_X e(TX)$$

  with $e(TX) \in H^{2n}(X)$

  (if $TX = l_1 \oplus l_2 \oplus \cdots \oplus l_n$ then

$$e(TX) = x_1 x_2 \cdots x_n;\ x_i = e(l_i))$$

- $\mu(X) = \mathrm{ind}(d + d^* : \text{even} \longrightarrow \text{odd alternating forms})$

Genera

6. $\mu(X) = \mu(Y)$ if there is an (orientable) W with

$$\partial W = X + (-Y); \qquad X \text{ is } \textbf{bordant} \text{ to } Y$$

Definition

A map with 1,3,4,5,6 is called a **genus**

$$\mu : \underbrace{\{\text{closed oriented mfds}\}/\text{bordism}}_{\Omega_{SO}} \xrightarrow{\text{ring map}} R$$

Example

- The Euler characteristic is not a genus but the map

$$\Omega^*_{SO} \longrightarrow \Omega^0_{SO} \longrightarrow \mathbb{Z}$$

which counts points is.

Example

- $sig(X^{4n})$ = signature of the quadratic form

$$(x, y) \mapsto \int_X x \wedge y = \langle x \cup y, [X] \rangle$$

Theorem (Hirzebruch)

$$sig(X) = \int_X \prod_{i=1}^{2n} \frac{x_i}{tanh(x_i)}$$

Moreover,

$$\mathrm{sig}(X) = \mathrm{ind}(D^+ = d + d^* : \text{positive} \longrightarrow \text{negative forms})$$

The spin refinement

Suppose X is spin (obstructions: $\omega_1 = \omega_2 = 0$) , that is, the loop space

$$LX = \{s : S^1 \longrightarrow X \,|\, s \text{ smooth } \}$$

is oriented. Then we have a bundle of spinors $\Delta^{\pm}$ and a Dirac operator $\partial^+ : \Delta^+ \longrightarrow \Delta^-$ with the property

$$\mathrm{ind}(D^+) = \mathrm{ind}(\partial^+ \otimes (\Delta^+ \oplus \Delta^-))$$

Theorem (Atiyah-Singer)

The index of ∂^+ is a genus

$$\hat{A} : \Omega^*_{Spin} \longrightarrow \mathbb{Z}$$

Moreover, we have the formula

$$ind(\partial^+) = \int_X \prod_{i=1}^{2n} \frac{x_i}{2sinh(x_i/2)}$$

The loop space

Witten's analysis

What is the index of ∂^+ on the loop space LX? Suppose LX is Spin, that is, suppose X is String (obstructions: $\omega_1 = \omega_2 = p_1/2 = 0$). Then we have

$$\begin{aligned} T_\gamma LX &\cong \{\text{vector fields along } \gamma\} \\ &\cong \Gamma(\gamma^* TX) \\ &\cong L(T_pX) \text{ if } \gamma \text{ is constant } p. \end{aligned}$$

This gives for the fix point set $X = (LX)^{S^1}$

$$T(LX)_{|X} \cong L(TX)_{|X} \cong TX \oplus \widehat{\bigoplus_{k=1}^{\infty}}(TX \otimes \mathbb{C})q^k$$

by Fourier expansion. The equivariant fix point formula of Atiyah and Segal applied to this infinite dimensional situation hence gives

$$\mathrm{ind}_{S^1}(\partial^+) = \int_X \prod_{i=1}^{2n} \frac{x_i}{2\sinh(x_i/2)} \prod_{k=1}^{\infty} \frac{(1-q^k)^2}{(1-q^k e^{x_i})(1-q^k e^{-x_i})}$$

The Witten genus

Definition

This integral power series

$$W(X) \in \mathbb{Z}[[q]]$$

with $q = e^{2\pi i\tau}$ is called the Witten genus of X.
It is an integral modular form, that is, an invariant of the pair $(\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}, dz)$:

$$W : \Omega^*_{String} \longrightarrow \mathbb{Z}[c_4, c_6, \Delta]/(c_4^3 - c_6^2 - 1728\Delta)$$

with the Eisenstein series

$$c_4 = 1 + 240 \sum_{n \geq 1} (\sum_{d|n} d^3) q^n$$

$$c_6 = 1 - 504 \sum_{n \geq 1} (\sum_{d|n} d^5) q^n$$

Family versions of genera

Pontryagin numbers

Natural multiplicative transformations of cohomology theories such as

$$\begin{array}{ccc} \Omega^*_{SO}(X) & \longrightarrow & H^*(X;\mathbb{Z}) \\ \left[M \xrightarrow{f} X\right] & \mapsto & f_!(1) \\ \left[X \xrightarrow{\sigma} TX\right] & \mapsto & \sigma_!(1)_{|X} = e(TX) = x_1 \cdots x_n \end{array}$$

give a system of $(H\mathbb{Z}-)$ Pontryagin classes which can be integrated to Pontryagin numbers.

Theorem (Thom, Milnor, Novikov, Wall)

M, N are oriented bordant iff all $H\mathbb{Z}$-, $H\mathbb{Z}/2$-Pontryagin numbers coincide.

KO-theory

Theorem (Anderson, Brown, Peterson)

M, N Spin are bordant iff all $H\mathbb{Z}/2$ and KO-Pontryagin numbers coincide.

The *KO*-numbers come from the natural transformation

$$\hat{A} : \Omega_{Spin} \longrightarrow KO$$

where *KO* is real *K*-theory

$$KO(X) = \{\text{vector bundles over } X + \text{formal inverses}\}.$$

Topological modular forms

The string bordism ring is not known.

Theorem (Hopkins et al.)

The Witten genus generalizes to a map of spectra

$$W : \Omega_{String} \longrightarrow tmf$$

The coefficient ring

$$tmf^*(*) = \{\text{topological modular forms}\}$$

maps into the ring of integral modular forms.

Conjecture:
M, N are String bordant iff all $H\mathbb{Z}/2$-, TMF-Pontryagin numbers coincide (with $TMF = tmf\left[\Delta^{-1}\right]$).
Note: This would lead to an understanding of the bordism ring Ω^*_{string} and of a great chunk of $\Omega^*_{fr} = \pi^*_{st}$.

Construction of *tmf*

Weierstrass equation:

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with $a_1, a_2, \ldots, a_6 \in R,\ \Delta \in R^\times$.
If $1/6 \in R$ then E can be written in the form

$$y^2 = x^3 - 27c_4x - 54c_6.$$

The equation gives the universal curve over

$$(\mathbb{Z}[c_4, c_6, \Delta]/c_4^3 - c_6^2 - 1728\Delta)\,[1/6]\left[\Delta^{-1}\right] = mf\left[(6\Delta)^{-1}\right]$$

Its formal group law is classified by a ring map from the Lazard ring L.

Theorem (Quillen)

Let Ω^*_U be the unitary bordism ring. Then there is an isomorphism of rings

$$L \cong \Omega^*_U.$$

Set

$$TMF\,[1/6]^*\,X = \Omega^*_U(X) \otimes_{\Omega^*_U} mf\left[(6\Delta)^{-1}\right]$$

To include $p = 2, 3$ take the best possible approximation to the universal object in the derived sense:

$$TMF = \operatorname{holim} E; \text{ with} E \text{ an } E_\infty \text{elliptic spectrum}$$

Problems:

- construction of the functor "E"
- relation to analysis on loop spaces
- a geometric interpretation

Field Theories

The $(0,1)$-dimensional case

Let $\mathcal{B}X^1$ be the category with objects the points in X and morphisms

$$\mathcal{B}(x_0, x_1) = \{\text{paths in } X \text{ from } x_0 \text{ to } x_1\}.$$

Suppose V is a vector bundle with connection. Then V defines a continuous functor

$$\begin{aligned} F : \mathcal{B}X^1 &\longrightarrow \text{vector spaces} \\ x &\mapsto V_x \\ \gamma &\mapsto (\text{parallel transport } : V_{x_0} \to V_{x_1}) \end{aligned}$$

F is a $(0,1)$-dimensional field theory.
Idea: Use 1-dimensional field theories to describe K-theory.

The *KO*-Hurewicz class

Example

Let V be the spinor bundle over a spin manifold M. Set

$$H = L^2(M; V).$$

Then the field theory associated to the *KO*-Hurewicz class is

$$\begin{array}{ccc} \mathcal{B}(*)^1 & \longrightarrow & \text{vector spaces} \\ * & \mapsto & H \\ I_t & \mapsto & e^{-t\partial^2} + \underbrace{\theta\partial e^{-t\partial^2}}_{\text{super coordinate}} \end{array}$$

The $(-1, 0)$-dimensional case

Let $\mathcal{B}X^0$ be the category with the object $\emptyset$ and the super point $* = \mathbb{R}^{0|1}$ as morphism. Then we have

Field Theories	de Rham cohomology
$* \to M$	exterior form
functor $F : \mathcal{B}X^0 \to v.s.$	differential form
euclidian, invariant	closed, even forms
up to concordance	even cohomology class
with Clifford grading	all cohomology classes

The $(1,2)$-dimensional case

Let $\mathcal{B}X^2$ be the category whose objects are 1-dimensional (super,...-) manifolds in X and with morphisms

$$\mathcal{B}X^2(S_0, S_1) = \{\text{surfaces in } X \text{ with boundray } S_0 + (-S_1)\}$$

(and eventually some extra structure). Consider continuous (or holomorphic?) functors

$$F : \mathcal{B}X^2 \longrightarrow \text{Hilbert spaces + trace class maps}$$

G. Segal formulates a contraction property: If A_q is an annulus for some $q = e^{2\pi i\tau}$ then the trace of the operator

$$F(A_q) : F(S^1) \longrightarrow F(S^1)$$

should only depend on the glued closed object Σ_τ and its metric. This implies that $F(\Sigma_\tau)$ is a modular form.

Stolz and Teichner noticed that the Mayer-Vietoris property can only be satisfied if one allows manifolds with boundaries, that is, intervals as objects. The field theories then should satisfy

$$F(\gamma_0 \cup \gamma_1) = F(\gamma_0) * F(\gamma_1)$$

where the right multiplication is Connes' fusion product. The main result is a construction of the Euler class in this context, which in a relative version, can lead to a Thom class and thus to the desired map of spectra

$$W : M\,String \longrightarrow tmf.$$