Isodiametric Inequality for the Steklov Eigenvalues

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September 2, 2017

Abstract

This is the report of a six week long internship at the *Université de Neuchâtel* done in the context of studies at the *École Normale Supérieure de Rennes* and under the supervision of Bruno COLBOIS.

The objective is to study and attempt to generalize the isodiametric inequality for the Steklov eigenvalues of B. Bogosel, D. Bucur and A. Giacomini [2]. The first chapter contains a general introduction to geometric spectral theory and to the geometric study of the Steklov eigenvalues. The second and third chapter are dedicated to the proof of the isodiametric inequality. Finally, the fourth and fifth chapters are an account of the attempts to generalize the inequality to the domains of the hyperbolic space or to an inequality involving the intrinsic diameter.

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Acknowledgments

I wish to thank Bruno Colbois for the time he has dedicated to my internship. We have spent hours on end discussing spectral geometry and he has provided continuous support for the writing of this report. Unfortunately, many mistakes (at best) remain, which are my own.

Many thanks go to all the staff and researchers of the institute for their warm welcome and their answering my questions, even when totally disconnected from their research interests.

I thank my Swiss family who has hosted me for the duration of the internship and without whom nothing would have been possible : my uncle and aunt Roland and Monique Benoit and my grand-parents Raymond and Mauricette Favre-Bulle. Their support has made these few weeks very special. I also am very grateful to Roland for originally throwing me into Mathematics, which has been a great delight.

Finally, I thank all my professors for the knowledge and love of Science they have given me, in particular Jean-Luc Joly, Denis Choimet, Patrick Séguin, Jacques Renault and Romain Presle (to name but a few).

Forwards

Let $\mathcal{M}$ be a compact Riemannian manifold of dimension $d \geq 2$ with smooth boundary $\partial\mathcal{M}$. The Steklov problem on $\mathcal{M}$ is finding those $\sigma \in \mathbb{C}$ for which there is a non trivial solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{M} \\ \partial_n u = \sigma u & \text{on } \partial\mathcal{M}, \end{cases} \tag{1}$$

where Δ is the Laplace-Beltrami operator on $\mathcal{M}$ and ∂_n is the outward normal derivative along the boundary $\partial\mathcal{M}$. This problem was posed by the Russian mathematician V. A. Steklov and describes (among other phenomena) the equilibrium heat flow through homogenous material, electrical impedance tomography or the behavior of fluids. An account of Steklov's life and some of his works is given in [13].

Problem (1) has a discrete and real spectrum $0 = \sigma_0(\mathcal{M}) < \sigma_1(\mathcal{M}) \leq \sigma_2(\mathcal{M}) \leq \cdots \to +\infty$ given by the min-max principle :

$$\sigma_k(\Omega) = \inf_{\dim(E)=k+1} \max_{u \in E, u \neq 0} \frac{\int_{\mathcal{M}} |\nabla u|^2 \mathrm{d}vol_d}{\oint_{\partial\mathcal{M}} u^2 \mathrm{d}vol_{d-1}}, \tag{2}$$

where the infimum ranged though all subspaces E of $H^1(\mathcal{M})$ of dimension $k+1$. This allows to prove geometric inequalities for the eigenvalues by using (2) with functions u of particular geometric relevance.

The main subject of the document is the study of the recent isodiametric inequality proved by B. Bogosel, D. Bucur and A. Giacomini in [2] : if $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain,

$$\sigma_k(\Omega) \leq C(d) \frac{k^{1+\frac{2}{d}}}{\mathrm{diam}(\Omega)}, \tag{3}$$

where $C(d)$ depends only on the dimension and $\text{diam}(\Omega) = \sup_{x,y\in\Omega} |x - y|$ is the diameter of Ω. The proof relies on the construction of functions u, supported in annuli, who, when used in the min-max principle, give information about the diameter of Ω. Doing this requires using relative isoperimetric inequalities in annuli that are uniform with respect to their width.

These relative isoperimetric inequalities are usually proved by using functions with bounded variation (BV functions), whose derivative are Radon measures. This allows to generalize the notion of perimeter to sets who don't have a Lipschitz boundary and provides powerful compactness results (used in [2]). See [8] for more on BV functions. An important part of the work was to adapt the relative inequalities proved using BV functions to the much simpler theory of smooth domains.

In the last part of this text, we attempt (rather unsuccessfully) to generalize (3) to domains of the hyperbolic space $\Omega \subset \mathbb{H}^d$ and to an inequality using the intrinsic diameter $\text{D}(\Omega)$ instead of the extrinsic diameter $\text{diam}(\Omega)$.

The text is organized as follows :

1. The first chapter contains introductory material on Geometric Spectral Theory and its methods, as well as a presentation of some selected works on the Steklov spectrum (including some recent research). This chapter is intended to be easily read as the discussion is kept at an informal level.

2. The second chapter contains the proof of the isodiametric inequality (3). It requires a good understanding of how to use the min-max principle and Rayleigh quotients to construct appropriate test functions.

3. The third chapter is dedicated to the proof of a relative isoperimetric inequality in annuli, uniformly to their width.

4. The fourth chapter is an account of an attempt to generalize (3) to domains of the hyperbolic space. Calculations become more complicated and require a good understanding of the two previous chapters.

5. The last and fifth chapter describes an attempt to adapt the method of [2] to obtain an inequality for the intrinsic diameter.

Notations

Throughout the whole text, we use use the following notations.

Euclidean space : when working in the Euclidean space $\mathbb{R}^d$ of dimension $d \geq 2$, we note $|\,.\,|$ the Euclidean norm and $\mathrm{d}x$ the Lebesgue measure. Therefore, if $\Omega \subset \mathbb{R}^d$ is a measurable set and $f \in L^1(\Omega)$, we note

$$\int_\Omega f = \int_\Omega f \mathrm{d}x = \int_\Omega f(x)\mathrm{d}x.$$

We also note $|\Omega|$ the Lebesgue measure of Ω.

Derivatives : unless otherwise mentioned, all derivatives are weak derivatives, *i.e.* derivatives in the sense of distributions. When needed, we use the Schwartz notation $\mathcal{D}(\mathbb{R}^d)$ for the space of smooth and compactly supported functions.

Sobolev spaces : we note $W^{1,p}(\Omega)$, $H^1(\Omega)$ and $H^1_0(\Omega)$ the Sobolev spaces

$$W^{1,p}(\Omega) = \{f \in L^p(\Omega) | \quad \nabla f \in L^p(\Omega)\},$$

$$H^1(\Omega) = W^{1,2}(\Omega),$$

$$H^1_0(\Omega) = \{f \in H^1(\Omega) | \quad f = 0 \text{ on } \partial\Omega\}.$$

Of course, all these notations adapt to the case where Ω is a Riemannian manifold with or without (smooth) boundary $\partial\Omega$.

Surface measures : if $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain (or more generally a Riemannian manifold with a nonempty boundary), we note S the Euclidean (Riemannian) surface measure and, when we believe it provides better legibility, we note $\oint$ the integral on the boundary. If $f \in L^1(\partial\Omega)$, we note

$$\oint_{\partial\Omega} f = \oint_{\partial\Omega} f \mathrm{d}S = \oint_{\partial\Omega} f(x)\mathrm{d}S(x).$$

Chapter 1

Introduction to Geometric Spectral Theory

There's nothing like deduction. We've determined everything about our problem but the solution.[1]

In this chapter, we give a short introduction to the topic of geometric spectral theory. The intent is to show the kind of results and methods that naturally occur in the subject so that the following chapters will be better understandable to the reader who is unfamiliar with it.

As this is only an introductory chapter and as the main focus of this document are isodiametric inequalities for the Steklov eigenvalues, the discussion is kept from being too formal and most of the statements are given without proof unless the aforementioned proofs are believed to help understanding the rest of the document.

The chapter is divided into three parts. In the first are presented the type of questions which interest spectral geometers and how they naturally arise. In the second part, we discuss some of the methods that are used in spectral geometry, namely Rayleigh quotients. The last and third part is specifically concerned with the Steklov spectrum and geometric inequalities for the Steklov eigenvalues.

1.1 Background

1.1.1 Thermodynamical Introduction

This paragraph is intended to show how the topic of geometric spectral theory naturally arises from a physical problem. Rigorous discussion is avoided in order to keep the explanation from being too complex or formal.

Let us introduce the geometric study of the spectra of differential operators with the following physical problem : let Ω be an open domain of $\mathbb{R}^d$ representing

[1] Isaac Asimov. *Runaround.*

a homogenous solid with temperature distribution being a function of space and time $u : \Omega \times \mathbb{R}_+ \longrightarrow \mathbb{R}$. Then u satisfies the heat equation[2] in Ω :

$$\Delta u = \partial_t u \qquad \text{in } \Omega. \tag{1.1}$$

In order for this problem to be well posed, it is necessary to specify a boundary condition on u, as well as initial conditions (see [3] chapter 10). Suppose, for instance, that the solid Ω has fixed temperature on the boundary, $u = 0$ on $\partial\Omega$, and that the temperature is initially $u(x, t = 0) = U_0(x)$. Then the following problem is well posed :

$$\begin{cases} \Delta u = \partial_t u \qquad \text{in } \Omega \\ u = 0 \qquad \text{on } \partial\Omega \\ u(t = 0) = U_0 \qquad \text{in } \Omega. \end{cases} \tag{1.2}$$

Not let us take the temporal Laplace transform[3] of u in order to "separate the variables":

$$\mathcal{L}u(x, \lambda) = \int_0^{+\infty} u(x, t) e^{-\lambda t} \mathrm{d}t, \tag{1.3}$$

so that problem (1.2) becomes

$$\begin{cases} -\Delta \mathcal{L}u = \lambda \mathcal{L}u \qquad \text{in } \Omega \\ \mathcal{L}u = 0 \qquad \text{on } \partial\Omega \\ u(t = 0) = U_0 \qquad \text{in } \Omega. \end{cases} \tag{1.4}$$

We hence are interested by those $\lambda \in \mathbb{C}$ such that the following problem has a non trivial solution

$$\begin{cases} -\Delta f = \lambda f \qquad \text{in } \Omega \\ f = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

The set of such λ is the *spectrum* of the Laplace operator on the space of functions $f : \Omega \longrightarrow \mathbb{R}$ that vanish on the boundary $\partial\Omega$. The following result holds (see [3] theorem 9.31 pp. 311-312 for a proof) :

Theorem 1.1. *Suppose Ω is connected and has a smooth enough boundary. Then problem* (1.5) *has a discrete and real*[4] *spectrum* $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \to +\infty$. *Moreover, the corresponding eigenfunctions $f_1, f_2, \ldots$ are C^∞ smooth, vanish on the boundary $\partial\Omega$ and form an (orthonormal) Hilbert basis of the space $L^2(\Omega)$.*

Then the decomposition

$$U_0(x) = \sum_{k=1}^{\infty} a_k f_k(x) \tag{1.6}$$

[2] In an appropriate set of units, all constants occurring in the heat equation can be set to unit value 1.

[3] We take the Laplace transform of u instead of taking the Fourier transform, as the heat equation is well posed only for positive times.

[4] So that all eigenfunctions are real-valued.

provides the following solution for the original problem (1.2) :

$$u(x,t) = \sum_{k=0}^{\infty} a_k e^{-t\lambda_k(\Omega)} f_k(x). \tag{1.7}$$

The eigenvalues $\lambda_k(\Omega)$ can therefore be seen as the inverse[5] of *cooling times* for the solid Ω : the "energy" left[6] in Ω after a period of t is, for large times and real u,

$$\mathcal{E}(t) = \int_\Omega u(x,t)^2 \mathrm{d}x = \sum_{k=0}^{\infty} a_k^2 e^{-2\lambda_k(\Omega)t} \sim a_1^2 e^{-2\lambda_1(\Omega)t}. \tag{1.8}$$

We expect[7] the heat loss to be greater if the surface $|\partial\Omega|$ is greater relatively to the volume $|\Omega|$. Therefore, in order to minimize heat loss from a solid of volume $|\Omega| = 1$, we wish to choose Ω with the smallest surface $|\partial\Omega|$ possible. Such a choice of Ω is possible : we take Ω to be the ball $B \subset \mathbb{R}^d$ of unit volume[8]. As the cooling time is closely linked to the eigenvalues (and especially to $\lambda_1(\Omega)$) through (1.8), we state the following :

Conjecture : For all $\Omega \subset \mathbb{R}^d$ of unit volume $|\Omega| = 1 = |B|$ we have :

$$\lambda_1(B) \leq \lambda_1(\Omega). \tag{1.9}$$

This result is in fact true and was independently proved by Faber and Krahn in the 1920s (see [5] pp. 78-81 for a proof). This is an example of result where the geometry of Ω (namely the volume $|\Omega|$) gives information concerning the spectrum (here a lower bound). Such problems are called *direct* problems.

The converse problems, determining information about the geometry of Ω based on the spectrum, are called *inverse* problems.

One of the main goals of *Geometric Spectral Theory* is the establishment of such direct or inverse properties for the spectra of different differential operators acting on spaces of functions on domains Ω, or more generally on manifolds.

1.1.2 Examples of Boundary Conditions

The choice of the boundary condition, $u = 0$ on $\partial\Omega$, was decisive in the previous example. The $\lambda_k(\Omega)$ were the eigenvalues of the Laplacian *on the space of functions vanishing on* $\partial\Omega$. Were the boundary conditions to be different, the Laplacian would act on a different space of functions and hence we would look at the spectrum *of a different operator*.

We give four examples of recurring boundary conditions. We suppose Ω is a connected subset of $\mathbb{R}^d$ with Lipschitz boundary.

[5] A physicist would say the quantities $1/\lambda_k$ are homogenous to times.

[6] What we here call energy is not what a physicist would. The thermodynamic energy$E$ is proportional to the temperature so that $E(t) = \int_\Omega u(x,t)\mathrm{d}x$, whereas what we call "energy" is merely the (squared) L^2_x norm $\mathcal{E}(t) = \int_\Omega u(x,t)^2\mathrm{d}x$.

[7] For no other reason than cooking experiments : chips and fries cook faster than whole potatoes.

[8] This result is called the *isoperimetric* property of the ball. It shall be investigated later.

Dirichlet Boundary Conditions

They correspond the following problem :

$$\begin{cases} -\Delta f = \lambda f & \text{in } \Omega \\ f = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

The Dirichlet eigenvalues form a discrete set

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \to +\infty, \tag{1.11}$$

where eigenvalues are repeated according to multiplicity. As indicated above, the $\lambda_k(\Omega)$ can be understood as being cooling times for a solid.

Neumann Boundary Conditions

Let n be the exterior normal to the boundary. The Neumann eigenvalues correspond to the following problem :

$$\begin{cases} -\Delta f = \mu f & \text{in } \Omega \\ \partial_n f = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where we note ∂_n the outwards normal derivative along the boundary $\partial\Omega$. The Neumann eigenvalues also form a discrete set

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \to +\infty. \tag{1.13}$$

Note that the first eigenvalue $\mu_0(\Omega)$ is zero and has multiplicity one. The associated eigenfunctions are the constant functions (forming a linear space of dimension 1).

Back to thermodynamics. The Neumann boundary conditions naturally arise, as above, when considering a piece of homogenous solid covered by an isothermal layer so that there is no heat exchange between the solid and the external environment. This gives (alongside Fourier's law[9]) the condition $\partial_n f = 0$ on the boundary. The problem is the following :

$$\begin{cases} \Delta u = \partial_t u & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega \\ u(t=0) = U_0 & \text{in } \Omega. \end{cases} \tag{1.14}$$

The $1/\mu_k(\Omega)$ can be interpreted as *relaxation times* : time elapsed before the temperature becomes uniform in the solid. If $\frac{1}{|\Omega|^{1/2}} = f_0, f_1, f_2, \ldots$ are the eigenfunctions associated with the Neumann eigenvalues $\mu_0(\Omega), \mu_1(\Omega), \mu_2(\Omega), \ldots$ (which form an orthonormal Hilbert basis of $L^2(\Omega)$), then

$$u(x,t) = \int_\Omega U_0(x)\mathrm{d}x + \sum_{k=1}^{\infty} a_k e^{-\mu_k(\Omega)t} f_k(x), \tag{1.15}$$

[9] Fourier's law states that the heat flux density j is given by (again, in the appropriate set of units) $j = -\nabla u$.

where the a_k are chosen so that

$$U_0(x) = \int_\Omega U_0(x)\mathrm{d}x + \sum_{k=1}^{\infty} a_k f_k(x). \tag{1.16}$$

For large times, the temperature is almost uniform :

$$u(x,t) \sim \int_\Omega U_0(x)\mathrm{d}x + a_1 e^{-\mu_1(\Omega)t} f_1(x). \tag{1.17}$$

We of course expect the Neumann eigenvalues to behave very differently from the Dirichlet eigenvalues when dependance on Ω is considered. For example, the Neumann eigenvalues are highly sensitive to the shape of the boundary $\partial\Omega$. We can give the intuition of such a phenomenon by considering disjoint domains connected by a thin passage. Take, for instance, any domain Ω and add small ball close to Ω and connected to it by a thin cylinder. See Figure 1.1 for a drawing of this.

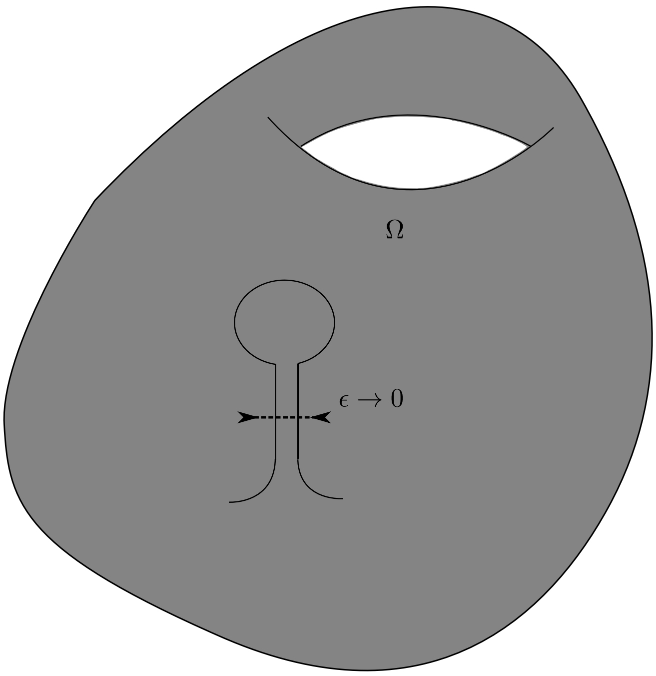

Figure 1.1 – Ball connected to Ω by a thin passage.

If we take U_0 such that the ball has initially high temperature and Ω low temperature, we expect a long time will be necessary to reach equilibrium, since the heat transfer from the ball to Ω can only take place in the thin passage connecting the two. We therefore expect $\mu_1(\Omega_\epsilon) \to 0$ as the width ϵ of the cylinder tends to zero, where Ω_ϵ is the union of Ω, the ball B and the cylinder.

This intuition is in fact justified, as we will see later on. This construction is an example of a small (local) deformation of the domain Ω (the ball can be taken as small as desired and the cylinder has width $\epsilon \to 0$) that has a massive impact on the first nonzero eigenvalue $\mu_1(\Omega)$.

Robin Boundary Conditions

The Robin boundary conditions are related to the following problem : let $\alpha \in \mathbb{R}$ be a fixed parameter, we seek the λ such that there is a non trivial solution to

$$\begin{cases} -\Delta f = \lambda f & \text{in } \Omega \\ \partial_n f + \alpha f = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.18}$$

The Robin boundary condition (also known as the *de Gennes* boundary condition by physicists) models a homogenous solid immersed in a gas of constant temperature : the heat loss from the solid is proportional to the difference of temperature between the solid and the gas, hence the condition $\partial_n u + \alpha u = 0$ where α depends on the physical characteristics of the gas.

The Robin problem has a discrete and real spectrum

$$\lambda_1(\Omega, \alpha) \leq \lambda_2(\Omega, \alpha) \leq \cdots \to +\infty. \tag{1.19}$$

Note that the Robin eigenvalues need not be nonegative !

Steklov Boundary Conditions

The Steklov boundary conditions, which are the main focus of the this document, correspond to the following problem : we seek the $\sigma \in \mathbb{C}$ such that there is a non trivial solution to

$$\begin{cases} -\Delta f = 0 & \text{in } \Omega \\ \partial_n f = \sigma f & \text{on } \partial\Omega. \end{cases} \tag{1.20}$$

It may seem less obvious, but the σ satisfying (1.20) are in fact the spectrum of some operator acting on the space of functions on $\partial\Omega$. Consider a (smooth) function $f : \partial\Omega \longrightarrow \mathbb{R}$ and note $F : \Omega \longrightarrow \mathbb{R}$ the (unique) harmonic extension of f to Ω :

$$\begin{cases} \Delta F = 0 & \text{in } \Omega \\ F = f & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Then note $Tf = \partial_n F$, thus defining a function on $\partial\Omega$. The operator T is called the Dirichlet to Neumann map and the σ are the spectrum of T. It can be shown that problem (1.20) has a discrete spectrum

$$0 = \sigma_0(\Omega) < \sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots \to +\infty. \tag{1.22}$$

We note that the first eigenvalue $\sigma_0(\Omega)$ is zero. Its corresponding eigenfunctions are the constant functions (which are in the kernel of T).

Remark 1.2. The Dirichlet to Neumann map T is not expressed as a combination of derivatives as the Laplace or the Laplace-Beltrami operators defined below are. It is therefore not, strictly speaking, a differential operator, but only a *pseudo-differential* operator. However, this makes absolutely no difference when studying the spectrum of T.

We go back one last time to thermodynamics. Suppose there is a fixed temperature distribution U_0 on the surface of the solid Ω. Then, in time-independent regime, the temperature u is the harmonic extension of U_0 to Ω :

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = U_0 & \text{on } \partial\Omega. \end{cases} \tag{1.23}$$

The function $TU_0 = \partial_n u$ is the heat flow from the external environment to the solid Ω. Unfortunately, this provides us with no obvious way of understanding the eigenvalues $\sigma_k(\Omega)$.

Boundary Conditions on Manifolds

We here consider a smooth Riemannian manifold $(\mathcal{M}, g)$ with boundary $\partial\mathcal{M}$. The Laplace-Beltrami operator associated to the metric g reads, in local coordinates $(x_i)_i$,

$$\Delta_g f(x) = \frac{1}{\sqrt{\det(g(x))}} \sum_{i,j} \partial_j \left(g^{ij}(x) \sqrt{\det(g(x))} \partial_i f(x) \right), \tag{1.24}$$

and if $(e_i(x))_i$ is the basis of the tangent space $\mathcal{T}_x\mathcal{M}$ associated with the coordinates $(x_i)_i$, the gradient operator reads

$$\nabla_g f(x) = \sum_{i,j} g^{ij}(x) \partial_j f(x) e_i(x). \tag{1.25}$$

We can then straitforwardly adapt all the preceding boundary conditions to functions on $\mathcal{M}$. For example, the Steklov problem on $\mathcal{M}$ is :

$$\begin{cases} -\Delta_g f = 0 & \text{in } \mathcal{M} \\ \partial_n f := \langle \nabla_g f | n \rangle_g = \sigma f & \text{on } \partial\mathcal{M}, \end{cases} \tag{1.26}$$

where n is the exterior normal to the boundary.

Note that the boundary $\partial\mathcal{M}$ can be empty, in which case $H_0^1(\mathcal{M}) = H^1(\mathcal{M})$. The corresponding problem is called the *closed* problem. For example, if $\Omega \subset \mathbb{R}^d$ is a smooth bounded domain, we can study the spectrum of the Laplace-Beltrami operator on the boundary $\partial\Omega$ seen as a Riemannian manifold equipped with the metric induced by the Euclidean structure of $\mathbb{R}^d$.

We refer to the third chapter of [14] for introductory material on Riemannian geometry.

1.2 Rayleigh Quotients and Test Functions

As one might expect, it is hard to study the spectrum when defined by a PDE problem as above. It is therefore crucial to find another characterization of the eigenvalues better suited to the establishment of links between the spectrum and the geometry of the space. Such a characterization is provided by the min-max principle introduced in this paragraph.

1.2.1 min-max Principle

First consider the Dirichlet eigenvalues. Problem (1.5) has the following weak formulation : we seek $\lambda \in \mathbb{C}$ and $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega), \qquad \int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega uv. \tag{1.27}$$

Therefore, if λ is a Dirichlet eigenvalue with associated eigenfunction u,

$$\lambda = \frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} := \mathcal{R}(u). \tag{1.28}$$

This quotient of two integrals is called the *Rayleigh quotient* $\mathcal{R}(u)$ of u (for the Dirichlet problem). This immediately gives a variational characterization of the lowest eigenvalue $\lambda_1(\Omega)$: decompose any function $u \in H_0^1(\Omega)$ on the Hilbert basis of orthonormal eigenfunctions $u_1, u_2, ...$ for the Dirichlet eigenvalues,

$$u = \sum_{k=1}^{\infty} a_k u_k, \tag{1.29}$$

we can then compute the Rayleigh quotient of u : the orthogonality of the u_k yeilds

$$\mathcal{R}(u) = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} = \sum_{k=1}^{\infty} \lambda_k(\Omega) \frac{a_k^2 \|u_k\|_{L^2}^2}{\|u\|_{L^2}^2} \geq \frac{\lambda_1(\Omega)}{\|u\|_{L^2}^2} \sum_{k=1}^{\infty} a_k^2 = \lambda_1(\Omega). \tag{1.30}$$

Equality in this last line is achieved if and only if $u \in \mathbb{R}u_1$ so that

$$\lambda_1(\Omega) = \min_{u \in H_0^1, u \neq 0} \mathcal{R}(u). \tag{1.31}$$

This is a particular case of the much more general min-max formula (theorem 4.5.5 p 94. of [14] -the pages 90-94 contain full explanations):

Theorem 1.3 (min-max principle for the Dirichlet Problem)**.** *For all $k \geq 1$, the Dirichlet eigenvalue $\lambda_k(\Omega)$ is given by*

$$\lambda_k(\Omega) = \inf_{dim(E)=k} \max_{u \in E, u \neq 0} \mathcal{R}(u), \tag{1.32}$$

where the infimum taken over all subspaces $E \subset H_0^1(\Omega)$ of dimension k and is achieved by

$$E_k = Vect\{u_1, u_2, ..., u_k\}, \tag{1.33}$$

where the u_k are the orthonormal eigenfunctions associated with the $\lambda_k(\Omega)$.

This can easily be generalized to the Neumann and the Steklov eigenvalues by adapting the Rayleigh quotient to one better suited for the purpose.

Neumann Eigenvalues

The Rayleigh quotient for the Neumann eigenvalues is the same as for the Dirichlet eigenvalues

$$\mathcal{R}(u) = \frac{\int_\Omega \|\nabla u\|^2}{\int_\Omega u^2} \tag{1.34}$$

but the functions u range in the larger space $H^1(\Omega)$ so that the appropriate min-max principle is as follows :

Theorem 1.4 (min-max principle for the Neumann Problem)**.** *For all $k \geq 0$, the Neumann eigenvalue $\mu_k(\Omega)$ is given by*

$$\mu_k(\Omega) = \inf_{dim(E)=k+1} \max_{u \in E, u \neq 0} \mathcal{R}(u), \tag{1.35}$$

where the infimum taken over all subspaces $E \subset H^1(\Omega)$ of dimension $k+1$ and is achieved by

$$E_k = Vect\{u_0, u_1, u_2, ..., u_k\}, \tag{1.36}$$

where the u_k are the orthonormal eigenfunctions associated with the $\mu_k(\Omega)$.

Remark 1.5. Note that since the lowest eigenvalue is $\mu_0(\Omega) = 0$, it is necessary to take subspaces $E \subset H^1(\Omega)$ of dimension $k+1$.

Robin Eigenvalues

The Rayleigh quotient for the Robin eigenvalues is

$$\mathcal{R}(u) = \frac{\int_\Omega |\nabla u|^2 + \alpha \oint_{\partial\Omega} u^2}{\int_\Omega u^2}, \tag{1.37}$$

and the functions u range in the larger space $H^1(\Omega)$ so that the appropriate min-max principle is as follows :

Theorem 1.6 (min-max principle for the Robin Problem)**.** *For all $k \geq 0$, the Robin eigenvalue $\lambda_k(\Omega, \alpha)$ is given by*

$$\lambda_k(\Omega, \alpha) = \inf_{dim(E)=k} \max_{u \in E, u \neq 0} \mathcal{R}(u), \tag{1.38}$$

where the infimum taken over all subspaces $E \subset H^1(\Omega)$ of dimension k and is achieved by

$$E_k = Vect\{u_1, u_2, ..., u_k\}, \tag{1.39}$$

where the u_k are the orthonormal eigenfunctions associated with the $\lambda_k(\Omega, \alpha)$.

Remark 1.7. Note that the apparition of the surface integral $\oint_{\partial\Omega} u^2$ makes the study of the Robin eigenvalues a bit different from that of the Neumann or the Dirichlet eigenvalues.

Steklov Eigenvalues

The Rayleigh quotient for the Steklov eigenvalues is

$$\mathcal{R}(u) = \frac{\int_\Omega |\nabla u|^2}{\oint_{\partial\Omega} u^2} \tag{1.40}$$

and the corresponding min-max principle is :

Theorem 1.8 (min-max principle for the Steklov Problem)**.** *For all $k \geq 0$, the Steklov eigenvalue $\sigma_k(\Omega)$ is given by*

$$\sigma_k(\Omega) = \inf_{dim(E)=k+1} \max_{u \in E, u \neq 0} \mathcal{R}(u), \tag{1.41}$$

where the infimum taken over all subspaces $E \subset H^1(\Omega)$ of dimension $k+1$ and is achieved by

$$E_k = Vect\{u_0, u_1, u_2, ..., u_k\} \tag{1.42}$$

where the u_k are the orthonormal eigenfunctions associated with the $\sigma_k(\Omega)$.

Remark 1.9. As for the Neumann problem, the lowest eigenvalue is $\sigma_0(\Omega) = 0$, so it is also necessary to take subspaces E of dimension $k+1$.

Remark 1.10. The surface integral $\oint_{\partial\Omega}$ appearing at the denominator of the Rayleigh quotient already indicates that the Steklov eigenvalues will behave very differently from those of the other problems. Especially, we expect the σ_k to be very sensitive to the boundary $\partial\Omega$ of the domain.

1.2.2 Test Functions

The min-max principle makes it fairly easy to give upper bounds for the eigenvalues : it sufficess to find a k-dimensional subspace E of $H_0^1(\Omega)$ (respectively a $k+1$-dimensional subspace of $H^1(\Omega)$) and majorize the Rayleigh quotient on that subspace to obtain an upper bound for $\lambda_k(\Omega)$ (respectively for $\mu_k(\Omega)$ and $\sigma_k(\Omega)$).

$$\lambda_k(\Omega) \leq \sup_{u\in E} \mathcal{R}(u) \leq \text{(upper bound)}. \tag{1.43}$$

Remark 1.11. Of course, bounding $\mathcal{R}(u)$ from above on any given subspace E can be somewhat tedious, unless one has an explicit $L^2(\Omega)$-orthonormal basis of E. This can be done by choosing E as a subspace of $H_0^1(\Omega)$ (or $H^1(\Omega)$) spanned by functions which are already known to be mutually orthogonal, for example functions with disjoint support. If $\phi_1, ..., \phi_k$ are such functions and $E = \text{Vect}\{\phi_1, ..., \phi_k\}$ then

$$\max_{u\in E, u\neq 0} \mathcal{R}(u) = \max_k \mathcal{R}(\phi_k) \tag{1.44}$$

As an example of this we shall prove our claim that the first nonzero Neumann eigenvalue $\mu_1(\Omega)$ is arbitrarily small on domains described by figure 1.1 in section 1.1.2. We call Ω_ϵ the domain formed by the union of Ω, the ball B and the cylinder of width ϵ.

Let m be the measure of the ball. Call x the coordinate on an axis directing the cylinder linking the ball B to Ω, as displayed in figure 1.2, and let ϕ be defined as follows :

On Ω : we set $\phi = -\frac{m}{|\Omega|}$.

On the ball : we set $\phi = 1$.

On the cylinder : we take $-\frac{m}{|\Omega|} \leq \phi \leq 1$ to be an affine function of x so that it is continuous everywhere.

The function ϕ is piecewise C^1-smooth so that $\phi \in H^1(\Omega)$. Moreover, the mean value of ϕ is 0 so that ϕ is orthogonal to the space of constant functions. We make the following choice for E :

$$E = \text{Vect}\{\phi, 1\} \qquad \text{so that} \qquad \dim(E) = 2. \tag{1.45}$$

Let $L > 0$ be the length of the cylinder (recall that the width is ϵ). The maximum value of $\mathcal{R}(u)$ for $u \in E$ is then $\mathcal{R}(\phi)$. On the one hand, since $\nabla\phi$ is supported in the cylinder, we have

$$\int_{\Omega_\epsilon} |\nabla\phi|^2 = CL\epsilon^{d-1}\left(\frac{m}{L|\Omega|}\right)^2, \tag{1.46}$$

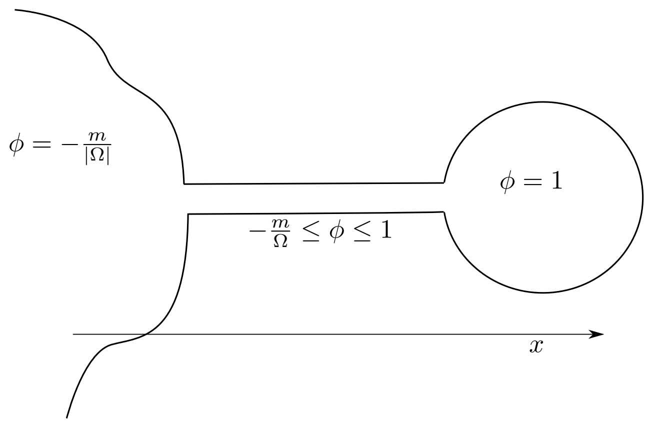

Figure 1.2 – Test function ϕ on disjoint domains connected by a thin passage.

where C is a constant depending only on the dimension d. And on the other hand,

$$\int_{\Omega_\epsilon} \phi^2 \geq \int_{\Omega} \phi^2 = |\Omega| \left(\frac{m}{|\Omega|}\right)^2. \tag{1.47}$$

Combining these two inequalities gives us the desired upper bound :

$$\mathcal{R}(\phi) = \frac{\int_{\Omega_\epsilon} |\nabla \phi|^2}{\int_{\Omega_\epsilon} \phi^2} \leq \frac{C}{L|\Omega|} \epsilon^{d-1} \longrightarrow 0 \tag{1.48}$$

so that

$$\mu_1(\Omega_\epsilon) \underset{\epsilon \to 0^+}{\longrightarrow} 0. \tag{1.49}$$

1.2.3 Asymptotic Theorems

Asymptotic study of the spectrum has been made in the domain of "high frequencies" $k \to +\infty$. These results are generically named *Weyl's law.*

Theorem 1.12 (Weyl's law for the Dirichlet and the Neumann problems)**.** *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension d (with or without boundary). We have the (same) following asymptotics for the Dirichlet, the Neumann (and the closed) eigenvalues as $k \to +\infty$,*

$$\lambda_k(\mathcal{M}, g) \sim \frac{(2\pi)^2}{\omega_d^{2/d}} \left(\frac{k}{V_g(\mathcal{M})}\right)^{2/d} \tag{1.50}$$

$$\mu_k(\mathcal{M}, g) \sim \frac{(2\pi)^2}{\omega_d^{2/d}} \left(\frac{k}{V_g(\mathcal{M})}\right)^{2/d} \tag{1.51}$$

where ω_d is the volume of the unit ball of $\mathbb{R}^d$ and V_g is the d-dimensional Riemannian measure on $(\mathcal{M}, g)$.

There also is a Weyl law for the Steklov eigenvalues.

Theorem 1.13 (Weyl's law for the Steklov spectrum)**.** *Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension d with nonempty boundary $\partial\mathcal{M}$. Then the following asymptotics hold for the Steklov spectrum : as $k \to +\infty$,*

$$\sigma_k(\Omega) \sim 2\pi \left(\frac{k}{\omega_{d-1} V_g(\partial\mathcal{M})} \right)^{\frac{1}{d-1}}, \tag{1.52}$$

where V_g is the $(d-1)$-dimensional Riemannian measure on $\mathcal{M}$.

Weyl's law shows that the spectrum automatically contains information about the geometry of the manifold $\mathcal{M}$, namely the volume of the manifold (or its boundary in the case of the Steklov eigenvalues). However, it must be noted that much more subtle and complex information can also be encoded in the eigenvalues : the asymptotic formulae gives no information about the first eigenvalues or the remainder terms !

For instance, we have seen (in section 1.2.2) that the presence of a small cylinder connecting a domain with a ball dramatically alters the first nonzero Neumann eigenvalue $\mu_1(\Omega_\epsilon)$. Adding a small ball and a thin cylinder doesn't affect the volume much $|\Omega_\epsilon| \approx |\Omega|$ so that the asymptotic formula (1.51) is barely affected.

1.3 The Steklov Spectrum

In this section, we list a few results concerning the Steklov spectrum (including some recent work on the subject) in order to better understand the Steklov eigenvalues and their dependency on the boundary of the manifold $\partial\mathcal{M}$. We then discuss the isodiametric control of the eigenvalues which is the focus of this document.

1.3.1 Upper Bounds and Isoperimetric Ratio

The following theorem ([6], theorem 1.3) links the Steklov eigenvalues of a subdomains $\Omega \subset \mathcal{M}$ of a Riemannian manifolds with its isoperimetric ratio, provided some condition on the curvature is fufilled.

Theorem 1.14 (B. Colbois, A. El Soufi, A. Girouard, 2011)**.** *Let $(\mathcal{M}, g_0)$ be a complete Riemannian manifold of dimension $d \geq 2$ with non-negative Ricci curvature $Ricc_{g_0} \geq 0$. Then there exists a constant $C(d)$ depending only on the dimension such that for any metric g in the conformal class $[g_0]$ of the metric g_0, for any bounded domain $\Omega \subset (\mathcal{M}, g)$ the following holds :*

$$\sigma_k(\Omega, g) \leq C(d) \frac{k^{2/d}}{I_g(\Omega)^{(d-2)/(d-1)}} \tag{1.53}$$

where $I_g(\Omega) = \frac{V_g(\partial\Omega)}{V_g(\Omega)^{(d-1)/d}}$ is the isoperimetric ratio related to the Riemannian d and $(d-1)$ dimensional measures both noted V_g.

Note that, since the hyperbolic space $\mathbb{H}^d$ is conformal to the Euclidean space $\mathbb{R}^d$, this result also holds when $\Omega \subset \mathbb{H}^d$. The existence of isoperimetric inequalities

$$|\Omega|^{(d-1)/d} \leq C_d |\partial\Omega| \tag{1.54}$$

on both spaces $\mathbb{R}^d$ and $\mathbb{H}^d$ assures us that the eigenvalues are always bounded from above :

$$\sigma_k(\Omega) \leq C(d)k^{2/d}. \tag{1.55}$$

Natural (and open) question : are there regular enough[10] domains of $\mathbb{R}^d$ which maximize the eigenvalues ? If so, can these domains be found explicitly ?

Theorem 1.14 has been extended to the case where the manifold $(\mathcal{M}, g_0)$ has a Ricci curvature bounded from below by $\mathrm{Ricc}_{g_0} \geq -(d-1)$. See [11], theorem 4.1.

Theorem 1.15 (A. Hassannezhad, 2011)**.** *Let $(\mathcal{M}, g_0)$ be a complete Riemannian manifold of dimension $d \geq 2$ with Ricci curvature satisfying $Ricc_{g_0} \geq -(d-1)$. Then, for any metric $g \in [g_0]$, for any C^1-smooth and bounded domain $\Omega \subset (\mathcal{M}, g)$,*

$$\sigma_k(\Omega, g)V_g(\partial\Omega)^{1/(d-1)} \leq \frac{1}{I_g(\Omega)^{(d-1)/(d-1)}} \left[C_1(d)V_{g_0}(\partial\Omega)^{2/d} + C_2(d)k^{2/d}\right] \tag{1.56}$$

Both these theorems show that the Steklov spectrum behaves very differently from the Dirichlet spectrum. In the case of Dirichlet eigenvalues, the Faber-Krahn inequality (1.9) shows the first eigenvalue $\lambda_1(\Omega)$ is maximized when the isoperimetric ratio of $\Omega \subset \mathbb{R}^d$ is minimal, whereas here, with the Steklov eigenvalues, a large isopermimetric ratio forces the eigenvalues to be small. (See figure 1.3 for an example of such an Ω)

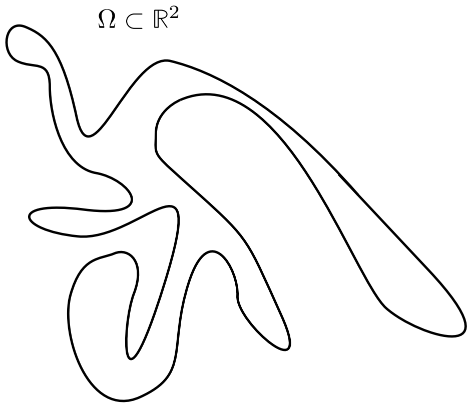

Figure 1.3 – Large Isoperimetric ratios induce small σ_k.

The intuition we get from these two theorems is that the more Ω has a complicated boundary the more we expect the eigenvalues to be small. Of course, in higher dimensions $d \geq 3$, there are a lot of ways for a domain to be "complicated" without necessarily having a small isoperimetric ratio. There even

[10]Regular enough that the Steklov spectrum is well defined.

are simple examples of domains with a large isoperimetric ratio *and* arbitrarily small eigenvalues. The following example[11] was published in a 2010 paper of A. Girouard and I. Polterovich ([10], section 2.2).

Set $d \geq 3$ and consider a domain $\Omega \subset \mathbb{R}^d$ with a thin collapsing passage, a cylinder for example. Note (as in section 1.2.2) L the length of the cylinder, ϵ its width and x a coordinate on an axis directing the cylinder (see figure 1.4).

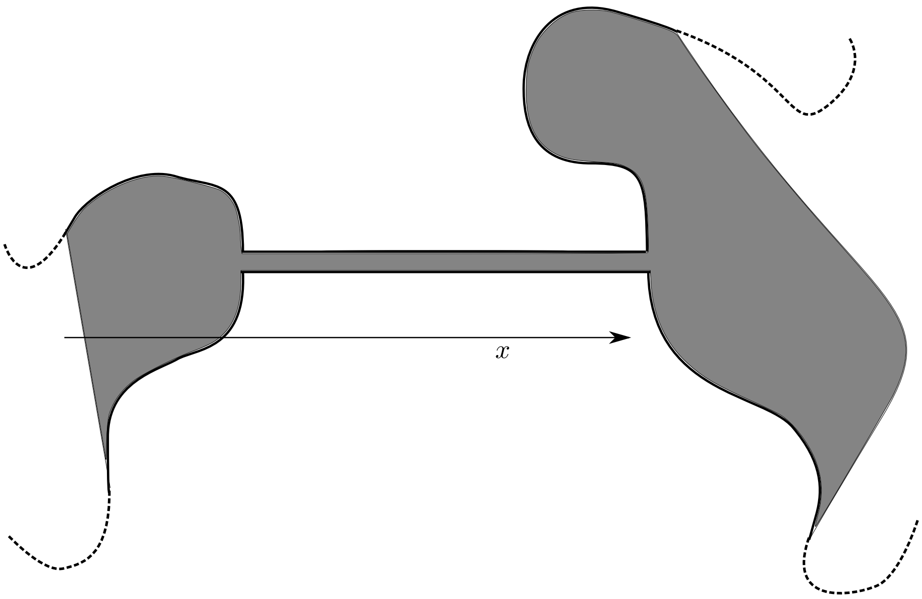

Figure 1.4 – A domain with a thin collapsing passage.

Note that, as we will see, the thin passage need not be connecting two disjoint parts of Ω as in section 1.1.2. It sufficess for Ω to possess such a cylinder as displayed in figure 1.4. See figure 1.6 for an example of that.

We now construct the appropriate test functions. Let $k \geq 1$. We consider $\chi \in C^\infty(\mathbb{R})$ with support $\text{supp}(\chi) \subset [0,1]$ satisfying $\chi \geq 0$. Define

$$\forall\ 0 \leq i \leq k, \qquad \phi_k(x) = \chi\left(kx/L - \frac{i}{kL}\right) \tag{1.57}$$

so that the $k+1$ functions $\phi_0, ..., \phi_k$ are disjointly supported in the cylinder (See figure 1.5).

We then compute the Rayleigh quotient of the ϕ_i. Call Ω_ϵ the domain described by figure 1.4. On the one hand, the numerator integral reads

$$\int_{\Omega_\epsilon} |\nabla \phi_i|^2 = C(d)\epsilon^d L \int_{i/k}^{(i+1)/k} k^2 \left|\chi'\left(kxL - \frac{iL}{k}\right)\right|^2 \mathrm{d}x, \tag{1.58}$$

where $C(d)$ is a constant depending only on the dimension d. And on the other

[11] The example presented here differs slightly from that shown in [10]. Instead of taking trigonometric test functions $\sin(\frac{2\pi n x}{\epsilon})$ as in done originally [10], A. Girouard pointed out that using disjointly supported test functions was just as simple and slightly easier to generalize.

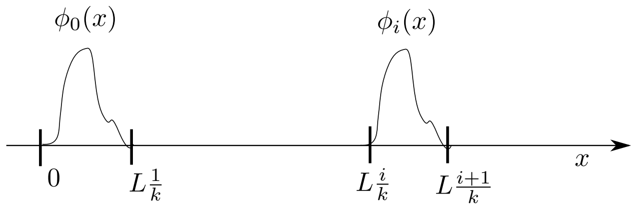

Figure 1.5 – The functions ϕ_i are disjointly supported in the cylinder.

hand, the denominator integral reads

$$\oint_{\partial\Omega_\epsilon} \phi_i^2 = C(d)\epsilon^{d-1} L \int_{i/k)}^{(i+1)/k} \chi\left(kxL - \frac{iL}{k}\right)^2 \mathrm{d}x, \tag{1.59}$$

where $C(d)$ is (another) constant depending only on the dimension d. The Rayleigh quotient then is, for each $0 \leq i \leq k$, for some function $A(k)$,

$$\mathcal{R}(\phi_i) = A(k)\epsilon \to 0. \tag{1.60}$$

This shows that as the width of the collapsing passage tends to zero, *all* the eigenvalues are small (though, obviously, not uniformly small) :

$$\forall k, \qquad \sigma_k(\Omega_\epsilon) \underset{\epsilon \to 0^+}{\longrightarrow} 0. \tag{1.61}$$

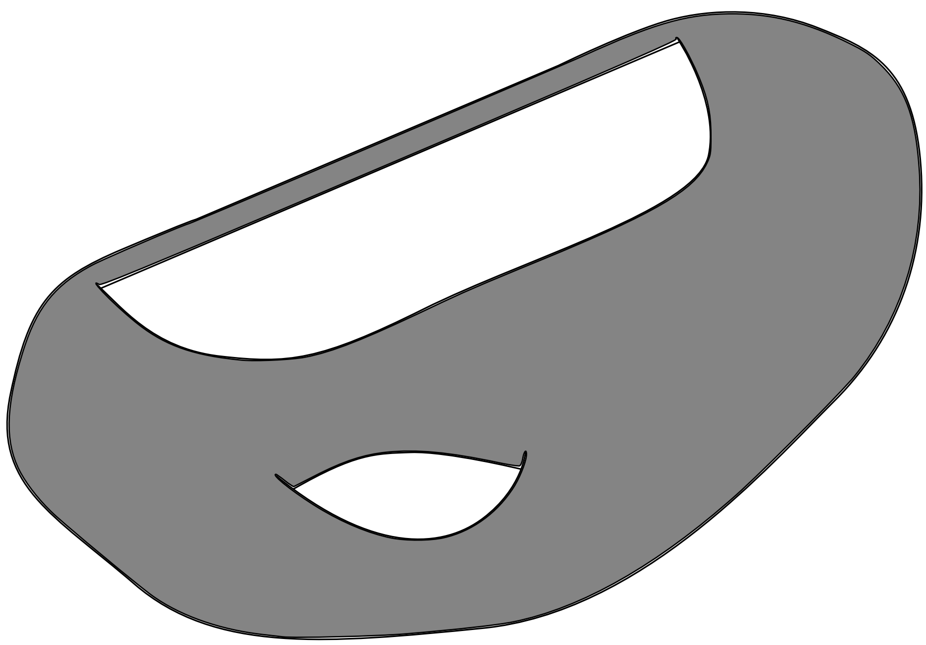

Figure 1.6 – The thin passage need not connect two disjoint parts of Ω.

Now, we have taken the dimension of the space to be at least 3 : $d \geq 3$. The consequence of this is that both the volume *and* the perimeter of the cylinder tend to zero with ϵ so that the isoperimetric ratio $I(\Omega_\epsilon)$ is close to that of the domain without the cylinder. The $\sigma_k(\Omega_\epsilon)$ are small but $I(\Omega_\epsilon)$ remains bounded. Hence the upper bounds (1.53) and (1.56) are blind so some geometric phenomenon which can have a massive impact on the spectrum.

The upper bounds (1.53) and (1.56) are consistent with the Weyl asymptotic law (1.52) in two ways. The first is that if the boundary $\partial\Omega$ has a large measure then both the upper bounds and the asymptotic formula are small. Secondly, the rate of growth $k^{2/d}$ of the upper bounds is larger than that predicted by the asymptotic formula $k^{2/(d-1)}$.

This last remark shows that the upper bounds (1.53) and (1.56) are probably not optimal. If $d \geq 3$, it is still an open question (June 2017) whether there are upper bounds of the form

$$\sigma_k \leq C(d)\left(\frac{k}{|\partial\Omega|}\right)^{1/(d-1)}. \tag{1.62}$$

In dimension $d = 2$, the problem has been solved in 1975 for simply connected $\Omega \subset \mathbb{R}^2$ by J. Hersch, L. E. Payne and M. M. Schiffer [12] :

$$\sigma_k(\Omega) \leq 2\pi\frac{k}{|\partial\Omega|}. \tag{1.63}$$

1.3.2 Steklov Spectrum on a Domain and Dirichlet Spectrum on the Boundary

We present a result (see [15]) which stresses the importance of the boundary when considering the Steklov spectrum. If $\Omega \subset \mathbb{R}^d$ is a bounded domain with a C^2 smooth boundary, we note

$$0 = \lambda_0(\partial\Omega) < \lambda_1(\partial\Omega) \leq \cdots \to +\infty \tag{1.64}$$

the eigenvalues of the Laplace-Beltrami operator Δ_{LB} on the boundary : $\Delta_{LB}u = \lambda u$ on $\partial\Omega$.

Theorem 1.16 (L. Provenzano, J. Stubbe, 2017)**.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with connected boundary $\partial\Omega$ at least C^2 smooth. Then there exists a constant $C(\Omega)$ such that, for all $k \geq 0$,*

$$\lambda_k(\partial\Omega) \leq \sigma_k(\Omega)^2 + 2C(\Omega)\sigma_k(\Omega) \tag{1.65}$$

$$\sigma_k(\Omega) \leq C(\Omega) + \sqrt{C(\Omega)^2 + \lambda_k(\partial\Omega)} \tag{1.66}$$

Moreover, the constant $C(\Omega)$ depends on the maximal possible size h of a tubular neighborhood about $\partial\Omega$ and on the maximal mean curvature of $\partial\Omega$ and can be explicitly given as a function of these parameters :

$$C(\Omega) = \frac{1}{2h} + \frac{d-1}{2}\bar{H}_\infty, \tag{1.67}$$

where, if κ_i are the principle curvatures on $\partial\Omega$,

$$\bar{H}_\infty = \left\|\frac{1}{d-1}\sum_{i=1}^{d-1}|\kappa_i(x)|\right\|_{L^\infty_x}. \tag{1.68}$$

This theorem allows, for example, to adapt geometric estimations for the Dirichlet boundary, which has been extensively studied over the years, to the Steklov problem, which is comparatively new in spectral geometry (see [15] for some corollaries).

However, this theorem is limited by the fact that the constant $C(\Omega)$ depends heavily on the geometry of Ω. This may prove to be a limiting factor when trying to use it for domains with a complex boundary.

1.3.3 Isodiametric Control of the Spectrum

We now discuss the main focus of this text : isodiametric control of the Steklov eigenvalues ([2] proposition 4.3). We note $\mathrm{diam}(\Omega)$ the diameter of a subset $\Omega \subset \mathbb{R}^d$:

$$\mathrm{diam}(\Omega) = \sup_{x,y\in\Omega} |x-y|. \tag{1.69}$$

Theorem 1.17 (B. Bogosel, D. Bucur, A. Giacomini, 2017)**.** *Let $d \geq 2$ and $\Omega \subset \mathbb{R}^d$ be a Lipschitz bounded domain. Then, for all $k \geq 1$,*

$$\sigma_k(\Omega) \leq C(d)\frac{k^{1+\frac{2}{d}}}{diam(\Omega)}, \tag{1.70}$$

where $C(d)$ is a constant depending only on the dimension d.

This theorem will be thoroughly proved in the next two chapters.

The isodiametric inequality (1.70) points out to a new range of geometric phenomena that will create small eigenvalues but to which the upper bounds (1.53) and (1.56) were blind. For instance, domains with long (and not necessarily thin) tubes will have low eigenvalues without having a large isoperimetric ratio. See figure 1.7 for an example of this.

The power $k^{1+\frac{2}{d}}$ is probably not optimal, since it is larger by a k factor from the Weyl asymptotic estimate. This indicates that geometric features that produce a large diameter without affecting too much the isoperimetric ratio, as in figure 1.7, will most affect the low frequencies[12] eigenvalues, and for high frequencies the isodiametric control (1.70) is much larger than the eigenvalues.

A few natural questions arise from this theorem.

1. This result is valid for domains of the Euclidean space. Is it possible to generalize to domains of, say, the hyperbolic space $\mathbb{H}^d$? Or to domains on more general manifolds ?
2. The notion of diameter used is that of *extrinsic* diameter, so inequality (1.70) says nothing of "curled-up" domains which will have a large *intrinsic* diameter[13], but a small extrinsic diameter (see figure 1.8 for an example of this). Is it possible to replace the extrinsic diameter with the intrinsic diameter in inequality (1.70) ?

[12] That is for small k.

[13] The intrinsic diameter is the diameter of Ω for the geodesic distance d : $d(x,y) = \inf_\gamma \mathrm{length}(\gamma)$ where γ ranges over all C^1 paths $\gamma : [0,1] \to \Omega$ linking x to y.

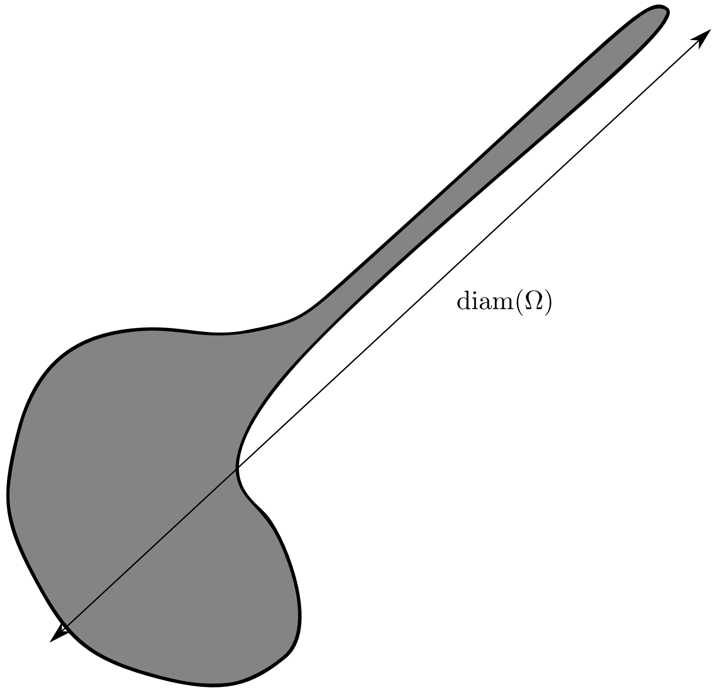

Figure 1.7 – Large diameters produce small eigenvalues without necessarily affecting the isoperimetric ratio.

Both these questions will be addressed in the last two chapters of this document.

Before moving onwards, we investigate whether there are similar results In other settings.

Steklov Problem on Manifolds

Interestingly enough, it turns out there is no isodiametric control for the Steklov eigenvalues on manifolds that aren't Euclidean domains. We display an example of a submanifold of $\mathbb{R}^d$ which has a arbitrarily large diameter but whose Steklov eigenvalues are not small.

Let $\Sigma \subset \mathbb{R}^d$ be an elongated "bag" (as displayed in figure 1.9) with Steklov boundary conditions on the boundary. Let $u_0, u_1, ...$ be an $L^2(\Sigma)$-orthonormal basis for the Steklov problem. Then,

$$\sigma_k(\Sigma) = \min \left\{ \mathcal{R}(u) \middle| \quad u \perp \{u_0, ..., u_{k-2}\} \right\}. \tag{1.71}$$

We now take a part $\Sigma' \subset \Sigma$ containing the whole of $\partial\Sigma$ (again, as in figure 1.9) and impose Neumann boundary conditions on the part of the boundary $\partial\Sigma'$ which does not intersect $\partial\Sigma$. Let $v_0, v_1, ...$ an $L^2(\Sigma')$-orthonormal for this mixed problem and $0 = \lambda_0(\Sigma') < \lambda_1(\Sigma') \leq \cdots$ the corresponding eigenvalues.

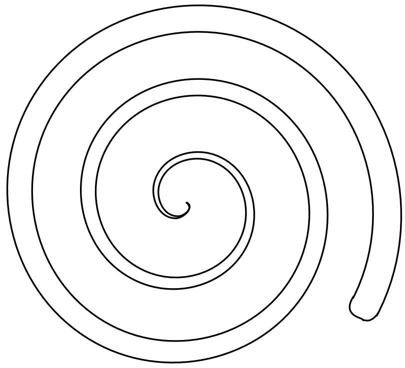

Figure 1.8 – Some domains have a large intrinsic diameter but a small extrinsic diameter.

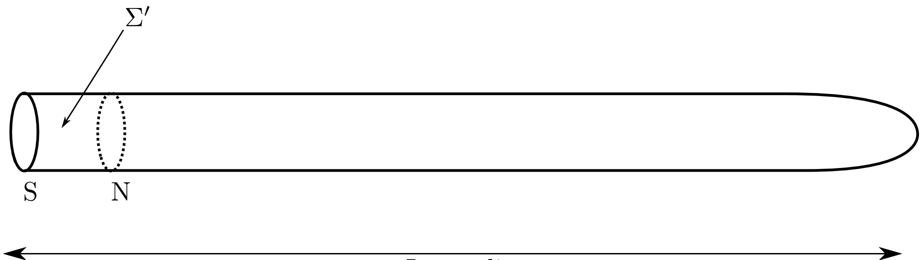

Figure 1.9 – An elongated "bag" has a large diameter but the Steklov eigenvalues aren't small. On the boundary $\partial\Sigma$ of the manifold, we impose Steklov boundary conditions, while on N we impose Neumann boundary conditions, thus considering a mixed problem on Σ'.

Now consider the linear map

$$\Phi : \begin{array}{c} \mathrm{Vect}\,\{v_0, ..., v_{k-2}\} \longrightarrow \mathbb{R}^{k-2} \\ v \longmapsto \left(\langle v|u_i\rangle_{L^2(\Sigma')}\right)_{i\leq k-2} \end{array} . \tag{1.72}$$

The linear map Φ necessarily has a kernel of dimension $\dim\ker(\Phi) \geq 1$, so that there is a $v \in \mathrm{Vect}\{v_0, ..., v_{k-2}\}$ such that $v \perp \mathrm{Vect}\{u_0, ..., u_{k-2}\}$ in $L^2(\Sigma')$, which we prolong by zero to all of Σ. Using first the min-max principle (with the fact that $v \perp 1$), and then (1.71), we have

$$0 < \lambda_1(\Sigma') \leq \mathcal{R}_{\Sigma'}(v) = \frac{\int_{\Sigma'} |\nabla v|^2}{\int_{\partial\Sigma} v^2} \leq \frac{\int_{\Sigma} |\nabla v|^2}{\int_{\partial\Sigma} v^2} = \sigma_k(\Sigma). \tag{1.73}$$

This lower bound is independent of the length of the diameter of the surface Σ so that there is no hope of there being an isodiametric inequality as in [2].

The reason for the breakdown of the isodiametric inequality in the case of such manifolds Σ seems to be that although Σ has a large diameter, the boundary $\partial\Sigma$ does not. We could legitimately wonder wether some inequality of the form

$$\sigma_k(\Omega) \leq C(d) \frac{k^\alpha}{\operatorname{diam}(\partial\Sigma)^\beta}, \qquad \alpha, \beta > 0, \tag{1.74}$$

holds, where the diameter $\operatorname{diam}(\partial\Sigma)$ is taken for the geodesic distance on Σ.

This whole question is avoided in the case of Euclidean domains since

$$\forall \Omega \subset \mathbb{R}^d, \qquad \operatorname{diam}(\Omega) = \operatorname{diam}(\partial\Omega). \tag{1.75}$$

Dirichlet Spectrum

As we saw, the Faber-Krahn inequality (1.9) implies that there can be no equivalent of theorem 1.17 for the Dirichlet spectrum. In fact, we can construct domains such that $\lambda_1 \to +\infty$ as the diameter becomes large.

Take, for $a > 0$, a rectangle thin of unit area $\Omega_a = [0, a] \times \left[0, \frac{1}{a}\right] \subset \mathbb{R}^2$ and of large diameter $\operatorname{diam}(\Omega_a) \geq a$ (see figure 1.10). Then the Dirichlet problem

$$\begin{cases} \Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.76}$$

can be explicitly solved giving eigenvalues of the form

$$\lambda_{ij} = \pi^2 \left(a^2 i^2 + \frac{j^2}{a^2} \right), \qquad \text{for } i, j \geq 1. \tag{1.77}$$

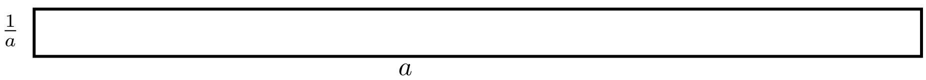

Figure 1.10 – A domain with large diameter and large Dirichlet eigenvalues.

The lowest eigenvalue is then (for sufficiently large a) :

$$\lambda_1(\Omega_a) = \pi^2 \left(a^2 + \frac{1}{a^2} \right) \underset{a \to +\infty}{\longrightarrow} +\infty. \tag{1.78}$$

The closed problem

Theorem 1.17 is also false when considering the closed problem, but the proof is much more subtle. It relies on the existence of *lower bounds* for the eigenvalues. Since the min-max principle gives the eigenvalues as an infimum, finding lower bounds for the eigenvalues is a lot less obvious than finding upper bounds.

One of the ways to find lower bounds is to avoid situations as described in section 1.1.2 where heat transfers take time because there is a thin passage connecting two *disjoint* parts of a domain or manifold. In order to express this in a rigorous way, we define the *Cheeger constant* of a manifold.

Definition 1.18 (Cheeger constant)**.** Let $(\mathcal{M}, g)$ be a compact Riemannian manifold of dimension d without boundary. We define the *Cheeger Constant* $h(\mathcal{M}, g)$ as

$$h(\mathcal{M}, g) = \inf_N \left\{ J(N), J(N) = \frac{V_g(N)}{\min\{V_g(M_1), V_g(M_2)\}} \right\}, \tag{1.79}$$

where N runs through all compact codimension one submanifolds dividing $\mathcal{M}$ into two disjoint connected open submanifolds M_1 and M_2 with common boundary $N = \partial M_1 = \partial M_2$.

The Cheeger constant is small when there is a way to cut $\mathcal{M}$ into two disjoint parts M_1 and M_2 separated by a thin passage N, hence making the heat transfer slow. (see figure 1.11)

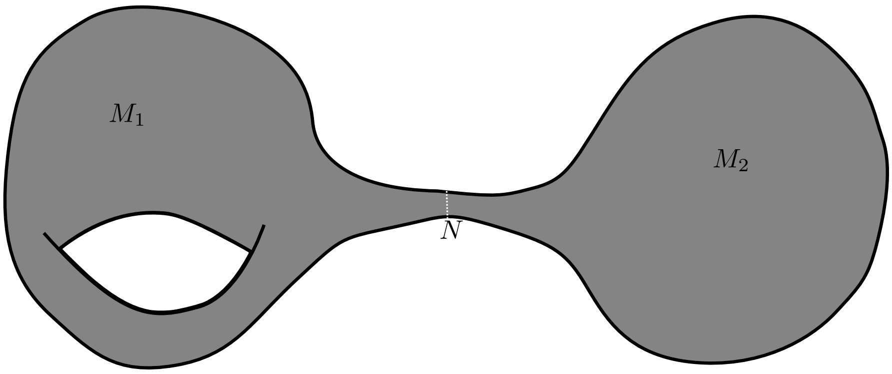

Figure 1.11 – A manifold with a small Cheeger constant has low eigenvalues.

Theorem 1.19 (Cheeger's inequality, 1978)**.** *The first nonzero eigenvalue* $\lambda_1(\mathcal{M}, g)$ *for the closed problem* $\Delta_g u = \lambda u$ *satisfies the following inequality :*

$$\lambda_1(\mathcal{M}, g) \geq \frac{1}{4} h(\mathcal{M}, g)^2. \tag{1.80}$$

In the light of this theorem, we are ready to construct a manifold with a large diameter *and* a Cheeger constant that does not tend to zero (see [9] for a similar example). Let $R \geq 1$ and $\chi_R : \mathbb{R} \to \mathbb{R}^+$ be a function satisfying the following conditions :

1. The function χ_R is supported in $[0, R]$.
2. For $x \geq 1$, the function χ_R is exponential : $\chi_R(x) = e^{-x+1}$.
3. The function χ_R has mass 1 on the interval $[0, 1]$:

$$\int_0^1 \chi_R(x) \mathrm{d}x = 1 \tag{1.81}$$

4. The function χ_R is C^∞ smooth on $]0, R[$.

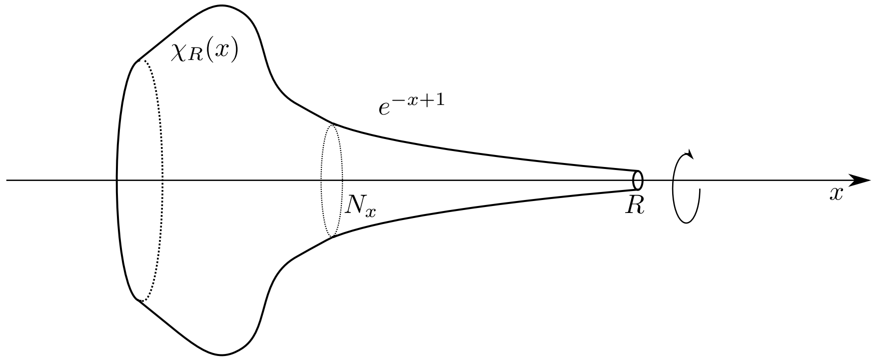

Figure 1.12 – A representation of the manifold $\mathcal{M}_R$.

Let us then consider $\Omega_R \subset \mathbb{R}^d$ the revolution surface based on the curve of χ, which we close at its extremities by a planar section (see figure 1.12). We call $\mathcal{M}_R$ the resulting manifold.

To compute the Cheeger constant of $\mathcal{M}_R$, we use the revolution symmetry of the surface : it suffices to consider in the infimum (1.79) submanifolds N which are also rotation invariant (see [1] proposition 6.5). We can index all such N by there x coordinate N_x, so we simply have to compute

$$h(\mathcal{M}_R) = \inf_{0 \leq x \leq R} J(N_x). \tag{1.82}$$

Because of the exponential term, the total surface $S(\mathcal{M}_R)$ of the manifold gets no larger than some constant $\limsup_R \mathcal{M}_R \leq \text{Cte}$ so that, as x becomes large, $J(N_x)$ is given by

$$J(N_x) = \frac{V_g(N_x)}{V_g(M_x)}, \tag{1.83}$$

where M_x is the part of $\mathcal{M}_R$ on the right hand of N_x. A direct computation then gives $\inf_{x \leq R} J(N_x) > 0$. We refer to [9] for more details on a similar example.

Finally, we mention an upper bound linking the eigenvalues of the closed problem to the diameter of the manifold.

Theorem 1.20 (Cheng, 1975)**.** *Let $(\mathcal{M}, g)$ be a compact d-dimensional Riemannian manifold without boundary. Let $a \geq 0$ such that $Ricc_g \geq -(d-1)a$. Then,*

$$\lambda_k(\mathcal{M}, g) \leq \frac{1}{4}(d-1)^2 a^2 + C(d)\frac{k}{diam(\mathcal{M}, g)^2}, \tag{1.84}$$

where $C(d)$ is a constant depending only on the dimension.

Note that when $(\mathcal{M}, g)$ is of positive curvature, we then have an upper bound involving the diameter of $\mathcal{M}$:

$$\lambda_k(\mathcal{M}, g) \leq C(d)\frac{k}{\text{diam}(\mathcal{M}, g)^2}. \tag{1.85}$$

This can be applied for example to the boundary of convex Euclidean domains.

Chapter 2

Isodiametric Inequality for the Spectrum

I do not think the forest would be so bright, nor the water so warm, nor love so sweet, if there were no danger in the lakes.[1]

In this chapter, we prove the isodiametric inequality for the Steklov eigenvalues.

Theorem 2.1 (B. Bogosel, D. Bucur, A. Giacomini, 2017)**.** *Let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz set. Suppose moreover that Ω is connected. The Steklov eigenvalues then satisfy*

$$\sigma_k(\Omega) \leq C(d)\frac{k^{1+2/d}}{diam(\Omega)}, \tag{2.1}$$

where $C(d)$ is a constant depending only on the dimension of the space.

As often, the geometry of the domain $\Omega \subset \mathbb{R}^d$ is used to construct disjointly supported test functions whose Rayleigh quotient provide an upper bound for the spectrum. This is lemma 2.2 which is proved in the first part of the chapter. The second and last part is dedicated to the proof of theorem 2.1.

Suppose Ω connected and let x_0 and y_0 be extremal points[2] of Ω. Since Ω is connected, each annulus

$$A_{x_0}(r, r+l) = \{x \in \mathbb{R}^d \quad |r < |x - x_0| < r + l\} \tag{2.2}$$

intersects Ω on a non-empty set as long as $r + l \leq \mathrm{diam}(\Omega)$. The key part of the proof, lemma 2.2, is to construct on each sector $\Omega \cap A_{x_0}(r, r+l)$ a test function ϕ such that the Rayleigh quotient

$$\mathcal{R}(\phi) = \frac{\int_\Omega |\nabla\phi|^2 \mathrm{d}x}{\oint_{\partial\Omega} \phi^2 \mathrm{d}S} \tag{2.3}$$

is not too great (see figure 2.1). How great precisely will depend on the width of the annulus.

[1] C. S. Lewis. *Out of the Silent Planet.*

[2] Points such that $|x_0 - y_0| = \mathrm{diam}(\Omega)$.

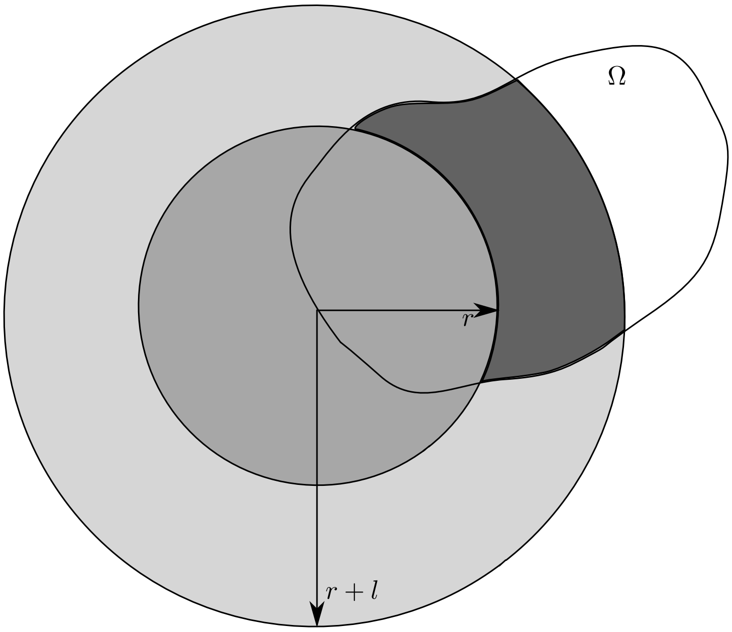

Figure 2.1 – The test function ϕ associated with the annulus $A(r, r+l)$ is supported in the darker area.

If the diameter diam(Ω) is large enough, a sufficient number of such disjoint annuli exist so that it is possible to construct the desired disjointly supported family of test functions. This is described in the second section of the chapter.

2.1 Construction of the Test Functions

We prove the following result.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded open Lipschitz set. Suppose moreover that Ω is connected.*

Let $m, \lambda > 0$. There exists a minimal width $L = L(m, \lambda, d)$ such that for every annulus[3] *$A = A(r, r+l)$ of width $l \geq L$ satisfying $0 < M := |A \cap \Omega| \leq m$, there exists a function $\phi \in H_0^1(A)$ with $\phi \neq 0$ in $L^2(\partial\Omega)$ satisfying*

$$\mathcal{R}(\phi) = \frac{\int_\Omega |\nabla \phi|^2 \, dx}{\oint_{\partial\Omega} \phi^2 \, dS} \leq \lambda. \tag{2.4}$$

Proof. Let $L > 0$ to be determined later and let $A = A(r, r+l)$ be an annulus of width $l \geq L$. Let us assume that there is no such test function as described in the lemma and infer a contradiction. Then, for all suitable ϕ, the Rayleigh quotient is bounded from below $\mathcal{R}(\phi) > \lambda$. Our goal will be to find an intermediate annulus $A(r+t, r+l-t) \subset A$ that splits Ω into two parts $\Omega \subset {}^c A(r+t, r+l-t)$,

[3]From now on, we omit writing the center x_0 of the annulus $A_{x_0}(r_1, r_2)$, which will stay the same throughout all the proof.

which is absurd since Ω is connected.

Set, for every $0 \le t < \frac{1}{2}l$,

$$\phi(x) = \min\left\{1, \frac{1}{t}\mathrm{d}\left(x, {}^cA(r, r+l)\right)\right\} \in H_0^1(A) \tag{2.5}$$

where $d(x, {}^cA)$ is the distance of x from the set ${}^cA$ (see figure 2.2).

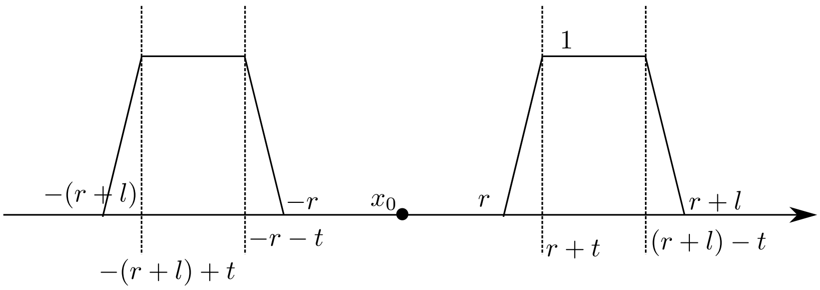

Figure 2.2 – The test function ϕ as a function of a coordinate on an axis passing through x_0.

On the one hand, the gradient $\nabla\phi$ is supported in the set $\Omega\cap(A(r, r+t) \cup A(r+l-t, r+l))$ and is worth $\nabla\phi(x) = \frac{x-x_0}{t|x-x_0|}$ on this set, so that

$$\int_\Omega |\nabla\phi|^2 \mathrm{d}x \le \frac{1}{t^2}m(t), \tag{2.6}$$

where $m(t)$ is the measure of the set $\Omega \cap (A(r, r+t) \cup A(r+l-t, r+l))$.

On the other hand, ϕ is equal to 1 on the set $\Omega \cap A(r+t, r+l-t)$ so that

$$\oint_{\partial\Omega} \phi^2 \mathrm{d}S \ge p(t), \tag{2.7}$$

where $p(t)$ is the measure of the set $\partial\Omega \cap \bar{A}(r+t, r+l-t)$. Note that since Ω is connected and $\Omega \cap A(r, r+l) \ne \emptyset$, we have $p(t) > 0$ for all $0 \le t < \frac{1}{2}l$. Figure 2.3 gives a representation of the situation.

These two inequalities and the initial assumption yield

$$\forall\ 0 \le t < \frac{1}{2}l, \qquad \lambda < \mathcal{R}(\phi) \le \frac{m(t)}{t^2 p(t)}. \tag{2.8}$$

In order to make use of this last inequality, it is necessary to link the measure $m(t)$ and the perimeter $p(t)$. This can be done using the following *isoperimetric inequality.*

Lemma 2.3 (Relative Isoperimetric Inequality in Annuli)**.** *Let $m > 0$ and $\Omega \subset \mathbb{R}^d$ be an open Lipschitz domain. There exists a minimal width $w = w(m, d) > 0$ and a constant $c = c(d)$ such that for every annulus $A = A(r, r+l)$ of width $l \ge w$ such that $|A \cap \Omega| \le m$ we have*

$$c|A \cap \Omega|^{d-1/d} \le |\partial\Omega \cap \bar{A}|. \tag{2.9}$$

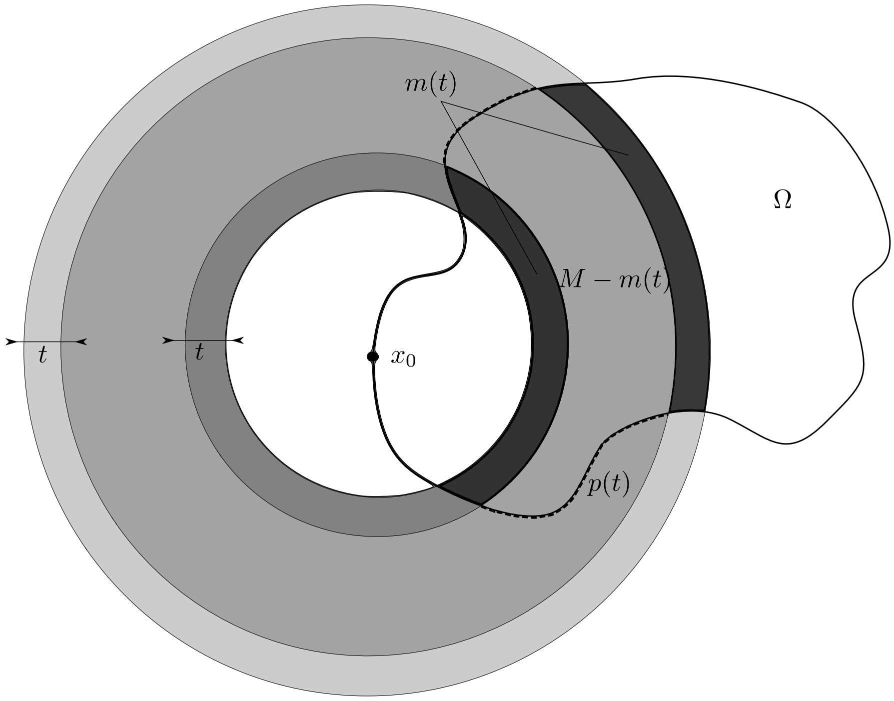

Figure 2.3 – Notations used when computing the Rayleigh quotient of the test function ϕ, $m(t)$ is the darker shaded area and $p(t)$ is the dotted part of the perimeter $\partial\Omega$.

The proof of this last lemma is long and difficult. It shall be postponed until the next chapter. In the meantime, we may note that this relative isoperimetric inequality is the key point of the proof. Any attempt to generalize the argument, to, say, domains of the hyperbolic space, is bound to contain a version of this lemma.

The relative isoperimetric inequality applied to m and to the ring $A(r+t, r+l-t)$ gives[4]

$$c(d)(M - m(t))^{d-1/d} \leq p(t) \tag{2.10}$$

as long as the width of the annulus $A(r+t, r+l-t)$ is sufficient :

$$(r + l - t) - (r + t) = l - 2t \geq w. \tag{2.11}$$

We therefore require $l \geq L > w$.

Combining both inequalities (2.8) and (2.10) gives a new inequality

$$c\lambda t^2 (M - m(t))^{d-1/d} \leq m(t). \tag{2.12}$$

This inequality is not sufficient achieve our goal of finding an intermediate annulus $A(r+t, r+l-t)$ splitting Ω into two parts. Nevertheless, a good choice of t remains possible.

[4]Recall that $M = |\Omega \cap A|$.

Claim : there exists $0 < t_1 < \frac{1}{2}(l-w)$ such that $m(t_1) = \frac{1}{2}M$. Note that the existence of a $t_1 < \frac{1}{2}l$ such that $m(t_1) = \frac{1}{2}M$ is trivial. However, we require t_1 to satisfy the stronger condition $t_1 < \frac{1}{2}(l-w)$ so that the intermediate annulus $A_1 = A(r+t_1, r+l-t_1)$ is of sufficient width to apply the isoperimetric inequality in A_1.

Since m is a continuous function and $m(t=0)=0$, if there were no such t_1 then we would have $m(t) < \frac{1}{2}M$ for all $0 \le t \le \frac{1}{2}(l-w)$, yielding

$$\frac{1}{2}M > m(t) \ge \lambda c t^2 (M - m(t))^{d-1/d} > \lambda c t^2 \left(\frac{M}{2}\right)^{d-1/d}. \tag{2.13}$$

As this holds for all $t < \frac{l-w}{2}$, we have

$$\frac{1}{2}M \ge \lambda c \left(\frac{l-w}{2}\right)^2 \left(\frac{M}{2}\right)^{d-1/d}, \tag{2.14}$$

or equivalently,

$$\frac{l-w}{2} \le \frac{M^{1/2d}}{\sqrt{\lambda c}} 2^{-1/2d}. \tag{2.15}$$

In order for this never to happen, we require

$$\frac{l-w}{2} \ge \frac{L-w}{2} > \frac{M^{1/2d}}{\sqrt{\lambda c}} 2^{-1/2d}. \tag{2.16}$$

Moreover, (2.12) allows us to bound t_1 from above :

$$t_1 \le \frac{M^{1/2d}}{\sqrt{\lambda c}} 2^{-1/2d}. \tag{2.17}$$

Claim : this process can be repeated by replacing the ring $A(r, r+l)$ by the middle ring $A_1 := A(r+t_1, r+l-t_1)$.

The ring A_1 is of width $l - 2t_1 \ge l - 2\frac{l-w}{2} \ge w$ so that the relative isoperimetric inequality contained in lemma 2.3 still applies in A_1. Therefore, if $0 \le t < \frac{1}{2}(l-w) - t_1$, the test function

$$\phi(x) = \min\left\{1, \frac{1}{t}\mathrm{d}\left(x, {}^cA_1\right)\right\} \in H_0^1(A_1) \subset H_0^1(A) \tag{2.18}$$

has Rayleigh quotient $\mathcal{R}(\phi) > \lambda$ and hence gives a new version of inequality (2.12),

$$m_1(t) \ge \lambda c t^2 \left(\frac{1}{2}M - m_1(t)\right)^{d-1/d} \tag{2.19}$$

where $m_1(t) = |\Omega \cap {}^cA(r+t_1+t, r+l-t_1-t)|$. We then try to find $0 \le t_2 < \frac{1}{2}(l-w) - t_1$ such that $m_1(t_2) = \frac{1}{4}M$. In the same way as before, if there were no such t_2, then we would have $m(t) < \frac{1}{4}M$ for all $t < \frac{1}{2}(l-w) - t_1$. Hence

$$\frac{l-w}{2} - t_1 < \frac{M^{1/2d}}{\sqrt{\lambda c}} 2^{-2/2d}. \tag{2.20}$$

In order for this never to happen, we require a stronger lower bound on L, namely

$$\frac{L-w}{2} > \frac{M^{1/2d}}{\sqrt{\lambda c}}\left(2^{-1/2d}+2^{-2/2d}\right) \geq \frac{M^{1/2d}}{\sqrt{\lambda c}}2^{-2/2d}+t_1. \tag{2.21}$$

Then (2.19) gives the following upper bound for t_2 :

$$t_1+t_2 \leq \frac{M^{1/2d}}{\sqrt{\lambda c}}\left(2^{-1/2d}+2^{-2/2d}\right) < \frac{l-w}{2}. \tag{2.22}$$

Induction : we repeat this process *ad infinitum*.

We require L to satisfy

$$\frac{L-w}{2} > \frac{M^{1/2d}}{\sqrt{\lambda c}}\sum_{k=0}^{\infty} 2^{-k/2d} \tag{2.23}$$

in order to construct $t_1, t_2, t_3, \ldots$ so that

$$|\Omega \cap A(r+t_1+\cdots+t_k, r+l-t_1-\cdots-t_k)| = \frac{M}{2^k} \tag{2.24}$$

and

$$t_1+\cdots+t_k < \frac{M^{1/2d}}{\sqrt{\lambda c}}\sum_{k=0}^{\infty} 2^{-k/2d} < \frac{L-w}{2} \leq \frac{l-w}{2} \tag{2.25}$$

Set $t_\infty = t_1+t_2+\cdots \leq \frac{1}{2}(l-w)$. Then

$$\Omega \cap A(r+t_\infty, r+l-t_\infty) = \emptyset \tag{2.26}$$

which is absurd, since Ω is connected. □

2.2 Proof of the Isodiametric Inequality for Steklov Spectrum

We recall the isodiametric inequality.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded Lipschitz set. Suppose moreover that Ω is connected. The Steklov eigenvalues then satisfy*

$$\sigma_k(\Omega) \leq C(d)\frac{k^{1+2/d}}{diam(\Omega)}, \tag{2.27}$$

where $C(d)$ is a constant depending only on the dimension of the space.

Proof. Let $k \geq 1$ be a fixed integer.

First note that the quantity $\sigma_k(\Omega)\,\mathrm{diam}(\Omega)$ is scale invariant :

$$\forall t>0, \qquad \sigma_k(t\Omega)\,\mathrm{diam}(t\Omega) = \sigma_k(\Omega)\,\mathrm{diam}(\Omega). \tag{2.28}$$

We may therefore suppose that $|\Omega| = 1$. Then, setting $m = |\Omega| = 1$ and $\lambda = 1$, lemma 2.2 provides a minimal width $L = L(d, m = 1, \lambda = 1) = L(d)$ such that test functions ϕ with bounded Rayleigh quotient $\mathcal{R}(\phi) \leq 1$ exist on any annuli of width $l \geq L$ intersecting Ω.

Let $x_0, y_0 \in \Omega$ be extremal points $|x_0 - y_0| = \text{diam}(\Omega)$ and set, for $i \geq 0$,

$$A_i = A_{x_0}(iL, (i+1)L) \tag{2.29}$$

If a sufficient number (namely $k+1$) of annuli A_i intersect Ω, then it shall be possible to construct $k+1$ disjointly supported test functions and obtain a control of $\sigma_k(\Omega)$. If not, we use the following isoperimetric control of the Steklov spectrum which is a direct immediate of theorem 1.14 :

Theorem 2.5 (B. Colbois, A. El Soufi, A. Girouard, 2011)**.** *The set Ω is the same as before. The following inequality holds for the Steklov eigenvalues :*

$$\sigma_k(\Omega) \leq C(d) \frac{k^{2/d}}{|\partial\Omega|^{1/d-1}}, \tag{2.30}$$

where $C(d)$ is a contant depending only on d.

We therefore consider two alternatives.

First case : $(k+1)L \geq \text{diam}(\Omega)$. Then

$$\sigma_k(\Omega)\,\text{diam}(\Omega) \leq \sigma_k(\Omega)(k+1)L = \sigma_k(\Omega)(k+1)L|\Omega|^{1/d} \tag{2.31}$$

The isoperimetric inequality $|\Omega|^{(d-1)/d} \leq C(d)|\partial\Omega|$ in the Euclidean space $\mathbb{R}^d$ gives[5]

$$\sigma_k(\Omega)\,\text{diam}(\Omega) \leq C(d)\sigma_k(\Omega)(k+1)L|\partial\Omega|^{\frac{1}{d-1}} \tag{2.32}$$

for some constant $C(d)$, so that the isoperimetric control of the spectrum (2.30) allows us to conclude : for some (other) constant $C(d)$,

$$\sigma_k(\Omega)\,\text{diam}(\Omega) \leq C(d)Lk^{2/d}(k+1) \leq C(d)k^{\frac{2}{d}+1}. \tag{2.33}$$

Second case : $(k+1)L < \text{diam}(\Omega)$. By applying an isometry if needed, we can suppose that $x_0 = 0$, thus making x_0 invariant under dilatations. Let $0 < t < 1$ such that $\text{diam}(t\Omega) = (k+1)L$. Then the $k+1$ annuli A_i defined in (2.29) all intersect $t\Omega$ and have width L. Lemma 2.2 provides $k+1$ test functions $\phi_i \in H_0^1(A_i)$ such that

$$\forall\ 0 \leq i \leq k, \qquad \mathcal{R}(\phi_i) = \frac{\int_{t\Omega} |\nabla\phi_i|^2 \text{d}x}{\oint_{\partial(t\Omega)} \phi_i^2 \text{d}S} \leq 1 \tag{2.34}$$

Taking the space $E = \text{Vect}\{\phi_0, ..., \phi_k\}$ generated by the disjointly supported (and therefore orthogonal) test functions ϕ_i in the min-max principle gives the upper bound

$$\sigma_k(t\Omega) = \frac{1}{t}\sigma_k(\Omega) \leq 1 \tag{2.35}$$

[5]Isoperimetric inequalities will be the main topic of the next chapter.

and finally, as $t = L\frac{k+1}{\text{diam}(\Omega)}$,

$$\sigma_k(\Omega) \leq L(d)\frac{k+1}{\text{diam}(\Omega)} \leq C(d)\frac{k^{\frac{2}{d}+1}}{\text{diam}(\Omega)}. \tag{2.36}$$

□

2.3 Optimal Shapes for the Steklov Eigenvalues

Before closing this chapter, let us say a word about the context in which theorem 2.1 was first sought then discovered. The goal of B. Bogosel, D. Bucur and A. Giacomini in their article is to show the existence of optimal shapes for the Steklov eigenvalues : for each k, finding a (or several) domain(s) $\Omega \subset \mathbb{R}^d$ which have unit volume and maximize $\sigma_k(\Omega)$.

The shapes Ω are searched in the range of sets of finite perimeter $|\partial\Omega| < +\infty$. This is an important point since theorem 2.5 shows that any maximizing sequence of shapes

$$\sigma_k(\Omega_n) \underset{n\to+\infty}{\longrightarrow} \sup_{|\partial\Omega|<+\infty} \sigma_k(\Omega) \tag{2.37}$$

cannot have a too large perimeter $|\partial\Omega|$ so that a "limit shape" Ω_∞ is also expected to have a finite perimeter. To show the existence of a limit shape, it remains "only" to find some form of compactness satisfied by the sequence $(\Omega_n)_n$. To do this, one more element is required : the isodiametric inequality which guaranties that the Ω_n will stay inside a fixed ball.

The result is the existence of sets Ω_∞ which maximize the eigenvalues in a "relaxed" sense, which allows to define the σ_k for highly unregular sets, but that are not necessarily sufficiently regular for the Steklov problem to be defined on Ω_∞.

It remains unknown (June 2017) if there are maximizing shapes that are regular enough for the Steklov problem to actually be defined on them. However, numerical computations for $d = 2$ displayed in [2] seem to indicate that the optimal shapes are smooth.

Chapter 3

Relative Isoperimetric Inequality in Annuli

Now for wrath, now for ruin and a red nightfall ![1]

As seen in the previous chapter, geometric inequalities lie at the heart of the proof of theorem 2.1 in the form of the relative isoperimetric inequality of lemma 2.2. In this chapter, we discuss at first "classical" isoperimetric inequalities, which are much weaker than that provided by lemma 2.2, before doing the same for "relative" isoperimetric inequalities.

The whole chapter is exclusively set in Euclidean geometry : we shall consider only domains Ω of $\mathbb{R}^d$. Attempts to generalize to hyperbolic geometry will be described in the next chapter.

3.1 Classical Isoperimetric Inequality

3.1.1 Definition

Let Ω be an bounded Lipschitz domain of $\mathbb{R}^d$ so that its volume $|\Omega|$ and perimeter $|\partial\Omega|$ are well defined (and finite). The domain Ω is said to have the *isoperimetric property* if any other bounded Lipschitz domain $E \subset \mathbb{R}^d$ of same volume has a larger perimeter. In scale invariant terms, for all bounded Lipschitz domain $E \subset \mathbb{R}^d$, regardless of its volume $|E|$,

$$\left(\frac{|E|}{|\Omega|}\right)^{(d-1)/d} \leq \frac{|\partial E|}{|\partial\Omega|} \tag{3.1}$$

The existence of such a domain Ω is of course far from being a trivial fact. It can be shown that of all regular domains, only the balls possess this isoperimetric property (see [5]). The consequence of this that all bounded Lipschitz domains satisfy an *isoperimetric inequality* :

$$|E|^{(d-1)/d} \leq C(d)|\partial E| \tag{3.2}$$

[1] J. R. R. Tolkien. *The Return of the King.*

where the constant $C(d)$ is given as a function of the geometry of the ball $B_d \subset \mathbb{R}^d$, which possesses the isoperimetric property: $C(d) = \frac{|B_d|^{(d-1)/d}}{|\partial B_d|}$.

However, in all that follows, we will not be concerned with the *value* of the optimal constant $C(d)$, but merely by its *existence*. This is not because the precise value of the optimal constant $C(d)$ is not relevant at all : as hinted by the Faber-Krahn inequality (1.9), this value is linked to the first Dirichlet eigenvalue $\lambda_1(\Omega)$, amongst other geometric quantities. Rather, it is because most of the inequalities used in the previous chapter are not expected to be optimal, and, furthermore, the proof therein produced has no reason to give an optimal inequality.

Most of the material of this chapter can be found in [8]. We now prove the isoperimetric inequality (3.2).

3.1.2 The Gagliardo-Nirenberg-Sobolev Inequality

One of the ways that leads to the isoperimetric inequality (3.2) is to use functional inequalities applied to characteristic functions of the domain studied. This will be described with detail in this section. In all that follows, Ω is an open bounded Lipschitz domain of $\mathbb{R}^d$.

The functional inequality we use is the following : ([8], theorem 1 p. 138)

Theorem 3.1 (Gagliardo-Nirenberg-Sobolev inequality)**.** *Let $d \geq 1$ be the dimension of the space $\mathbb{R}^d$ and $1 \leq p < d$. Define the Sobolev conjugate p^* of p by*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}. \tag{3.3}$$

Then there is a constant $C(d)$ such that for all function $u \in W^{1,p}(\mathbb{R}^d)$,

$$\left(\int_{\mathbb{R}^d} |u|^{p^*} \right)^{1/p^*} \leq C(d) \left(\int_{\mathbb{R}^d} |\nabla u|^p \right)^{1/p}. \tag{3.4}$$

We now take $p = 1$ and $p^* = \frac{d}{d-1}$. As explained above, we would like to take $u = \mathbb{1}_\Omega$ in this inequality. Unfortunately, this is not possible since the derivative $\nabla \mathbb{1}_\Omega$ is not an L^1 function. In order for this to work, we must take u to be an approximation $u \approx \mathbb{1}_\Omega$.

Let $\chi : \mathbb{R} \longrightarrow \mathbb{R}$ be a C^∞-smooth function such that (see figure 3.1)

1. The function χ is supported in $\mathbb{R}_+$.
2. For $x \geq 1$, $\chi(x) = 1$.
3. The function satisfies $0 \leq \chi \leq 1$.

We now define the approximation. For $\epsilon > 0$, set

$$u_\epsilon(x) = \chi\left(\frac{1}{\epsilon} d(x, \partial\Omega) \right) \mathbb{1}_\Omega(x), \tag{3.5}$$

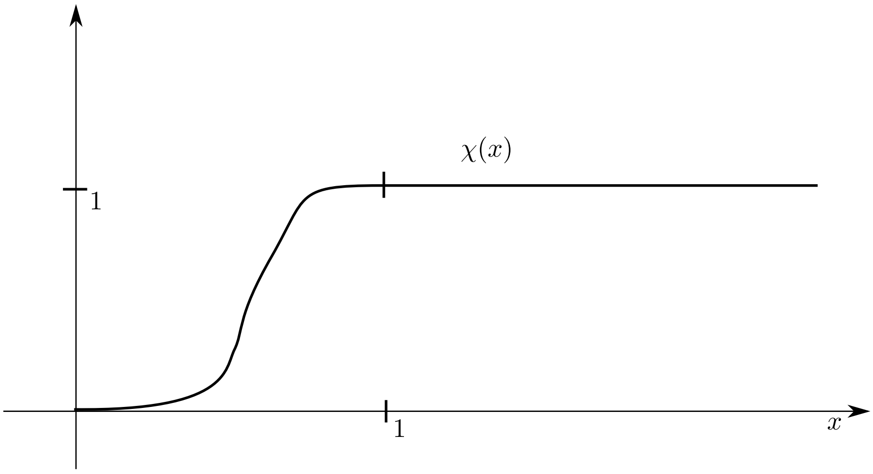

Figure 3.1 – The graph of the function χ.

where $d(x, \partial\Omega)$ is the distance of x from the boundary $\partial\Omega$:

$$d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|. \tag{3.6}$$

The resulting function u_ϵ (see figure 3.2) converges to the function $\mathbb{1}_\Omega$ in L^1. The derivative of u_ϵ is

$$\nabla u_\epsilon(x) = \frac{1}{\epsilon}\chi'\left(\frac{1}{\epsilon}d(x, \partial\Omega)\right)\nabla_x\left(d(x, \partial\Omega)\right). \tag{3.7}$$

In order to apply theorem 3.1 to u_ϵ, we must verify that the derivative ∇u_ϵ is as well a L^1 function. We recall that the distance from a subset $d(x, \partial\Omega)$ is a 1-Lipschitz function and use the following theorem ([8], theorem 5 p. 131).

Theorem 3.2. *Let $U \subset \mathbb{R}^d$ be open. A measurable function function $f : U \longrightarrow \mathbb{R}$ is Lipschitz if and only if $\nabla f \in L^\infty(U)$. Moreover, the Lipschitz constant of f is $\|\nabla f\|_{L^\infty}$.*

Therefore, ∇u_ϵ is supported in the set $\{x \in \mathbb{R}^d \mid 0 \leq d(x, \partial\Omega) \leq \epsilon\}$ and is of norm

$$|\nabla u_\epsilon(x)| \leq \frac{1}{\epsilon}\|\chi'\|_{L^\infty}, \tag{3.8}$$

so that $u_\epsilon \in W^{1,1}(\mathbb{R}^d)$. We can then apply the Gagliardo-Nirenberg-Sobolev (GNS) inequality (3.4) to u_ϵ. We then have

$$\left(\int_{\mathbb{R}^d} |u_\epsilon|^{d/(d-1)}\right)^{(d-1)/d} \leq C(d)\int_{\mathbb{R}^d} |\nabla u_\epsilon| \tag{3.9}$$

One the one hand, the first integral converges to the volume (with the appropriate power) :

$$\left(\int_{\mathbb{R}^d} |u_\epsilon|^{d/(d-1)}\right)^{(d-1)/d} \underset{\epsilon\to 0^+}{\longrightarrow} \left(\int_{\mathbb{R}^d} \mathbb{1}_\Omega^{d/(d-1)}\right)^{(d-1)/d} = |\Omega|^{(d-1)/d}. \tag{3.10}$$

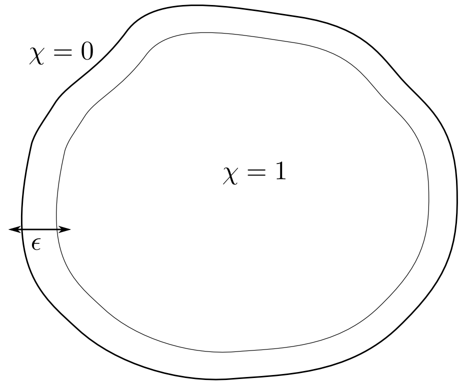

Figure 3.2 – The approximation function u_ϵ.

On the other hand, since the boundary $\partial\Omega$ has Lipschitz regularity, the second integral satisfies

$$\int_{\mathbb{R}^d} |\nabla u_\epsilon(x)|\mathrm{d}x \leq \|\chi'\|_{L^\infty} \frac{1}{\epsilon} \int_{0\leq d(x,\partial\Omega)\leq\epsilon} \mathrm{d}x \underset{\epsilon\to 0^+}{\longrightarrow} |\partial\Omega|, \tag{3.11}$$

and the combination of these two limits gives the desired inequality :

$$|\Omega|^{(d-1)/d} \leq C(d)|\partial\Omega|. \tag{3.12}$$

3.2 Relative Inequalities

In the previous section, he have dealt with the "classical" isoperimetric inequality : a control of the volume by the whole of the perimeter. This is very different from the *relative* control achieved by lemma 2.3, which we recall.

Lemma 3.3 (Relative Isoperimetric Inequality in Annuli)**.** *Let $m > 0$ and $\Omega \subset \mathbb{R}^d$ be an open Lipschitz domain. There exists a minimal width $w = w(m, d) > 0$ and a constant $c = c(d)$ such that for every annulus $A = A(r, r + l)$ of width $l \geq w$ such that $|A \cap \Omega| \leq m$ we have*

$$c|A \cap \Omega|^{d-1/d} \leq |\partial\Omega \cap \bar{A}|. \tag{3.13}$$

As we see here, the volume of the subset $A\cap\Omega$ is controlled by only *a part* of its perimeter $\partial(A\cap\Omega)$: the part $\partial\Omega\cap\bar{A}$ which is as well a part of the perimeter of Ω (see figure 3.3).

Remark 3.4. This relative isoperimetric inequality is slightly different of that proved and used by B. Bogosel, D. Bucur and A. Giacomini : instead of controlling the volume $|A \cap \Omega|$ by $|\partial\Omega \cap \bar{A}|$, they have the following inequality

$$c|A \cap \Omega| \leq \mathcal{P}(\Omega, A) \tag{3.14}$$

where $\mathcal{P}(\Omega, A)$ is the perimeter of Ω relatively to A :

$$\mathcal{P}(\Omega, A) = \sup\left\{\int_\Omega \mathrm{div}(\phi), \quad \phi \in C_c^\infty(A, \mathbb{R}^d), \|\phi\|_{L^\infty} \leq 1\right\}. \tag{3.15}$$

The differences between our inequality (3.58) and inequality (3.14) are twofold.

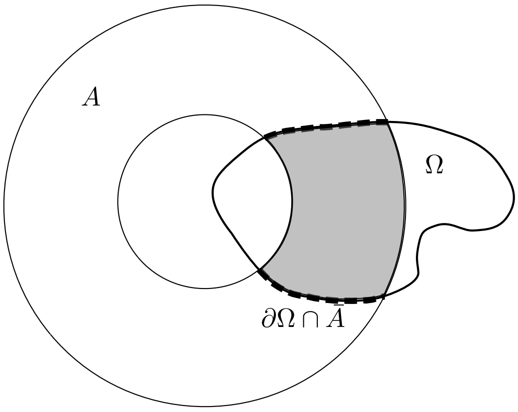

Figure 3.3 – The relative isoperimetric inequality in an annulus gives a control of the volume (the shaded area) $|\Omega \cap A|$ by only a part $|\partial\Omega \cap \bar{A}|$ of the perimeter $|\partial\Omega|$ (which is dotted).

1. First of all, inequality (3.14) is better than (3.58) since, for regular domains Ω,
$$\mathcal{P}(\Omega, A) = |\partial\Omega \cap \mathrm{Int}(A)| \leq |\partial\Omega \cap \bar{A}|. \tag{3.16}$$
Because these two quantities, $|\partial\Omega \cap \mathrm{Int}(A)|$ and $|\partial\Omega \cap \bar{(}A)|$, are similar, we will sometimes call $|\partial\Omega \cap \bar{A}|$ the *perimeter of* Ω *relative to* A.

2. Secondly, the setting used by B. Bogosel, D. Bucur and A. Giacomini is the much more general setting of functions of bounded variations $BV(\mathbb{R}^d)$ and sets of finite perimeter. The notion (3.15) of perimeter allows to define the perimeter of sets which do not have a Lipschitz boundary. While this is absolutely necessary for proving the existence of optimal shapes for the Steklov eigenvalues, it suffices to use the more restrictive setting of Lipschitz sets to prove the isodiametric inequality.

The proof of this relative inequality heavily relies on two functional inequalities : the GNS inequality (3.4) and the Poincaré inequality which we have not yet introduced. Both these functional inequalities allow to prove relative isoperimetric inequalities inside and outside a ball, these two results giving in turn lemma 3.3.

3.2.1 Relative Inequality Outside a Ball

We prove the following result.

Lemma 3.5. *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded open Lipschitz domain and* $B = B(r)$ *a ball of radius* $r > 0$. *Then,*
$$|\Omega \cap {}^cB|^{(d-1)/d} \leq C(d)|\partial\Omega \cap Adh({}^cB)|, \tag{3.17}$$
where $C(d)$ *is a constant depending only on* d.

Remark 3.6. The volume $|\Omega \cap {}^cB|$ is not controlled by all of its perimeter $|\partial(\Omega \cap {}^cB)|$ but only the perimeter of Ω *relative* to ${}^cB$, that is $|\partial\Omega \cap \mathrm{Adh}({}^cB)|$. For this reason, (3.17) is a *relative* isoperimetric inequality.

Proof. As in the proof of the classical isoperimetric inequality, we use the GNS inequality (3.4). Only this time, we do not take $u \approx \mathbb{1}_{\Omega\cap {}^cB}$ because then we would only end up with the classical isoperimetric inequality $|\Omega \cap {}^cB|^{(d-1)/d} \leq C(d)|\partial(\Omega \cap {}^cB)|$.

Step 1 : let $u \in \mathcal{D}(\mathbb{R}^d)$ (which is to be an approximation of $\mathbb{1}_\Omega$). We first prove that

$$\left(\int_{^cB} |u(x)|^{d/(d-1)}\mathrm{d}x\right)^{(d-1)/d} \leq C(d)\left[\int_{^cB} |\nabla u(x)|\mathrm{d}x + \oint_{\partial B} |u(x)|\mathrm{d}S(x)\right]. \tag{3.18}$$

Let $\chi \in C^\infty(\mathbb{R})$ be the function described in figure 3.1. Define the following approximation for $\mathbb{1}_{^cB}$:

$$f_\epsilon(x) = \chi\left(\frac{|x| - r}{\epsilon}\right) \in C^\infty. \tag{3.19}$$

Then the GNS inequality applied to uf_ϵ with $p = 1$ and $p^* = \frac{d}{d-1}$ gives :

$$\left(\int_{\mathbb{R}^d} |f_\epsilon u|^{d/(d-1)}\right)^{(d-1)/d} \leq C(d)\int_{\mathbb{R}^d} |\nabla(f_\epsilon u)| \tag{3.20}$$

$$\leq C(d)\left[\int_{\mathbb{R}^d} |u||\nabla f_\epsilon| + \int_{\mathbb{R}^d} f_\epsilon|\nabla u|\right]. \tag{3.21}$$

We look at each integral separately.

1. Since $f_\epsilon \to \mathbb{1}_{^cB}$ in $L^1(\mathbb{R}^d)$,

$$\left(\int_{\mathbb{R}^d} |f_\epsilon u|^{d/(d-1)}\right)^{(d-1)/d} \underset{\epsilon\to 0^+}{\longrightarrow} \left(\int_{^cB} |u|^{d/(d-1)}\right)^{(d-1)/d}. \tag{3.22}$$

2. The function ∇f_ϵ is supported in the set $\{r \leq |x| \leq r + \epsilon\}$ and is bounded by

$$|\nabla f_\epsilon(x)| \leq \frac{1}{\epsilon}\|\chi'\|_{L^\infty}, \tag{3.23}$$

as in equation (3.8). Therefore, we similarly have, by noting S_r the Euclidean surface measure on the sphere $\partial B(r)$,

$$\int_{\mathbb{R}^d} |u(x)||\nabla f_\epsilon(x)|\mathrm{d}x \leq \|\chi'\|_{L^\infty}\frac{1}{\epsilon}\int_{r\leq|x|\leq r+\epsilon} |u(x)|\mathrm{d}x \tag{3.24}$$

$$\underset{\epsilon\to 0^+}{\longrightarrow} \|\chi'\|_{L^\infty}\oint_{\partial B} |u(x)|\mathrm{d}S_r(x). \tag{3.25}$$

3. Since $f_\epsilon \to \mathbb{1}_{^cB}$ in L^1 we also have

$$\int_{\mathbb{R}^d} f_\epsilon|\nabla u| \underset{\epsilon\to 0^+}{\longrightarrow} \int_{^cB} |\nabla u| \tag{3.26}$$

The combination of these three points is (3.18). Now note that all the integrals are continuous quantities with respect to u in the $W^{1,1}(\mathbb{R}^d)$ topology (by the GNS inequality) so that (3.18) holds for any $u \in W^{1,1}(\mathbb{R}^d)$.

Step 2 : we now apply (3.18) to an approximation $u \approx \mathbb{1}_\Omega$.

Take χ as before and set (see figure 3.2 for a drawing)

$$u_\epsilon(x) = \chi\left(\frac{1}{\epsilon}d(x, \partial\Omega)\right)\mathbb{1}_\Omega(x). \tag{3.27}$$

As in the proof of the classical isoperimetric inequality of section 3.1.2, the function u_ϵ has bounded derivative ∇u_ϵ supported in $\{x \in \Omega |\ 0 \leq d(x, \partial\Omega) \leq \epsilon\}$ so that $u_\epsilon \in W^{1,1}(\mathbb{R}^d)$. Then applying (3.18) to the approximation u_ϵ gives

$$\left(\int_{^cB} |u_\epsilon(x)|^{d/(d-1)}\mathrm{d}x\right)^{(d-1)/d} \leq C(d)\left[\int_{^cB} |\nabla u_\epsilon(x)|\mathrm{d}x + \oint_{\partial B} |u_\epsilon(x)|\mathrm{d}S_r(x)\right] \tag{3.28}$$

The first two integrals in this inequality are handled as in (3.9) in section 3.1.2.

$$\left(\int_{^cB} |u_\epsilon(x)|^{d/(d-1)}\mathrm{d}x\right)^{(d-1)/d} \underset{\epsilon\to 0^+}{\longrightarrow} |\Omega \cap {}^cB|^{(d-1)/d} \tag{3.29}$$

and

$$\int_{^cB} |\nabla u_\epsilon(x)|\mathrm{d}x \leq \frac{1}{\epsilon}\|\chi'\|_{L^\infty} \int_{x\in\Omega\cap^cB;0\leq d(x,\partial\Omega)\leq\epsilon} \mathrm{d}x \tag{3.30}$$

$$\underset{\epsilon\to 0^+}{\longrightarrow} \|\chi'\|_{L^\infty} |\partial\Omega \cap \mathrm{Adh}(^cB)|. \tag{3.31}$$

Only the last integral $\oint_{\partial B} |u_\epsilon(x)|\mathrm{d}S(x)$ has to be handled differently.

Step 3 : we prove that

$$\oint_{\partial B} |u_\epsilon(x)|\mathrm{d}S_r(x) \leq \int_{^cB} |\nabla u_\epsilon(x)|\mathrm{d}x \tag{3.32}$$

To prove this last inequality, we make use of polar coordinates, reducing the problem to a one dimensional one. Let $\alpha \in \mathbb{R}^d$ with $|\alpha| = r$ and set for $t \geq 0$

$$\phi(t) = u_\epsilon(\alpha t) \in W^{1,1}(t > 0). \tag{3.33}$$

Then, using the Cauchy-Schwarz inequality in $\mathbb{R}^d$,

$$|u(\alpha)| = |\phi(1)| = \left|\int_1^{+\infty} \phi'(t)\mathrm{d}t\right| = \left|\int_1^{+\infty} \nabla u_\epsilon(\alpha t)\cdot\alpha\ \mathrm{d}t\right| \tag{3.34}$$

$$\leq \int_1^{+\infty} r|\nabla u_\epsilon(\alpha t)|\mathrm{d}t \tag{3.35}$$

$$\underset{\rho=rt}{=} \int_r^{+\infty} \left|\nabla u_\epsilon\left(\rho\frac{\alpha}{r}\right)\right|\mathrm{d}\rho \tag{3.36}$$

We now integrate this inequality over $\partial B = \{\alpha |\ |\alpha| = r\}$. For all $\rho > 0$, we note S_ρ the Euclidean surface measure on the sphere $\partial B(\rho) = \{|x| = \rho\}$.

$$\oint_{\partial B(r)} |u(\alpha)|\mathrm{d}S_r(\alpha) \leq \oint_{|\alpha|=r}\int_r^{+\infty} \left|\nabla u_\epsilon\left(\rho\frac{\alpha}{r}\right)\right|\mathrm{d}\rho\,\mathrm{d}S_r(\alpha) \tag{3.37}$$

In order to obtain an integral over all ${}^cB(r)$ as in (3.32), we do the following change of variables : for $\rho \geq r$,

$$\mathrm{d}S_r(\alpha) = r^{d-1}\mathrm{d}S_1\left(\frac{\alpha}{r}\right) = \left(\frac{r}{\rho}\right)^{d-1}\mathrm{d}S_\rho\left(\rho\frac{\alpha}{r}\right) \leq \mathrm{d}S_\rho\left(\rho\frac{\alpha}{r}\right) \tag{3.38}$$

This change of variable is a dilatation and is therefore a C^∞ diffeomorphism. We deduce

$$\oint_{|\alpha|=r}\int_r^{+\infty}\left|\nabla u_\epsilon\left(\rho\frac{\alpha}{r}\right)\right|\mathrm{d}\rho\,\mathrm{d}S_r(\alpha) \leq \int_r^{+\infty}\oint_{|\alpha|=r}\left|\nabla u_\epsilon\left(\rho\frac{\alpha}{r}\right)\right|\mathrm{d}S_\rho\left(\rho\frac{\alpha}{r}\right)\mathrm{d}\rho \tag{3.39}$$

$$\underset{\beta=\rho\alpha/r}{=} \int_r^{+\infty}\oint_{|\beta|=\rho}|\nabla u_\epsilon(\beta)|\mathrm{d}S_\rho(\beta)\mathrm{d}\rho \tag{3.40}$$

$$= \int_{{}^cB(r)}|\nabla u_\epsilon(x)|\mathrm{d}x \tag{3.41}$$

Finally, equations (3.30) and (3.31) give

$$\limsup_\epsilon \oint_{\partial B}|u_\epsilon(x)|\mathrm{d}S_r(x) \leq \|\chi'\|_{L^\infty}|\partial\Omega \cap \mathrm{Adh}({}^cB)|. \tag{3.42}$$

Step 4 : end of the proof. The combination of (3.29), (3.31) and (3.42) ends the proof.

□

3.2.2 Relative Inequality Inside a Ball

In this section, we prove a relative isoperimetric inequality for sets inside a ball $B(r)$.

If $\Omega \subset \mathbb{R}^d$, we would like to control the volume $|\Omega \cap B(r)|$ by the perimeter relative to the ball $|\partial\Omega \cap \bar{B}|$.

However tempting the prospect, there is no such inequality : it is easy to construct counter-examples. If $B \subset \mathrm{Int}\Omega$, then $\partial\Omega \cap \bar{B} = \emptyset$ so that there can be no control of $\Omega \cap B$ by the relative perimeter $\partial\Omega \cap \bar{B}$.

Even when less obvious examples, we can see that things are not much better. Take Ω to be the complement of a small ball $\Omega = {}^cB(\epsilon) \subset B(r)$. Then the perimeter relative to the ball $|\partial\Omega \cap \bar{B}(r)| = |\partial B(\epsilon)|$ is small, whereas the volume $|\Omega \cap B(r)| = |B(r)\setminus B(\epsilon)| \approx |B(r)|$ is large (see figure 3.4).

The problem is that of the *compacity* of the ball $B(r)$. The necessarily large complement of a small subset and the aforementioned subset share the same boundary but have very different volumes. Hence an isoperimetric inequality applies to the small ball $|B(\epsilon)|^{(d-1)/d} \leq C(d)|\partial B(\epsilon)|$ whereas no such inequality can possibly apply to the complement ${}^cB(\epsilon)$.

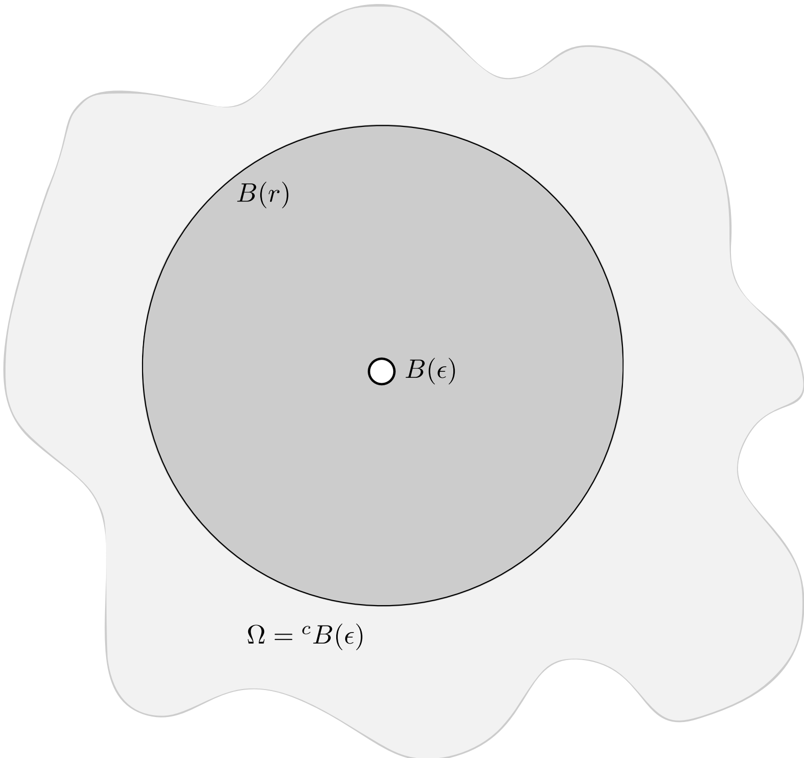

Figure 3.4 – The complement of a small ball has small perimeter but large volume.

Let us look at the previous proof, for the relative inequality outside of the ball. The non compacity of the complement ${}^cB(r)$ is used in step 3 : the proof inequality (3.32).

$$\oint_{\partial B} |u(x)|\mathrm{d}S_r(x) \leq \int_{{}^cB} |\nabla u(x)|\mathrm{d}x \tag{3.43}$$

This inequality is obviously false if we replace the complement ${}^cB$ with the ball $B(r)$: take u to be a constant function, then $\nabla u = 0$ whereas the first integral is nonzero. Unfortunately, step 3, inequality (3.32), is a very crucial part of the proof. The integral $\oint_{\partial B} |u_\epsilon(x)|\mathrm{d}S_r(x)$ having as limit $|\partial B \cap \bar{\Omega}|$, we would be left with

$$|\Omega \cap {}^cB|^{(d-1)/d} \leq C(d)\left[|\partial\Omega \cap \mathrm{Adh}({}^cB)| + |\partial B \cap \bar{\Omega}|\right] = C(d)|\partial\Omega|, \tag{3.44}$$

which is only the classical isoperimetric inequality. This destroys any hope of obtaining some kind of relative isoperimetric inequality inside the ball by adapting *mutatis mutandi* the previous proof.

The solution lies in another functional inequality : Poincaré's inequality ([8], theorem 2 p. 141).

Theorem 3.7 (Poincaré's Inequality)**.** *Let $1 \leq p < d$ and $r > 0$ and let $B = B(r)$ a ball of radius r. There exists a constant $C(p,d)$ such that, for all $u \in$*

$W^{1,p}(B(r))$,

$$\left(\frac{1}{|B(r)|}\int_{B(r)}|u-(u)|^{p^*}\right)^{1/p^*}\leq rC(p,d)\left(\frac{1}{|B(r)|}\int_{B(r)}|\nabla u|^p\right)^{1/p}, \quad (3.45)$$

where p^* is the Sobolev conjugate of p and (u) is the mean value of u

$$(u)=\frac{1}{|B(r)|}\int_{B(r)}f. \quad (3.46)$$

We can reformulate inequality (3.45) so that the constants do not depend on the radius r : as $|B(r)| = \mathrm{Cte}\, r^d$ and $p^* = \frac{dp}{d-p}$,

$$\left(\int_{B(r)}|u-(u)|^{p^*}\right)^{1/p^*}\leq C(p,d)rr^{d\frac{d-p}{dp}}r^{d/p}\left(\int_{B(r)}|\nabla u|^p\right)^{1/p}=C(p,d)\left(\int_{B(r)}|\nabla u|^p\right)^{1/p}, \quad (3.47)$$

and therefore

$$\left(\int_{B(r)}|u-(u)|^{p^*}\right)^{1/p^*}\leq C(p,d)\left(\int_{B(r)}|\nabla u|^p\right)^{1/p} \quad (3.48)$$

We are now prepared to state and prove a relative isoperimetric inequality inside the ball.

Lemma 3.8. *Let $\Omega\subset\mathbb{R}^d$ be an open bounded Lipschitz set and $B=B(r)$ a ball of radius $r>0$. Then, for some constant $C(d)$ depending only on d,*

$$\min\{|\Omega\cap B|,|B\setminus\Omega|\}^{(d-1)/d}\leq C(d)|\partial\Omega\cap\bar{B}|. \quad (3.49)$$

Remark 3.9. Note that the presence of the minimum in the right hand term of the inequality settles the problem posed by subsets Ω that are the complement of small balls $B(\epsilon)$. Inequality (3.49) is an isoperimetric inequality for the subset of $B(r)$ of maximal volume.

Proof. We apply the Poincaré inequality to an approximation of $\mathbb{1}_\Omega$. Let χ be the function described in figure 3.1. Set (see figure 3.2)

$$u_\epsilon(x)=\chi\left(\frac{1}{\epsilon}d(x,\partial\Omega)\right)\mathbb{1}_\Omega(x) \quad (3.50)$$

We apply the Poincaré inequality (3.48) to the function u_ϵ with $p=1$ and $p^*=\frac{dp}{d-p}$. Then,

$$\left(\int_{B(r)}|u_\epsilon-(u_\epsilon)|^{d/(d-1)}\right)^{(d-1)/d}\leq C(d)\int_{B(r)}|\nabla u_\epsilon|. \quad (3.51)$$

Step 1 : first integral. As $u_\epsilon\to\mathbb{1}_\Omega$ in $L^1(\mathbb{R}^d)$,

$$\left(\int_{B(r)}|u_\epsilon-(u_\epsilon)|^{d/(d-1)}\right)^{(d-1)/d}\underset{\epsilon\to0^+}{\longrightarrow}\left(\int_{B(r)}|\mathbb{1}_\Omega-(\mathbb{1}_\Omega)|^{d/(d-1)}\right)^{(d-1)/d} \quad (3.52)$$

The mean value of $\mathbb{1}_\Omega$ is $\frac{|\Omega\cap B|}{|B|}$ and $\mathbb{1}_\Omega$ can take only two values, 0 and 1. Therefore,

$$\begin{aligned}&\left(\int_{B(r)} |\mathbb{1}_\Omega - (\mathbb{1}_\Omega)|^{d/(d-1)}\right)^{(d-1)/d} \\ &= \left[|\Omega\cap B|\left(1-\frac{|\Omega\cap B|}{|B|}\right)^{d/(d-1)} + |B\diagdown\Omega|\left(\frac{|\Omega\cap B|}{|B|}\right)^{d/(d-1)}\right]^{(d-1/d)}\end{aligned} \tag{3.53}$$

By bounding the terms $|\Omega\cap B|$ and $|B\diagdown\Omega|$ from below, we get

$$\begin{aligned}&\left(\int_{B(r)} |\mathbb{1}_\Omega - (\mathbb{1}_\Omega)|^{d/(d-1)}\right)^{(d-1)/d} \\ &\geq \min\left\{|\Omega\cap B|, |B\diagdown\Omega|\right\}^{(d-1)/d}\left[\left(1-\frac{|\Omega\cap B|}{|B|}\right)^{d/(d-1)} + \left(\frac{|\Omega\cap B|}{|B|}\right)^{d/(d-1)}\right]^{(d-1/d)}\end{aligned} \tag{3.54}$$

By setting, for $0\leq x\leq 1$,

$$\psi(x) = \left[(1-x)^{d/(d-1)} + x^{d/(d-1)}\right]^{(d-1/d)} \geq C(d) > 0, \tag{3.55}$$

where $C(d)$ is some positive constant, we get

$$\liminf_{\epsilon}\left(\int_{B(r)} |u_\epsilon - (u_\epsilon)|^{d/(d-1)}\right)^{(d-1)/d} \geq C(d)\min\left\{|\Omega\cap B|, |B\diagdown\Omega|\right\}^{(d-1)/d}. \tag{3.56}$$

Step 2 : second integral. We proceed as in al the previous proofs. The function ∇u_ϵ is supported in the set $\{x\in\Omega|\ 0\leq d(x,\partial\Omega)\leq\epsilon\}$ and has a bounded from above derivative $|\nabla u_\epsilon| \leq \frac{1}{\epsilon}\|\chi'\|_{L^\infty}$. Then,

$$\int_{B(r)} |\nabla u_\epsilon| \leq \|\chi'\|_{L^\infty}\frac{1}{\epsilon}\int_{x\in\Omega; 0\leq d(x,\partial\Omega)\leq\epsilon} \mathrm{d}x \underset{\epsilon\to 0^+}{\longrightarrow} C(d)|\partial\Omega\cap\bar{B}| \tag{3.57}$$

Step 3 : the combination of (3.56) and (3.57) gives the result.

□

3.2.3 Relative Inequalities in an Annulus

We recall one last time the relative inequality in an annulus.

Lemma 3.10 (Relative Isoperimetric Inequality in Annuli). *Let $m > 0$ and $\Omega\subset\mathbb{R}^d$ be an open Lipschitz domain. There exists a minimal width $w = w(m,d) > 0$ and a constant $c = c(d)$ such that for every annulus $A = A(r, r+l)$ of width $l\geq w$ such that $|A\cap\Omega|\leq m$ we have*

$$c|A\cap\Omega|^{d-1/d} \leq |\partial\Omega\cap\bar{A}|. \tag{3.58}$$

With the relative inequalities inside and outside the ball, we are ready to prove lemma 3.3. The main idea of the proof is to separate the annulus $A = A(r, r+l)$ into two disjoint pieces : the inside of an intermediate ball B and the outside. We then apply lemmas 3.5 and 3.8 to $\Omega \cap B$ and $\Omega \cap {}^c B$ relatively to the balls $B(r)$ and $B(r+l)$ (see figure 3.5).

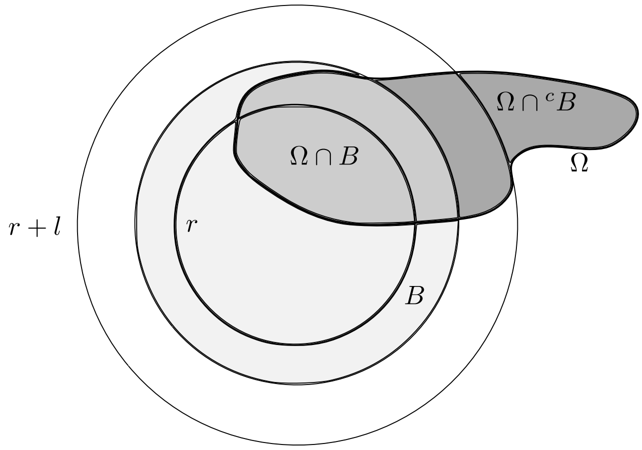

Figure 3.5 – The ball B separates the annulus into two disjoint pieces to which the relative isoperimetric inequalities are applied.

Proof of lemma 3.3. The proof is a two step one.

Step 1 : we first show the result for $\Omega \cap A$ small enough. More precisely, we prove that there exist $\epsilon > 0$ and $C > 0$ which depend only on d such that, for all annuli $A(r, r+l)$ of width $l \geq 1$, for all open bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ such that $|\Omega \cap A| \leq \epsilon$, the following relative isoperimetric inequality holds :

$$|\Omega \cap A|^{(d-1)/d} \leq C(d)|\partial\Omega \cap \bar{A}|. \tag{3.59}$$

We first seek an appropriate ball B to separate A into two pieces. Using polar coordinates, we obtain the following expression for $|A \cap \Omega|$,

$$|A \cap \Omega| = \int_A \mathbb{1}_\Omega(x)\mathrm{d}x = \int_r^{r+l} \oint_{|\alpha=1|} \mathbb{1}_\Omega(t\alpha)\mathrm{d}S_t(t\alpha)\mathrm{d}t, \tag{3.60}$$

where S_t is the Euclidean surface measure on the sphere $\partial B(t)$. Since $|\Omega \cap \partial B(t)| = \oint_\alpha \mathbb{1}_\Omega(t\alpha)\mathrm{d}S_t(t\alpha)$,

$$|A \cap \Omega| = \int_r^{r+l} |\Omega \cap \partial B(t)|\mathrm{d}t. \tag{3.61}$$

We deduce from this that there must be a $l_0 \in]0, l[$ such that $|\Omega \cap \partial B(r+l_0)| \leq |\Omega \cap A|$. We then set

$$\Omega_1 = \Omega \cap B(r+l_0) \tag{3.62}$$

$$\Omega_2 = \Omega \setminus B(r+l_0) \tag{3.63}$$

We now apply the relative isoperimetric inequalities to Ω_1 and Ω_2.

1. We apply lemma 3.5 to Ω_1 relatively to ${}^cB(r)$. We then have

$$|\Omega_1 \cap \mathrm{Adh}({}^cB(r))|^{(d-1)/d} \leq C(d)|\partial\Omega_1 \cap \mathrm{Adh}({}^cB(r))| \tag{3.64}$$

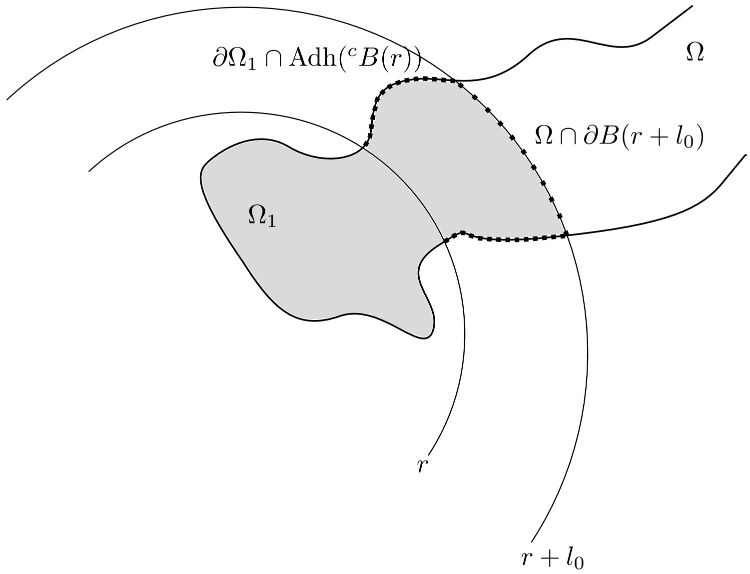

Figure 3.6 – The perimeter of Ω_1 relative to ${}^cB(r)$ has two parts. The first one is $\partial\Omega_1 \cap \mathrm{Adh}({}^cB(r))$ (regularly dotted) and the second is $\Omega \cap \partial B(r)$ (sparsely dotted).

We separate the relative perimeter $\partial\Omega_1 \cap \mathrm{Adh}({}^cB(r))$ into two parts : the part of $\partial\Omega_1$ lying on the sphere $\partial B(r+l_0)$ and the rest (see figure 3.6). Therefore,

$$|\Omega_1 \cap \mathrm{Adh}({}^cB(r)) \cap A|^{(d-1)/d} \leq C(d)\left[|\partial\Omega \cap \bar{A}(r, r+l_0)| + |\Omega \cap \partial B(r+l_0)|\right] \tag{3.65}$$

And, since we chose l_0 so that $|\Omega \cap \partial B(r+l_0)| \leq |\Omega \cap A|$, we have the following upper bound :

$$|\Omega_1 \cap \mathrm{Adh}({}^cB(r)) \cap A|^{(d-1)/d} \leq C(d)\left[|\partial\Omega \cap \bar{A}(r, r+l_0)| + |\Omega \cap A|\right]. \tag{3.66}$$

2. We apply lemma 3.8 to Ω_2 relatively to $B(r+l)$. We then have

$$\min\left\{|\Omega_2 \cap B(r+l)|, |B(r+l) \setminus \Omega_2|\right\}^{(d-1)/d} \leq C(d)|\partial\Omega_2 \cap \bar{B}(r+l)| \tag{3.67}$$

We proceed in the same way as before.

The perimeter of Ω_2 relative to the ball $B(r+l)$ is divided into two parts : $\partial\Omega \cap B(r+l)$ and $\Omega \cap \partial B(r)$ (see figure 3.7). Since $|\Omega \cap \partial B(r)| \leq |\Omega \cap A|$, we have

$$\min\left\{|\Omega_2 \cap B(r+l)|, |B(r+l) \setminus \Omega_2|\right\}^{(d-1)/d} \leq C(d)\left[|\partial\Omega \cap \bar{A}(r+l_0, r+l)| + |\Omega \cap A|\right]. \tag{3.68}$$

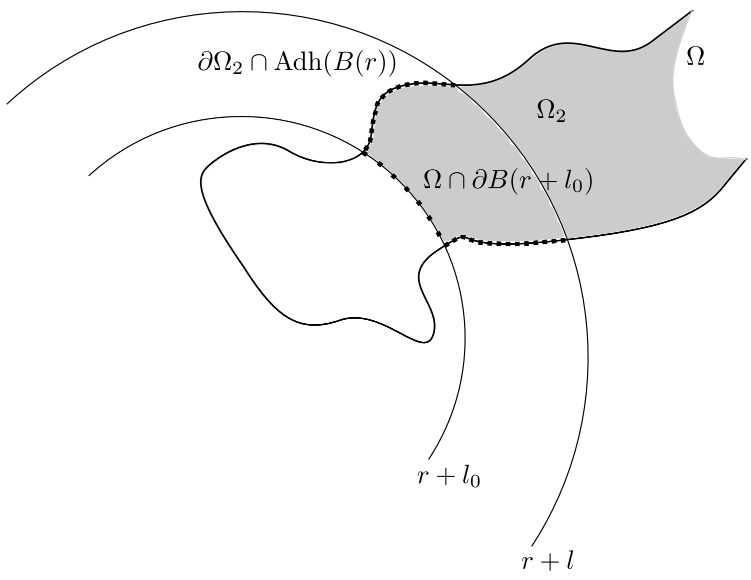

Figure 3.7 – The perimeter of Ω_2 relative to $B(r+l)$ has two parts. The first one is $\partial\Omega \cap B(r+l)$ (regularly dotted) and the second is $\Omega \cap \partial B(r)$ (sparsely dotted).

To make this last inequality practical, we must get rid of the minimum. This is where we make a choice for ϵ. We have

$$|\Omega_2 \cap B(r+l)| \leq |\Omega \cap A| \leq \epsilon, \tag{3.69}$$

and since we supposed that $l \geq 1$, we also have

$$|B(r+l) \smallsetminus \Omega| \geq |B(r+l)| - \epsilon \geq B(1) - \epsilon. \tag{3.70}$$

The combination of these two last lines assures that for ϵ small enough,

$$\min\{|\Omega_2 \cap B(r+l)|, |B(r+l) \smallsetminus \Omega_2|\} = |\Omega_2 \cap B(r+l)|. \tag{3.71}$$

We have obtained the following upper bound :

$$|\Omega_2 \cap B(r+l)| \leq C(d)\left[|\partial\Omega \cap \bar{A}(r+l_0, r+l)| + |\Omega \cap A|\right]. \tag{3.72}$$

The combination of both inequalities (3.66) and (3.72) gives

$$\left(\frac{|\Omega \cap A|}{2}\right)^{(d-1)/d} \leq |\Omega_1 \cap {}^c B(r)|^{(d-1)/d} + |\Omega_2 \cap B(r+l)|^{(d-1)/d} \tag{3.73}$$

$$\leq C(d)\left[|\partial\Omega \cap \bar{A}| + |\Omega \cap A|\right]. \tag{3.74}$$

In other words,

$$1 \leq C(d)\left[\frac{|\partial\Omega \cap \bar{A}|}{|\Omega \cap A|^{(d-1)/d}} + |\Omega \cap A|^{1/d}\right]. \tag{3.75}$$

Since the volume $|\Omega \cap A|^{1/d} \to 0^+$ as $\epsilon \to 0^+$, this last inequality implies that, by taking ϵ even smaller if necessary, there is a constant $C(d)$ such that if $|\Omega \cap A| \leq \epsilon$,

$$\left(\frac{|\Omega \cap A|}{2}\right)^{(d-1)/d} \leq C(d)|\partial\Omega \cap \bar{A}|. \tag{3.76}$$

This ends step 1.

Step 2 : We can prove the lemma in its general form by using dilatations.

Let $w = w(m,d)$ such that $w^d \epsilon = m$. Let $\Omega \subset \mathbb{R}^d$ such that $|\Omega \cap A| \leq m$ and let $l \geq w$. Set $\Omega' = \frac{1}{w}\Omega$.

Then the annulus $\frac{1}{w}A = A(r/w, (r+l)/w)$ is of width $l/w \geq 1$ and $|\Omega' \cap \frac{1}{w}A| \leq \epsilon$. Step 1 provides the following inequality :

$$\left(\frac{1}{w^d}|\Omega \cap A|\right)^{(d-1)/d} \leq C(d)\left|\partial\Omega' \cap \frac{1}{w}\bar{A}\right| = \frac{C(d)}{w^{d-1}}|\partial\Omega \cap \bar{A}|. \tag{3.77}$$

Rearranging the terms provides the inequality we covet :

$$c|A \cap \Omega|^{d-1/d} \leq |\partial\Omega \cap \bar{A}|. \tag{3.78}$$

□

Chapter 4

Discussion in the Hyperbolic Space

Non relinquetur hic lapis super lapidem qui non destruatur.[1]

Warning : the material contained in this chapter is the result of a small research work done during the internship, and is therefore liable to be heavily burdened with mistakes and errors of all kinds.

This chapter contains an attempt to generalize the isodiametric inequality, theorem 2.1, to domains of the hyperbolic space by replicating the proof of [2]. This can only be done if an equivalent of some sort of the relative isoperimetric inequality in annuli holds in the hyperbolic space.

To attempt proving such an inequality, we study the GNS and Poincaré inequalities in the hyperbolic space and the relative inequalities induced.

4.1 The Poincaré Ball Model

This part is dedicated to introducing the definitions and notations which will be used throughout the chapter.

In this chapter, we use the Poincaré ball model of the hyperbolic space. Consider the open unit ball $\mathbb{B} \subset \mathbb{R}^d$ and equip it with the following Riemannian metric :

$$\forall x \in \mathbb{B}, \qquad g(x) = \frac{4}{(1-|x|)^2} g_0(x), \tag{4.1}$$

where $g_0 = I$ is the Euclidean metric on $\mathbb{B}$. This metric is *conformal* to the Euclidean metric on the ball : it is deduced from the Euclidean metric by multiplication by a positive factor. The resulting Riemannian manifold is noted $\mathbb{H}^d$ and has constant scalar curvature $R = -1$.

[1] Mt. 24:2.

The hyperbolic space is homogenous. In other words, two balls of same volume are always isometric. This allows us, when conducting calculations in balls of $\mathbb{H}^d$ (for the geodesic distance), to suppose that we are working in the ball centered at the origin $0 \in \mathbb{B}$, which is very convenient, since the geodesics spanned from 0 are strait lines $\{t\alpha | t \geq 0\}$ (see figure 4.1).

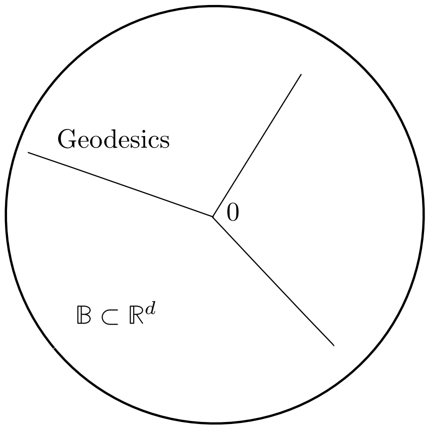

Figure 4.1 – The geodesics of $\mathbb{H}^d$ spanned from the center $0 \in B$ are strait lines.

For this reason, it will be practical to use polar coordinates on $\mathbb{H}^d$. We express the Euclidean metric in polar coordinates $\rho, \theta_1, ..., \theta_{d-1}$ with $0 \leq \rho < 1$:

$$g_0(\rho, \theta) = \text{diag}\left(1, \frac{1}{\rho^2} g_1(\theta), ..., \frac{1}{\rho^2} g_{d-1}(\theta)\right), \tag{4.2}$$

where $g_i(\theta)$ are functions of $\theta_1, ..., \theta_{d-1}$. This gives the following form for the hyperbolic metric :

$$\mathrm{d}s^2 = \frac{4}{(1 - |x|^2)^2} \left[\mathrm{d}\rho^2 + \frac{1}{\rho^2} \sum_{i=1}^{d-1} g_i(\theta) \mathrm{d}\theta_i^2\right] \tag{4.3}$$

We note δ the geodesic distance on $\mathbb{H}^d$ and, for all $r > 0$, we note $B(r)$ the open (geodesic) ball centered in $0 \in \mathbb{B}$ and of radius r :

$$B(r) = \{x \in \mathbb{H}^d | \delta(0, x) < r\}. \tag{4.4}$$

We note ν the Riemannian d-dimensional and $(d-1)$-dimensional measures on $\mathbb{H}^d$ and, for all $r > 0$, we note S_r the Riemannian $(d-1)$-dimensional measure on the sphere

$$\partial B(r) = \{x \in \mathbb{H}^d | \delta(0, x) = r\}. \tag{4.5}$$

Finally, if $f : \mathbb{H}^d \longrightarrow \mathbb{R}$, we note $\nabla f(x) = \nabla_g f(x) \in \mathcal{T}_x \mathbb{H}^d$ the gradient of f with respect to the metric, as in (1.25), and $|\nabla f| = |\nabla_g f|_g$ its norm, thus omitting the dependency in the metric g.

4.2 Functional Inequalities

4.2.1 The GNS Inequality

The hyperbolic space supports an isoperimetric inequality. See [4] pp. 86 and following for a geometric proof of this fact.

Theorem 4.1 (Isoperimetric Inequality in $\mathbb{H}^d$)**.** *Let $\Omega \subset \mathbb{H}^d$ be an open bounded Lipschitz domain. Then,*

$$\nu(\Omega)^{(d-1)/d} \leq C(d)\nu(\partial\Omega), \tag{4.6}$$

where $C(d)$ is a constant depending only on the dimension d.

As we have seen in the Euclidean space $\mathbb{R}^d$, the isoperimetric inequality is implied by the GNS inequality. In fact, the converse is also true : the euclidean isoperimetric inequality implies the GNS inequality (see [8] pp. 192-193). This operation can also be done in the hyperbolic space $\mathbb{H}^d$, in exactly the same way.

Theorem 4.2 (Hyperbolic GNS inequality)**.** *Let $f \in W^{1,1}(\mathbb{H}^d)$. Then*

$$\left(\int_{\mathbb{H}^d} |f|^{(d-1)/d} d\nu\right)^{(d-1)/d} \leq C(d) \int_{\mathbb{H}^d} |\nabla f| d\nu. \tag{4.7}$$

where $C(d)$ is the isoperimetric constant of theorem 4.1.

Proof. As all quantities in the GNS inequality (4.7) are continuous with respect to f in the $W^{1,1}(\mathbb{H}^d)$ topology, so it is sufficient to prove (4.7) for $f \in \mathcal{D}(\mathbb{H}^d)$.

Assume first that $f \geq 0$ and note $E_t = \{f > t\}$. Then the coarea formula yields

$$\int_{\mathbb{H}^d} |\nabla f| \mathrm{d}\nu = \int_{-\infty}^{+\infty} \nu\left(\partial E_t\right) \mathrm{d}t = \int_0^{+\infty} \nu\left(\partial E_t\right) \mathrm{d}t \tag{4.8}$$

The isoperimetric inequality (4.6) allows us to bound the perimeter $\nu(\partial E_t)$ from below : $\nu(\partial E_t) \geq \frac{1}{C}\nu(E_t)^{(d-1)/d}$, where $C = C(d)$ is the isoperimetric constant.

$$\int_{\mathbb{H}^d} |\nabla f| \mathrm{d}\nu \geq \frac{1}{C} \int_0^{+\infty} \nu(E_t)^{(d-1)/d} \mathrm{d}t \tag{4.9}$$

We now apply this last inequality to an auxiliary funxtion. Set

$$f_t(x) = \min\{t, f(x)\} \tag{4.10}$$

$$\chi(t) = \left(\int_{\mathbb{H}^d} f_t(x)^{d/(d-1)} \mathrm{d}\nu(x)\right)^{(d-1)/d} \tag{4.11}$$

Then $f_0 = 0$ and $f_t \to f$ in $L^1(\mathbb{H}^d)$ as $t \to +\infty$, since f is very regular $f \in \mathcal{D}(\mathbb{H}^d)$. This implies that

$$\left(\int_{\mathbb{H}^d} f(x)^{d/(d-1)} \mathrm{d}\nu(x)\right)^{(d-1)/d} = \int_0^{+\infty} \chi'(t) \mathrm{d}t \tag{4.12}$$

We study the derivative χ'. For $h > 0$, the triangular inequality in the $L^{d/(d-1)}$ norm yields

$$0 \le \chi(t+h)-\chi(t) \le \left(\int_{\mathbb{H}^d} |f_{t+h}(x) - f_t(x)|^{d/(d-1)} \mathrm{d}\nu(x)\right)^{(d-1)/d} \le h\nu(E_t)^{(d-1)/d} \tag{4.13}$$

Therefore, the derivative χ' is bounded from above :

$$\chi'(t) \le \nu(E_t)^{(d-1)/d}. \tag{4.14}$$

Integrating for $t \ge 0$, with (4.9), gives the desired result :

$$\left(\int_{\mathbb{H}^d} f(x)^{d/(d-1)} \mathrm{d}\nu(x)\right)^{(d-1)/d} = \int_0^{+\infty} \chi'(t)\mathrm{d}t \tag{4.15}$$

$$\le \int_0^{+\infty} \nu(E_t)^{(d-1)/d}\mathrm{d}t \tag{4.16}$$

$$\le C \int_{\mathbb{H}^d} |\nabla f(x)| \mathrm{d}\nu(x). \tag{4.17}$$

□

Remark 4.3. The GNS inequality for $p = 1$ implies the inequality for all other p, as shown in part 2. of the proof of [8] p. 140 which can be immediately adapted to the hyperbolic space.

Remark 4.4. Nothing in the previous proof is specific to the hyperbolic space, so that the GNS inequality is true on any manifold supporting an isoperimetric inequality. In fact, the following theorem holds (see [5]) :

Theorem 4.5. *Let $(\mathcal{M}, g)$ be a Riemannian manifold and note V_g the d and $(d-1)$-dimensional Riemannian measures on $\mathcal{M}$. Define the isoperimetric constant of $\mathcal{M}$,*

$$\mathcal{I}(\mathcal{M}) = \sup_{\Omega} \frac{V_g(\Omega)^{(d-1)/d}}{V_g(\partial\Omega)}, \tag{4.18}$$

and the Sobolev constant of $\mathcal{M}$,

$$\mathcal{S}(\mathcal{M}) = \sup_{f} \frac{\left(\int_{\mathcal{M}} |f|^{d/(d-1)}\, dV_g\right)^{(d-1)/d}}{\int_{\mathcal{M}} |\nabla_g f|_g\, dV_g} \tag{4.19}$$

where the first supremum is taken over the open bounded Lipschitz domains $\Omega \subset \mathcal{M}$ and the second one over the $f \in \mathcal{D}(\mathcal{M})$ and where V_g si the d-dimensional Riemannian metric.

Then $\mathcal{I}(\mathcal{M}) = \mathcal{S}(\mathcal{M})$.

4.2.2 Poincaré's Inequality

Proving an equivalent of the Poincaré inequality (3.45) in the hyperbolic space is a very long and difficult process. What is proposed in this section is replicating the Euclidean proof of [8] pp. 140-142 by adapting each part to the hyperbolic geometry. The goal is to prove an inequality of the following form : if $B(r) \subset \mathbb{H}^d$ is a ball, we seek to prove that

$$\left(\int_{B(r)} |f - (f)|^{p^*} \mathrm{d}\nu\right)^{1/p^*} \leq A(r)C(d,p)\left(\int_{B(r)} |\nabla f|^p \mathrm{d}\nu\right)^{1/p} \tag{4.20}$$

where $A(r)$ is a function of the radius of the ball that has to be determined. Of course, the exact form of $A(r)$ will have an impact on the induced isoperimetric inequalities and hence our ability to prove (or not) an isodiametric control of the spectrum.

The Euclidean proof of the Poincaré inequality for $f : B(r) \longrightarrow \mathbb{R}$ relies on three important steps, which we will replicate in the hyperbolic space.

1. First, an *extension theorem* ([8] theorem 1. p. 135) allows to extend f into a function[2] $Ef : \mathbb{R}^d \to \mathbb{R}$ with bounded Sovolev norm, and in particular,

$$\|\nabla(Ef)\|_{L^p(\mathbb{R}^d)} \leq A_1(r,d,p)\left(\|\nabla f\|_{L^p(B(r))} + \|f\|_{L^p(B(r))}\right). \tag{4.21}$$

2. Then, we can apply the GNS inequality to the function $Eg = E(f - (f))$ to get an inequality of the form

$$\left(\int_{B(r)} |f - (f)|^{p^*} \mathrm{d}\nu\right)^{1/p^*} \leq \left(\int_{\mathbb{H}^d} |Eg|^{p^*} \mathrm{d}\nu\right)^{1/p^*} \tag{4.22}$$

$$\leq C(d,p)\left(\int_{\mathbb{H}^d} |\nabla(Eg)|^p \mathrm{d}\nu\right)^{1/p} \tag{4.23}$$

$$\leq A_2(r,d,p)\left[\left(\int_{B(r)} |\nabla f|^p \mathrm{d}\nu\right)^{1/p} + \left(\int_{B(r)} |g|^p \mathrm{d}\nu\right)^{1/p}\right]. \tag{4.24}$$

3. Finally, we seek to obtain an inequality of the form

$$\int_{B(r)} |g|^p \mathrm{d}\nu \leq A_3(r,d,p)\int_{B(r)} |\nabla f|^p \mathrm{d}\nu, \tag{4.25}$$

   which allows us to conclude.

We first prove the extension theorem (in the next section) before finishing the proof of the Poincaré inequality.

4.2.3 Extension Operators on Hyperbolic Balls

Of those three steps, the first one is the most difficult and implies the most involved computations. Even if it is a widely known result that some bounded extension operator E exists on any ball $B(r)$, our personal research has not allowed us to find any documentation on the study of its norm. We have proven the following theorem :

[2] By extending, we mean that $Ef = f$ on $B(r)$.

Theorem 4.6 (Extension Operator on Balls)**.** *Let $r_0 \geq 1$ and $f \in C^\infty(\bar{B}(r_0))$. Then there is an extension operator E such that the function $Ef : \mathbb{H}^d \longrightarrow \mathbb{R}$ is C^∞-smooth, compactly supported in $\mathbb{H}^d$ and satisfies*

$$\|\nabla(Ef)\|_{L^p(\mathbb{H}^d)} \leq C(d,p)\left(\|\nabla f\|_{L^p(B(r_0))} + \|f\|_{L^p(B(r_0))}\right). \tag{4.26}$$

Remark 4.7. In other words, $A_1(r) = C(p,d)$.

Remark 4.8. The proof of this theorem is very simple in the Euclidean space, since it supports dilatations : by replacing $f(x)$ by $(1/r_0)f(r_0 x)$, we may suppose that $r_0 = 1$.

Proof. The main idea of the proof is to use a symmetrization process over the boundary $\partial B(r_0)$ to construct the extended function Ef. We then verify that inequality (4.26) is satisfied.

The hyperbolic space is homogenous so, by using an isometry if necessary, we may suppose that the ball $B(r_0)$ is centered in $0 \in \mathbb{B}$. Throughout the proof, we use polar coordinates $\rho, \theta_1, ..., \theta_{d-1}$ (where $0 \leq \rho < 1$) on the unit ball $\mathbb{B} \subset \mathbb{R}^d$ so that the hyperbolic metric is given by

$$\mathrm{d}s^2 = \frac{4}{(1-|x|^2)^2}\left[\mathrm{d}\rho^2 + \frac{1}{\rho^2}\sum_{i=1}^{d-1} g_i(\theta)\mathrm{d}\theta_i^2\right], \tag{4.27}$$

and if $x = (\rho, \theta) \in \mathbb{B}$, we note $r = \delta(0,x) = \tanh^{-1}(\rho)$ the (geodesic) distance between x and the origin 0.

Let $\chi \in C^\infty(\mathbb{R})$ such that (see figure 4.2)

1. The function χ is identically equal to 1 around 0 : $\chi(t) \equiv 1$ for $0 \leq t \leq \frac{1}{2}$.
2. $0 \leq \chi \leq 1$.
3. The function χ is compactly supported in $\mathrm{supp}(\chi) \subset [-1,1]$.

We are ready to define the extension function Ef. See f as a function of the hyperbolic radius, $f(r,\theta)$, and let

$$Ef(r,\theta) = \begin{cases} f(r,\theta) & \text{if } r \leq r_0, \\ f\left(2r_0 - r, \theta\right)\chi\left(r - r_0\right) & \text{if } r \leq 2r_0, \\ 0 & \text{elsewhere.} \end{cases} \tag{4.28}$$

We now verify that the extension (4.28) satisfies inequality (4.26).

Since

$$\nabla\left(Ef(r,\theta)\right) = f(2r_0 - r,\theta)\nabla\left(\chi(r-r_0)\right) + \chi(r-r_0)\nabla\left(f(2r_0 - r)\right), \tag{4.29}$$

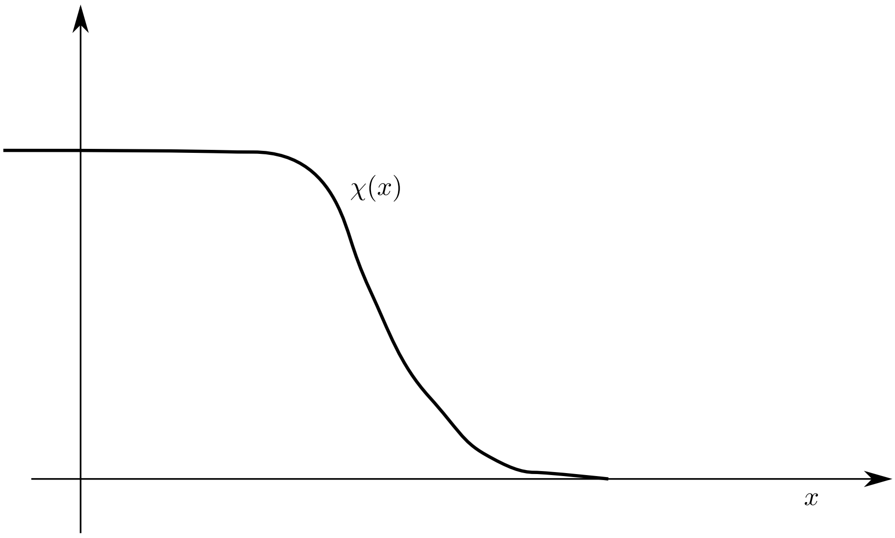

Figure 4.2 – The function χ is used to provide a compact support for the extension Ef.

the triangular inequality for the L^p norm provides us with

$$\left[\int_{\mathbb{H}^d} |\nabla(Ef)(x)|^p \mathrm{d}\nu(x)\right]^{1/p} \leq \left[\int_{r\leq r_0} |\nabla f(x)|^p \mathrm{d}\nu(x)\right]^{1/p}$$
$$+ \left[\int_{r\geq r_0} |f(2r_0 - r, \theta)\nabla\left(\chi(r-r_0)\right)|^p \mathrm{d}\nu(r,\theta)\right]^{1/p}$$
$$+ \left[\int_{r\geq r_0} |\chi(r-r_0)\nabla\left(f(2r_0 - r, \theta)\right)|^p \mathrm{d}\nu(r,\theta)\right]^{1/p}. \quad (4.30)$$

We then calculate the gradients $\nabla(.) = \sum_{i,j} g^{ij}\partial_i(.)e_j$ using the metric (4.27) in order to bound from above the two last integrals.

First integral $\int_{r\geq r_0} |f(2r_0 - r, \theta)\nabla\left(\chi(r-r_0)\right)|^p \mathrm{d}\nu(r,\theta)$.

We recall that $r = \mathrm{th}^{-1}(\rho)$. The derivative of the term $\chi(r - r_0)$, which is a radial function, is

$$\nabla\left(\chi(r-r_0)\right) = (1-\rho^2)^2\left[\partial_\rho\left(\chi(\mathrm{th}^{-1}(\rho) - r_0)\right)e_\rho + \frac{1}{\rho^2}\sum_{i=1}^{d-1}\frac{1}{g_i(\theta)}\partial_{\theta_i}\left(\chi(r-r_0)\right)e_{\theta_i}\right] \quad (4.31)$$
$$= (1-\rho^2)\chi'(r-r_0)e_\rho. \quad (4.32)$$

The norm of a vector $u \in \mathcal{T}\mathbb{H}^d$ is $|u| = \sqrt{\sum_{i,j} g_{ij}u^i u^j}$ so that the norm of the previous derivative is

$$|\nabla\left(\chi(r-r_0)\right)| = |\chi'(r-r_0)|(1-\rho^2)|e_\rho| = |\chi'(r-r_0)|. \quad (4.33)$$

This allows us to give an upper bound for the integral : since the function $\chi(r-r_0)$ is supported in the set $\{r_0 \leq r \leq r_0+1\}$,

$$\int_{r\geq r_0} |f(2r_0-r,\theta)\nabla\left(\chi(r-r_0)\right)|^p \mathrm{d}\nu(r,\theta) \leq \|\chi'\|^p_{L^\infty} \int_{r_0\leq r\leq r_0+1} |f(2r_0-r,\theta)|^p \mathrm{d}\nu(r,\theta) \tag{4.34}$$

We then use the coarea formula to separate radial and angular variables. We remind that S_r is the $(d-1)$-dimensional Riemannian measure on the sphere $\partial B(r)$. The isotropy of the hyperbolic space implies that $\mathrm{d}S_r(\theta) = C(d)\,\mathrm{sh}(r)^{d-1}\mathrm{d}S_1(\theta)$, where $C(d)\,\mathrm{sh}(r)^{d-1}$ is the measure of the sphere $\nu(\partial B(r))$. We use this fact to make a change of variables.

$$\int_{r_0\leq r\leq r_0+1} |f(2r_0-r,\theta)|^p \mathrm{d}\nu(r,\theta) = \int_{r_0}^{r_0+1}\int_\theta |f(2r_0-r,\theta)|^p \mathrm{d}S_r(\theta)\mathrm{d}r \tag{4.35}$$

$$= \int_{r_0}^{r_0+1}\int_\theta |f(2r_0-r,\theta)|^p \left(\frac{\mathrm{sh}(r)}{\mathrm{sh}(2r_0-r)}\right)^{d-1} \mathrm{d}S_{2r_0-r}(\theta)\mathrm{d}r \tag{4.36}$$

$$\underset{2r_0-r\to r}{=} \int_{r_0-1}^{r_0}\int_\theta |f(r,\theta)|^p \left(\frac{\mathrm{sh}(2r_0-r)}{\mathrm{sh}(r)}\right)^{d-1} \mathrm{d}S_r(\theta)\mathrm{d}r \tag{4.37}$$

To bound this from above by an integral of f only, we must get rid of the fraction. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{sh}(2r_0-t)}{\mathrm{sh}(t)}\right) = \frac{1}{\mathrm{sh}(t)^2}\left[-\mathrm{ch}(2r_0-t)\,\mathrm{sh}(t) - \mathrm{ch}(t)\,\mathrm{sh}(2r_0-t)\right] < 0 \tag{4.38}$$

so that the fraction $\left(\frac{\mathrm{sh}(2r_0-r)}{\mathrm{sh}(r)}\right)^{d-1}$ is a decreasing function of r and thus attains its maximum value on $r_0 - \leq r \leq r_0$ at $r = r_0-1$:

$$\left(\frac{\mathrm{sh}(2r_0-r)}{\mathrm{sh}(r)}\right)^{d-1} \leq \left(\frac{\mathrm{sh}(r_0+1)}{\mathrm{sh}(r_0-1)}\right)^{d-1} \underset{r_0\to+\infty}{=} O(1). \tag{4.39}$$

Hence the following upper bound :

$$\int_{r\geq r_0} |f(2r_0-r,\theta)\nabla\left(\chi(r-r_0)\right)|^p \mathrm{d}\nu(r,\theta) \leq C(p,d)\int_{r_0-1}^{r_0}\int_\theta |f(r,\theta)|^p \mathrm{d}S_r(\theta)\mathrm{d}r \tag{4.40}$$

$$\leq C(p,d)\int_{B(r_0)} |f|^p \mathrm{d}\nu. \tag{4.41}$$

Second integral : $\left[\int_{r\geq r_0} |\chi(r-r_0)\nabla\left(f(2r_0-r,\theta)\right)|^p \mathrm{d}\nu(r,\theta)\right]^{1/p}$. We proceed in the same way. Only this time, we must deal with angular derivatives.

The derivative of $f(2r_0-r,\theta)$ is, with respect to the hyperbolic metric (4.27),

$$\nabla\left(f(2r_0-r,\theta)\right) = (1-\rho^2)^2\left[\partial_\rho\left(f(2r_0-\mathrm{th}^{-1}(\rho))\right)e_\rho + \frac{1}{\rho^2}\sum_{i=1}^{d-1}\frac{1}{g_i(\theta)}\partial_{\theta_i}\left(f(2r-\mathrm{th}^{-1}(\rho),\theta)\right)e_{\theta_i}\right] \tag{4.42}$$

$$= (1-\rho^2)\partial_\rho f(2r_0-r,\theta)e_\rho + \frac{(1-\rho^2)^2}{\rho^2}\sum_{i=1}^{d-1}\frac{1}{g_i(\theta)}\partial_{\theta_i} f(2r_0-r,\theta)e_{\theta_i}. \tag{4.43}$$

We compute the norm of this vector :

$$|\nabla\left(f(2r_0-r,\theta)\right)|^p = \left[(\partial_\rho f(2r_0-r,\theta))^2 + \sum_{i=1}^{d-1}\left(\frac{(1-\rho^2)}{\rho}\frac{1}{\sqrt{g_i(\theta)}}\partial_{\theta_i} f(2r_0-r,\theta)\right)^2\right]^{p/2} \tag{4.44}$$

As before, we use again the coarea formula to separate radial and angular variables.

$$\begin{aligned}
&\int_{r\geq r_0} |\chi(r-r_0)\nabla\left(f(2r_0-r,\theta)\right)|^p \mathrm{d}\nu(r,\theta) \\
&\quad\leq \|\chi\|_{L^\infty}^p \int_{r_0\leq r\leq r_0+1} |\nabla\left(f(2r_0-r,\theta)\right)|^p \mathrm{d}\nu(r,\theta) \qquad (4.45)\\
&\quad\leq C(p)\int_{r_0}^{r_0+1}\int_\theta \left[(\partial_\rho f(2r_0-r,\theta))^2 + \sum_{i=1}^{d-1}\left(\frac{1-\rho^2}{\rho}\frac{1}{\sqrt{g_i(\theta)}}\partial_{\theta_i} f(2r_0-r,\theta)\right)^2\right]^{p/2} \mathrm{d}S_r(\theta)\mathrm{d}r \qquad (4.46)
\end{aligned}$$

We now make the change of variable $2r_0 - r \to r$ in this integral. Of course, we must remember that $\rho = \mathrm{th}(r)$ is a function of r and hence it affected by the change of variable. The last integral in (4.46) is, after the change of variable,

$$\int_{r_0-1}^{r_0}\int_\theta \left[(\partial_\rho f(r,\theta))^2 + \sum_{i=1}^{d-1}\left(\frac{1-\mathrm{th}(2r_0-r)^2}{\mathrm{th}(2r_0-r)}\frac{1}{\sqrt{g_i(\theta)}}\partial_{\theta_i} f(r,\theta)\right)^2\right]^{p/2} \times\left(\frac{\mathrm{sh}(2r_0-r)}{\mathrm{sh}(r)}\right)^{d-1}\mathrm{d}S_r(\theta)\mathrm{d}r \tag{4.47}$$

In order to compare it to the integral of $|\nabla f|$, we must replace the $\mathrm{th}(2r_0-r)$ appearing in (4.47) by $\mathrm{th}(r)$. We therefore seek to compare

$$\frac{1-\mathrm{th}(2r_0-r)^2}{\mathrm{th}(2r_0-r)} \quad\text{and}\quad \frac{1-\mathrm{th}(r)^2}{\mathrm{th}(r)}. \tag{4.48}$$

The function th is an increasing one and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1-t^2}{t^2}\right) = -\frac{1-t^2}{t^2} - 2 < 0. \tag{4.49}$$

Since $2r_0 - r$ varies in the interval $[r_0, r_0+1]$, we write

$$\frac{1-\mathrm{th}(2r_0-r)^2}{\mathrm{th}(2r_0-r)} \leq \frac{1-\mathrm{th}(r_0+1)^2}{\mathrm{th}(r_0+1)} \leq \frac{1-\mathrm{th}(r_0+1)^2}{\mathrm{th}(r_0+1)}\left(\frac{1-\mathrm{th}(r)^2}{\mathrm{th}(r)}\frac{\mathrm{th}(r_0)}{1-\mathrm{th}(r_0)^2}\right). \tag{4.50}$$

To use this inequality, we bound the terms depending on r_0. This can be done easily be noticing that $\mathrm{th}(t) = 1 + O(e^{-t})$ for $t \to +\infty$. Hence,

$$\frac{1-\mathrm{th}(r_0+1)^2}{\mathrm{th}(r_0+1)}\frac{1-\mathrm{th}(r)^2}{\mathrm{th}(r)}\frac{\mathrm{th}(r_0)}{1-\mathrm{th}(r_0)^2} \underset{t\to+\infty}{=} \frac{O(e^{-(r+1)})}{1+O(e^{-(r+1)})}\frac{1+O(e^{-r})}{O(e^{-r})} = O(1). \tag{4.51}$$

This yields the following upper bound :

$$\frac{1-\mathrm{th}(2r_0-r)^2}{\mathrm{th}(2r_0-r)} \leq C\frac{1-\mathrm{th}(r)^2}{\mathrm{th}(r)}, \tag{4.52}$$

where $C > 0$ is an absolute constant. Using this last upper bound and (4.39) in (4.47) gives the following inequality :

$$\int_{r \geq r_0} |\chi(r - r_0)\nabla (f(2r_0 - r, \theta))|^p \mathrm{d}\nu(r, \theta)$$

$$\leq C(p,d) \int_{r_0-1}^{r_0} \int_{\theta} \left[(\partial_\rho f(r,\theta))^2 + \sum_{i=1}^{d-1} \left(\frac{1 - \mathrm{th}(r)^2}{\mathrm{th}(r)} \frac{1}{\sqrt{g_i(\theta)}} \partial_{\theta_i} f(r,\theta) \right)^2 \right]^{p/2} \mathrm{d}S_r(\theta)\mathrm{d}r. \tag{4.53}$$

And this last integral is that of $|\nabla f|^p$ on $\{r_0 \leq r \leq r_0\}$ so that we have

$$\int_{r \geq r_0} |\chi(r - r_0)\nabla (f(2r_0 - r, \theta))|^p \mathrm{d}\nu(r, \theta) \leq C(p,d) \int_{B(r_0)} |\nabla f|^p \mathrm{d}\nu. \tag{4.54}$$

The combination of inequalities (4.41) and (4.54) in inequality (4.30) finally gives the desired result :

$$\left[\int_{\mathbb{H}^d} |\nabla(Ef)(x)|^p \mathrm{d}\nu(x)\right]^{1/p} \leq (1 + C(p,d)) \left[\int_{r \leq r_0} |\nabla f|^p \mathrm{d}\nu\right]^{1/p} + C'(p,d) \left[\int_{r \leq r_0} |f|^p \mathrm{d}\nu\right]^{1/p} \tag{4.55}$$

$$\leq C(p,d) \left[\|\nabla f\|_{L^p(B(r_0))} + \|f\|_{L^p(B(r_0))} \right]. \tag{4.56}$$

□

This result is limited to balls with radius $r_0 \geq 1$. However, the ball $B(1) \subset \mathbb{H}^d$ is quasi-isometric to the Euclidean ball $B(1) \subset \mathbb{R}^d$: there is a constant $\epsilon(d) > 0$ and a a diffeomorphism $F : B_{\mathbb{H}^d}(1) \longrightarrow B_{\mathbb{R}^d}(1)$ such that

$$1 - \epsilon(d) \leq |\nabla F| \leq 1 + \epsilon(d). \tag{4.57}$$

Using this quasi-isometry allows us to transfer any extension operator on Euclidean balls of radius $r \leq 1$ to hyperbolic balls of radius $r \leq 1$ (see remark 4.8). Therefore, the following theorems holds :

Theorem 4.9 (Extension Operator on Balls)**.** *Let $r_0 > 0$ and $f \in C^\infty(\bar{B}(r_0))$. Then there is an extension operator E such that the function $Ef : \mathbb{H}^d \longrightarrow \mathbb{R}$ is C^∞-smooth, compactly supported in $\mathbb{H}^d$ and satisfies*

$$\|\nabla(Ef)\|_{L^p(\mathbb{H}^d)} \leq C(d,p) \left(\|\nabla f\|_{L^p(B(r_0))} + \|f\|_{L^p(B(r_0))}\right). \tag{4.58}$$

4.2.4 Proof of Poincaré's Inequality

We are ready to prove Poincaré's inequality in the hyperbolic space.

Theorem 4.10 (Poincarés Inequality in $\mathbb{H}^d$)**.** *Let $r_0 > 0$ the the radius of the ball $B(r_0) \subset \mathbb{H}^d$, $1 \leq p < n$, and let $f \in W^{1,p}(B(r_0))$. Then*

$$\left(\int_{B(r_0)} |f(x) - (f)|^{p^*} d\nu(x)\right)^{1/p^*} \leq C(p,d)(r_0^p + 1)e^{(d-1)r_0} \int_{B(r_0)} |\nabla f(x)| d\nu(x), \tag{4.59}$$

where (f) is the mean value of f on the ball $B(r_0)$.

Proof. As all the integrals in (4.59) are continuous with respect to f in the $W^{1,p}(B(r_0))$ topology, it suffices to prove the theorem for $f \in C^\infty(B(r_0))$.

Set $g = f - (f)$ and $p^* = \frac{d}{d-1}$. The extension theorem 4.9 provides us with an extension of g on the whole space $Eg : \mathbb{H}^d \longrightarrow \mathbb{R}$ with controlled Sobolev norm

$$\|\nabla(Eg)\|_{L^1(\mathbb{H}^d)} \leq C(d) \left(\|\nabla g\|_{L^1(B(r_0))} + \|g\|_{L^1(B(r_0))}\right). \tag{4.60}$$

We apply the (hyperbolic) GNS inequality (4.7) to the function Eg :

$$\left(\int_{B(r)} |f-(f)|^{p^*} \mathrm{d}\nu\right)^{1/p^*} \leq \left(\int_{\mathbb{H}^d} |Eg|^{p^*} \mathrm{d}\nu\right)^{1/p^*} \tag{4.61}$$

$$\leq C(d) \left(\int_{\mathbb{H}^d} |Eg|^p \mathrm{d}\nu\right)^{1/p} \tag{4.62}$$

$$\leq C(d) \left[\left(\int_{B(r_0)} |g|^p \mathrm{d}\nu\right)^{1/p} + \left(\int_{B(r_0)} |\nabla g|^p \mathrm{d}\nu\right)^{1/p}\right] \tag{4.63}$$

$$= C(d) \left[\left(\int_{B(r_0)} |f-(f)|^p \mathrm{d}\nu\right)^{1/p} + \left(\int_{B(r_0)} |\nabla f|^p \mathrm{d}\nu\right)^{1/p}\right] \tag{4.64}$$

To conclude, we must bound the integral $\left(\int_{B(r_0)} |f-(f)|^p \mathrm{d}\nu\right)^{1/p}$ from above by an integral involving ∇f. We use the following technical lemma :

Lemma 4.11. *Let $r > 0$, $x \in \mathbb{H}^d$ and $z \in B(x,r)$. Recall that δ notes the geodesic distance on $\mathbb{H}^d$. Then,*

$$\int_{B(x,r)} |f(y)-f(z)|^p \, d\nu(y) \leq C(p,d) r^{p-1} e^{2(d-1)r} \int_{B(x,r)} |\nabla f(w)|^p \, sh(\delta(w,z))^{1-d} \, d\nu(w) \tag{4.65}$$

We postpone the proof of this lemma and finish proving the Poincaré inequality. Set $B(r_0) = B(x, r_0)$.

$$\int_{B(r_0)} |f(y)-(f)| \mathrm{d}\nu(y) = \int_{B(r_0)} \left| f(y) - \frac{1}{\nu(B(r_0))} \int_{B(r_0)} f(z) \mathrm{d}\nu(z) \right| \mathrm{d}\nu(y) \tag{4.66}$$

$$\leq \frac{1}{\nu(B(r_0))} \int_{y \in B(r_0)} \int_{z \in B(r_0)} |f(y)-f(z)|^p \mathrm{d}\nu(z) \mathrm{d}\nu(y). \tag{4.67}$$

The technical lemma 4.11 provides us with the following upper bound :

$$\int_{B(r)} |f(y) - (f)| \mathrm{d}\nu(y)$$
$$\leq C(p,d) \frac{r_0^{p-1} e^{2(d-1)r_0}}{\nu(B(r_0))} \int_{y \in B(r_0)} \int_{z \in B(r_0)} |\nabla f(w)|^p \operatorname{sh}(\delta(z,w))^{1-d} \mathrm{d}\nu(z) \mathrm{d}\nu(w) \quad (4.68)$$
$$\leq C(p,d) \frac{r_0^{p-1} e^{2(d-1)r_0}}{\nu(B(r_0))} \int_{w \in B(r_0)} |\nabla f(w)|^p \left(\int_{z \in B(r_0)} \operatorname{sh}(\delta(z,w))^{1-d} \mathrm{d}\nu(z) \right) \mathrm{d}\nu(w) \quad (4.69)$$

The integral in z can be evaluated independently of w. We use polar coordinates $r, \theta_1, ..., \theta_{d-1}$ where $r = \delta(w,z)$.

$$\int_{z \in B(r_0)} \operatorname{sh}(\delta(z,w))^{1-d} \mathrm{d}\nu(z) \leq \int_0^{r_0} \int_\theta \operatorname{sh}(r)^{1-d} \mathrm{d}S_r(\theta) \mathrm{d}r, \quad (4.70)$$

and since $\frac{1}{\operatorname{sh}(r)^{d-1}} \mathrm{d}S_r(\theta) = \mathrm{d}S_1(\theta)$,

$$\int_0^{r_0} \int_\theta \operatorname{sh}(r)^{1-d} \mathrm{d}S_r(\theta) \mathrm{d}r = \int_0^{r_0} \int_\theta \mathrm{d}S_1(\theta) \mathrm{d}r = r_0 \nu(B(1)) = r_0 C(d). \quad (4.71)$$

And hence

$$\int_{B(r)} |f(y) - (f)| \mathrm{d}\nu(y) \leq C(p,d) \frac{r_0^p e^{2(d-1)r_0}}{\nu(B(r_0))} \int_{B(r_0)} |\nabla f|^p \mathrm{d}\nu. \quad (4.72)$$

Finally, we note that, for $r_0 \to +\infty$,

$$\nu(B(r_0)) = C(d) \int_0^{r_0} \operatorname{sh}(r)^{d-1} \mathrm{d}r \sim C(d) \int_0^{r_0} e^{(d-1)r} \mathrm{d}r \sim C(d) e^{(d-1)r} \quad (4.73)$$

so that

$$\int_{B(r)} |f(y) - (f)| \mathrm{d}\nu(y) \leq C(d,p) r_0^p e^{(d-1)r_0} \int_{B(r_0)} |\nabla f|^p \mathrm{d}\nu. \quad (4.74)$$

This and equation (4.64) ends the proof :

$$\left(\int_{B(r_0)} |f(x) - (f)|^{p^*} \mathrm{d}\nu(x) \right)^{1/p^*} \leq C(p,d)(r_0^p + 1) e^{(d-1)r_0} \int_{B(r_0)} |\nabla f(x)| \mathrm{d}\nu(x). \quad (4.75)$$

□

We now prove the technical lemma, mainly by using the mean value inequality on a geodesic curve linking z to y.

Proof of the technical lemma 4.11. Let $y \in B(x,r)$ and let $\gamma_{zy}(t)$ be the geodesic curve between z and y with $0 \leq t \leq 1$ so that γ_{zy} has constant speed equal to $\delta z, y$. Now set $\phi(t) = f(\gamma_{zy}(t))$. The derivative of ϕ is

$$\phi'(t) = \left\langle \nabla f(\gamma_{zy}(t)) | \gamma'_{zy}(t) \right\rangle. \quad (4.76)$$

The Cauchy-Schwarz inequality in the tangent space $\mathcal{T}_{\gamma_{zy}(t)}(\mathbb{H}^d)$ yields

$$|f(y) - f(z)|^p \le \left(\int_0^1 |\nabla f(\gamma_{zy}(t))| \delta(y, z) \mathrm{d}t \right)^p \tag{4.77}$$

$$\le \delta(z, y)^p \int_0^1 |\nabla f(\gamma_{zy}(t))|^p \mathrm{d}t. \tag{4.78}$$

The last inequality can be obtained by using Riemann sums for the integral, for example.

Let $s > 0$ and for $\rho > 0$ let S_ρ be the Riemannian measure on the sphere $\partial B(\rho)$. Then integrating (4.78) over $y \in B(x, r) \cap \partial B(z, s)$ yields

$$\int_{B(x,r)\cap\partial B(z,s)} |f(y) - f(z)|^p \mathrm{d}S_s(y) \le \int_0^1 s^p \int_{B(x,r)\cap\partial B(z,s)} |\nabla f(\gamma_{zy}(t))|^p \mathrm{d}S_s(y) \mathrm{d}t \tag{4.79}$$

As the hyperbolic space is isotropic, $\mathrm{d}S_s(y) = \left(\frac{\mathrm{sh}(s)}{\mathrm{sh}(st)}\right)^{d-1} \mathrm{d}S_{st}(\gamma_{zy}(t))$. Therefore, if we set $w = \gamma_{zy}(t)$, we get $\delta(w, z) = st$ and

$$\int_{B(x,r)\cap\partial B(z,s)} |f(y) - f(z)|^p \mathrm{d}S_s(y)$$

$$\le s^p \,\mathrm{sh}(s)^{d-1} \int_0^1 \int_{B(x,r)\cap\partial B(z,st)} |\nabla f(w)| \frac{1}{\mathrm{sh}(st)^{d-1}} \mathrm{d}S_{st}(w) \mathrm{d}t \tag{4.80}$$

$$\le s^p \,\mathrm{sh}(s)^{d-1} \int_0^1 \int_{B(x,r)\cap\partial B(z,st)} |\nabla f(w)| \,\mathrm{sh}(\delta(z, w))^{1-d} \mathrm{d}S_{st}(w) \mathrm{d}t \tag{4.81}$$

The coarea formula allows us to evaluate this last integral in terms of an integral on an open domain :

$$\int_0^1 \int_{B(x,r)\cap\partial B(z,st)} |\nabla f(w)| \,\mathrm{sh}(\delta(z, w))^{1-d} \mathrm{d}S_{st}(w) \frac{1}{s} \mathrm{d}(st)$$

$$= \frac{1}{s} \int_{B(x,r)\cap B(z,s)} |\nabla f(w)|^p \,\mathrm{sh}(\delta(z, w))^{1-d} \mathrm{d}\nu(w). \tag{4.82}$$

so that

$$\int_{B(x,r)\cap\partial B(z,s)} |f(y) - f(z)|^p \mathrm{d}S_s(y)$$

$$\le s^{p-1} \,\mathrm{sh}(s)^{d-1} \int_{B(x,r)\cap B(z,s)} |\nabla f(w)|^p \,\mathrm{sh}(\delta(z, w))^{1-d} \mathrm{d}\nu(w). \tag{4.83}$$

Finally, we integrate this over $s \in [0, 2r]$: the coarea formula gives once more

$$\int_0^{2r} \int_{B(x,r)\cap\partial B(z,s)} |f(y)-f(z)|^p \mathrm{d}S_s(y) = \int_{B(x,r)} |f(y)-f(z)|^p \mathrm{d}\nu(y) \quad (4.84)$$

$$\leq \int_0^{2r} s^{p-1}\,\mathrm{sh}(s)^{d-1} \int_{B(x,r)\cap B(z,s)} |\nabla f(w)|^p\,\mathrm{sh}(\delta(z,w))^{1-d}\mathrm{d}\nu(w)\mathrm{d}s \quad (4.85)$$

$$\leq \left(\int_{B(x,r)\cap B(z,s)} |\nabla f(w)|^p\,\mathrm{sh}(\delta(z,w))^{1-d}\mathrm{d}\nu(w)\right)\int_0^{2r} s^{p-1}\,\mathrm{sh}(s)^{d-1}\mathrm{d}s, \quad (4.86)$$

and we conclude by noticing that since $s^{p-1}\,\mathrm{sh}(s)^{d-1} \sim s^{p-1}e^{(d-1)r} \notin L^1$ for $r \to +\infty$,

$$\int_0^{2r} s^{p-1}s^{p-1}\,\mathrm{sh}(s)^{d-1}\mathrm{d}s \sim \int_0^{2r} s^{p-1}e^{(d-1)r}\mathrm{d}r \sim C(p)re^{2(d-1)r}. \quad (4.87)$$

□

4.3 Relative Isoperimetric Inequalities

We now dispose of a GNS and a Poincaré inequalities that are valid in the hyperbolic space. The Euclidean GNS inequality is very much the same as the hyperbolic one, so the relative isoperimetric inequality outside a ball will be similar to the one we have proved in the Euclidean space. However, the hyperbolic Poincaré inequality is not as good as the Euclidean one so that we cannot hope to have a relative isoperimetric inside balls or in annuli as good as the Euclidean one.

4.3.1 Relative Inequality Outside a Ball

The following relative isoperimetric inequality holds :

Lemma 4.12. *Let $r > 0$, $B = B(x_0, r_0)$ be a ball of $\mathbb{H}^d$ and $\Omega \subset \mathbb{H}^d$ be an open bounded Lipschitz domain. Then*

$$\nu(\Omega \cap {}^cB)^{(d-1)/d} \leq C(d)\nu(\partial\Omega \cap \bar{B}), \quad (4.88)$$

where $C(d)$ is a constant depending only on the dimension d.

Proof. The proof follows the same lines as the Euclidean one (lemma 3.5). We first prove using the GNS inequality that, for $u \in \mathcal{D}(\mathbb{H}^d)$,

$$\left(\int_{{}^cB} |u|^{d/(d-1)}\mathrm{d}\nu\right)^{(d-1)/d} \leq C(d)\left[\int_{{}^cB} |\nabla u|\mathrm{d}\nu + \oint_{\partial B} |u|\mathrm{d}S_r\right]. \quad (4.89)$$

To do this, we apply the GNS inequality (4.7) to an approximation $f = f_\epsilon u$ of $\mathbb{1}_{{}^cB}$ exactly as in the proof of lemma 3.5 : if $d(x, \partial B)$ is the distance separating x from the sphere ∂B and $\chi \in C^\infty(\mathbb{R})$ as described in figure 3.1,

$$f_\epsilon(x) = \chi\left(\frac{1}{\epsilon}d(x, \partial B)\right)\mathbb{1}_{{}^cB}(x). \quad (4.90)$$

We next need to take care of the surface integral $\oint_{\partial B} |u| \mathrm{d}S_r$. We show that

$$\oint_{\partial B} |u| \mathrm{d}S_r \leq \int_{{}^cB} |\nabla u| \mathrm{d}\nu. \tag{4.91}$$

We may suppose $x_0 = 0$ and we use (again) polar coordinates $r, \theta_1, ..., \theta_{d-1}$ with $r = \delta(x, 0)$:

$$\oint_{\partial B} |u| \mathrm{d}S_{r_0} \leq \int_{{}^cB} |\nabla u| \mathrm{d}\nu = \int_\theta \left| \int_{r_0}^{+\infty} \frac{\mathrm{d}}{\mathrm{d}r} u(r, \theta) \mathrm{d}r \right| \mathrm{d}S_{r_0}(\theta) \tag{4.92}$$

$$\leq \int_\theta \int_{r_0}^{+\infty} |\nabla u(r, \theta)| \mathrm{d}r \, \mathrm{d}S_{r_0}(\theta) \tag{4.93}$$

$$= \int_{r_0}^{+\infty} \int_\theta |\nabla u(r, \theta)| \left(\frac{\mathrm{sh}(r_0)}{\mathrm{sh}(r)} \right)^{d-1} \mathrm{d}S_r(\theta) \mathrm{d}r \tag{4.94}$$

Now since $r_0 \leq r$ and since sh is an increasing function,

$$\oint_{\partial B} |u| \mathrm{d}S_{r_0} \leq \int_{r_0}^{+\infty} \int_\theta |\nabla u(r, \theta)| \mathrm{d}S_r(\theta) \mathrm{d}r = \int_{{}^cB} |\nabla u| \mathrm{d}\nu. \tag{4.95}$$

Finally, we apply (4.89) to an approximation u_ϵ of $\mathbb{1}_\Omega$, namely

$$u_\epsilon(x) = \chi \left(\frac{1}{\epsilon} d(x, \partial\Omega) \right) \mathbb{1}_\Omega(x). \tag{4.96}$$

This leads to the result. □

4.3.2 Relative Inequality Inside a Ball

A similar relative inequality inside a ball holds. Note that the constant of the lemma depends on the radius of the ball : this can be understood by remembering that the part of the perimeter which intersects the sphere ∂B has a larger area than the inner part of the perimeter.

Lemma 4.13. *Let $r > 0$, $B = B(r_0)$ be a ball of $\mathbb{H}^d$ and $\Omega \subset \mathbb{H}^d$ be an open bounded domain. Then*

$$\min \{\nu(\Omega \cap B), \nu(B \setminus \Omega)\}^{(d-1)/d} \leq C(d)(r_0 + 1) e^{((d-1)r_0)} \nu(\partial\Omega \cap \bar{B}), \tag{4.97}$$

where $C(d)$ depends only on the dimension.

Proof. The proof is identical to the Euclidean one by replacing the constant of the Euclidean Poincaré inequality by the constant $C(d)(r_0 + 1)e^{((d-1)r_0)}$ of the hyperbolic inequality : We apply Poincaré's inequality to an approximation u_ϵ of $\mathbb{1}_B$. See the proof of lemma 3.8 for more details. □

4.3.3 Relative Inequality in Annuli

The proof of a relative inequality in annuli, even if it follow the same lines as that in the Euclidean space (see the proof in section 3.2.3), leads, as we will see, to a very different result. We will note the differences as they appear.

Let $A = A(r, r+l)$ be an an annulus and $\Omega \subset \mathbb{H}^d$ be open Lipschitz domain. We seek an intermediate sphere $\partial B(r+l_0) \subset A$ that does not intersect Ω too much. The coarea formula gives

$$\nu(\Omega \cap A) = \int_r^{r+l} \nu(\Omega \cap \partial B(t)) \mathrm{d}t. \tag{4.98}$$

Now, the existence of a l_0 such that $\nu(\Omega \cap \partial B(r+l_0)) \leq \nu(A \cap \Omega)$ is guaranteed as long as the width of the annulus is large enough $l \geq 1$. We therefore suppose that this is the case.

Difference 1 : we suppose the width l of the annulus A to be at least unity $l \geq 1$.

We set

$$\Omega_1 = \Omega \cap B(r+l_0) \tag{4.99}$$

$$\Omega_2 = \Omega \cap {}^cB(r+l_0) \tag{4.100}$$

Applying lemma 4.12 to the domain Ω_1 relatively to the complement of the ball ${}^cB(r)$ gives

$$\nu(\Omega_1 \cap B(r+l_0))^{(d-1)/d} \leq C(d) \left[\nu(\partial\Omega \cap \bar{A}(r, r+l_0)) + \nu(\Omega \cap A)\right] \tag{4.101}$$

And applying lemma 4.13 to the domain Ω_2 relatively to the ball $B(r+l)$ gives

$$\min\{\nu(\Omega_2 \cap B(r+l))^{(d-1)/d}, \nu(B(r+l) \setminus \Omega_2)\} \leq C(d)(1+r+l)e^{(d-1)(r+l)} \left(\nu(\partial\Omega \cap \bar{A}(r+l_0, r+l))\right). \tag{4.102}$$

To get rid of the minimum term, we suppose that $\nu(\Omega \cap A) \leq \frac{1}{2}\nu(B(r+l))$ and therefore,

$$\nu(\Omega_2 \cap B(r+l))^{(d-1)/d} \leq C(d)(1+r+l)e^{(d-1)(r+l)} \left(\nu(\partial\Omega \cap \bar{A}(r+l_0, r+l))\right). \tag{4.103}$$

Difference 2 : we suppose that Ω is of not too great measure : $\nu(\Omega \cap A) \leq \frac{1}{2}\nu(B(r+l))$.

The combination of inequalities (4.101) and (4.103) gives

$$\left(\frac{1}{2}\nu(\Omega \cap A)\right)^{(d-1)/d} \leq \nu(\Omega_1 \cap B(r+l_0))^{(d-1)/d} + \nu(\Omega_2 \cap B(r+l))^{(d-1)/d} \tag{4.104}$$

$$\leq C(d)(1+r+l)e^{(d-1)(r+l)} \left[\nu(\partial\Omega \cap \bar{A}) + \nu(\Omega \cap A)\right], \tag{4.105}$$

and from this

$$1 \leq C(d)(1+r+l)e^{(d-1)(r+l)} \left[\frac{\nu(\partial\Omega \cap \bar{A})}{\nu(\Omega \cap A)^{(d-1)/d}} + \nu(\Omega \cap A)^{1/d}\right]. \tag{4.106}$$

We will get an isoperimetric inequality of the desired form provided that the term $\nu(\Omega \cap A)^{1/d}$ is sufficiently small.

Difference 3 : more precisely, we request that $(1+r+l)e^{(d-1)(r+l)}\nu(\Omega\cap A)^{1/d}$ be small when compared to 1 : for some constant $\alpha(d)$,

$$\nu(\Omega \cap A)^{1/d} \leq \frac{\alpha(d)}{(1+r+l)e^{(d-1)(r+l)}}. \tag{4.107}$$

We then have the relative isoperimetric inequality

$$\nu(\Omega \cap A) \leq C(d)(1+r+l)e^{(d-1)(r+l)}\nu(\partial\Omega \cap \bar{A}). \tag{4.108}$$

We have proven the following lemma :

Lemma 4.14. *There are constants $\alpha(d), c(d) > 0$ such that for all open Lipschitz domain $\Omega \subset \mathbb{H}^d$ and for all annulus $A = A(r, r+l)$ satisfying the following conditions,*

1. *the width l of the annulus is greater than unity : $l \geq 1$,*
2. *the domain $\Omega \cap A$ is not to large when compared to that of the annulus : $\nu(\Omega \cap A) \leq \frac{1}{2}\nu(B(r+l))$,*
3. *we have $\nu(\Omega \cap A)^{1/d} \leq \frac{\alpha(d)}{(1+r+l)e^{(d-1)(r+l)}}$,*

the following relative isoperimetric inequality holds :

$$c(d)\nu(\Omega \cap A) \leq (1+r+l)e^{(d-1)(r+l)}\nu(\partial\Omega \cap \bar{A}). \tag{4.109}$$

Obviously, this lemma is a lot less useful than its Euclidean equivalent. However it is possible to continue replicating the proof of [2] and obtain some form of geometric control of the Steklov eigenvalues.

4.4 Test Functions

In this section, we use the relative isoperimetric inequality above to construct test functions. As stated before, lemma 4.14 is not as good as its Euclidean counterpart, so it will not be possible to construct test functions on any annulus as we did in chapter 2. Only *some* annuli will support adequate test functions : our goal is to determine which ones do.

As in section 2.1, we take an annulus $A = A_0 = A(r, r+l)$, an open Lipschitz domain $\Omega \subset \mathbb{H}^d$, which we suppose to be connected, and a $\lambda > 0$. We suppose that there are no $\phi \in H_0^1(A)$ such that $\phi \neq 0$ in $L^2(\Omega)$ and such that

$$\mathcal{R}(\phi) = \frac{\int_\Omega |\nabla\phi|^2 \mathrm{d}\nu}{\oint_{\partial\Omega} \phi^2 \mathrm{d}\nu} \leq \lambda. \tag{4.110}$$

Let $0 < t < \frac{1}{2}l$ and set

$$\phi_0(x) = \min\left\{1, \frac{1}{t}d(x, {}^cA)\right\} \in H_0^1(A). \tag{4.111}$$

Our hypothesis applied to ϕ_0 yields

$$\lambda < \frac{m(t)}{t^2 p(t)}, \tag{4.112}$$

where $m(t)$ and $p(t)$ are defined exactly as in section 2.1 :

$$m(t) = \nu(\Omega \cap (A(r, r+t) \cap A(r+l-t, r+l))), \tag{4.113}$$

$$p(t) = \nu(\partial\Omega \cap \bar{A}(r+t, r+l-t)). \tag{4.114}$$

We now wish to use the relative isoperimetric inequality in order to link $m(t)$ and $p(t)$. To do this, we insure that all three conditions of lemma 4.14 are satisfied for $A_t = A(r+t, r+l-t)$ and Ω. In fact, we make the more general suppositions

1. the width $l - 2t$ of A_1 is at least 1, or equivalently $t \leq \frac{1}{2}(l-1)$,
2. $\nu(\Omega \cap A) \leq \frac{1}{2}\nu(B(r+l))$,
3. $\nu(\Omega \cap A)^{1/d} \leq \frac{\alpha(d)}{(1+r+l)e^{(d-1)(r+l)}}$.

Note that the fulfillment of the last two conditions for A imply their fulfillment for *any* intermediate annulus $A_t = A(r+t, r+l-t)$. Therefore if we suppose that $A = A_0$ and Ω satisfy these conditions, we need only care that the first one is satisfied in A_t to apply the relative inequality of lemma 4.14 in any A_t.

Using this and setting $M = \nu(\Omega \cap A)$ yields

$$\lambda c t^2 (M - m(t))^{1/d} \leq (1+r+l)e^{(d-1)(r+l)} m(t), \tag{4.115}$$

We now seek t_1 such that $m(t_1) = \frac{1}{2}M$ and $t_1 \leq \frac{1}{2}(l-1)$. If there were no such t_1, then we would have $m(t) < \frac{1}{2}M$ for $t \leq \frac{1}{2}(l-1)$ and hence

$$\lambda c \left(\frac{l-1}{2}\right)^2 \left(\frac{M}{2}\right)^{1/d} \leq (1+r+l)e^{(d-1)(r+l)} \frac{M}{2} \tag{4.116}$$

or equivalently,

$$\frac{l-1}{2} > 2^{-1/2d} \sqrt{\frac{M^{1/d}}{\lambda c}(1+r+l)e^{(d-1)(r+l)}}. \tag{4.117}$$

In order for this never to happen, we suppose

$$\frac{l-1}{2} > 2^{-1/2d} \sqrt{\frac{M^{1/d}}{\lambda c}(1+r+l)e^{(d-1)(r+l)}}. \tag{4.118}$$

Moreover, using (4.115) again provides us with an upper bound for t_1 :

$$t_1 \leq 2^{-1/2d} \sqrt{\frac{M^{1/d}}{\lambda c}(1+r+l)e^{(d-1)(r+l)}}. \tag{4.119}$$

Let $t_1 \leq \frac{l-1}{2}$ such that $m(t_1) = \frac{1}{2}M$. We now construct in the same way, and in the annulus $A_1 = A_{t_1}$, a t_2 such that $t_1 + t_2 \leq \frac{l-1}{2}$ and $m(t_2) = \frac{1}{2^2}M$. In order to insure the existence of such a t_2, we must be able to use lemma 4.14

in the annulus A_1. This is possible because the width $1-2t_1$ of the annulus satisfies $1-2t_1 \leq \frac{l-1}{2}$ in virtue of (4.119).

Let, for $t \leq \frac{l-1}{2} - t_1$,

$$\phi_1(x) = \min\left\{1, \frac{1}{t} d(x, {}^c A_1)\right\} \in H_0^1(A). \tag{4.120}$$

Then, as before, if $m_1(t) = \nu(\Omega \setminus A(r+t_1+t, r+l-t_1-t))$ and $M_1 = \frac{1}{2}M = \nu(\Omega \cap A_1)$,

$$\lambda c t^2 (M_1 - m_1(t))^{1/d} \leq (1+r+l) e^{(d-1)(r+l)} m_1(t). \tag{4.121}$$

We then have $t_2 \leq \frac{l-1}{2} - t_1$ such that $m_2(t_2) = \frac{1}{2^2} M$ as long as we suppose that

$$\frac{l-1}{2} > (2^{-1/2d} + 2^{-2/2d}) \sqrt{\frac{M^{1/d}}{\lambda c} (1+r+l) e^{(d-1)(r+l)}}. \tag{4.122}$$

We proceed in that way infinitely many times by an induction : provided we suppose that

$$\frac{l-1}{2} > \sum_{k=1}^{\infty} 2^{-k/2d} \sqrt{\frac{M^{1/d}}{\lambda c} (1+r+l) e^{(d-1)(r+l)}}, \tag{4.123}$$

then we construct $t_\infty = t_1 + t_2 + \cdots$ such that

$$t_\infty \leq \sum_{k=1}^{\infty} 2^{-k/2d} \sqrt{\frac{M^{1/d}}{\lambda c} (1+r+l) e^{(d-1)(r+l)}} < \frac{l-1}{2} \tag{4.124}$$

and

$$\nu(A(r+t_\infty, r+l-t_\infty) \setminus \Omega) = 0, \tag{4.125}$$

which is absurd, since we have supposed Ω to be connected.

We sum up this discussion. We have shown that if the three following conditions are satisfied,

1. $\frac{l-1}{2} > A(d) \sqrt{\frac{\nu(A\cap\Omega)^{1/d}}{\lambda} (1+r+l) e^{(d-1)(r+l)}}$, where $A(d) = \frac{1}{\sqrt{c(d)}} \sum_{k\geq 1} 2^{-k/2d}$,
2. $\nu(\Omega \cap A) \leq \frac{1}{2} \nu(B(r+l))$
3. $\nu(\Omega \cap A)^{1/d} \leq \frac{\alpha(d)}{(1+r+l)e^{(d-1)(r+l)}}$,

then there is a function $\phi \in H_0^1(A)$ such that $\mathcal{R}(\phi) \leq \lambda$. Now suppose condition 3. is satisfied. Then condition 1. is implied by the simpler (and stronger) condition

$$\frac{l-1}{2} > \frac{\beta(d)}{\sqrt{\lambda}}, \tag{4.126}$$

where $\beta(d) = A(d)\alpha(d)$.

Proposition 4.15. *There are two constants $\alpha(d), \beta(d) > 0$ such that for any annulus $A = A(r, r+l)$ and any $\Omega \subset \mathbb{H}^d$ that satisfies :*

1. $\nu(\Omega \cap A) \leq \frac{1}{2}\nu(B(r+l))$
2. $\nu(\Omega \cap A)^{1/d} \leq \frac{\alpha(d)}{(1+r+l)e^{(d-1)(r+l)}}$,

there is a function $\phi \in H_0^1(A)$ with $\phi \neq 0 \in L^2(\partial\Omega)$ and

$$\mathcal{R}(\phi) = \frac{\int_\Omega |\nabla\phi|^2 \, d\nu}{\oint_{\partial\Omega} \phi^2 \, d\nu} \leq \frac{\beta(d)}{(l-1)^2}. \tag{4.127}$$

4.5 Conclusion

Proposition 4.15 is obviously not as good as its Euclidean counterpart, lemma 2.2. Nevertheless, we may obtain some geometric control of the spectrum by using it.

For example, obtaining an upper bound for the first (nonzero) eigenvalue $\sigma_1(\Omega)$ requires two test functions. Therefore, to use proposition 4.15, we need to find two annuli which satisfy the conditions therein. Note $\delta = \text{diam}(\Omega)$ and suppose that

$$\nu(\Omega) \leq \min\left\{\frac{1}{2}\omega_d \,\text{sh}(\delta)^{d-1}, \left(\frac{\alpha}{(1+\delta)e^{(d-1)\delta}}\right)^d\right\}, \tag{4.128}$$

where $\omega_d \,\text{sh}(r)^{d-1}$ is the volume of a hyperbolic ball of radius r. Then both the annuli $A_1 = A(0, \delta/2)$ and $A_2 = A(\delta/2, \delta)$ have width $\frac{1}{2}\delta$ and satisfy the conditions of proposition 4.15. Therefore, there are two functions $\phi_1 \in H_0^1(A_1), \phi_2 \in H_0^1(A_2)$ with disjoint support and such that

$$\mathcal{R}(\phi_i) \leq \frac{\beta(d)}{(\delta/2-1)^2} \leq \frac{C(d)}{\delta^2}, \qquad i = 1, 2, \tag{4.129}$$

so that

$$\sigma_1(\Omega) \leq \frac{C(d)}{\delta^2}. \tag{4.130}$$

In fact, we can generalize this to any number of annuli $A_i = A\left(\frac{i\delta}{k+1}, \frac{(i+1)\delta}{k+1}\right)$, $i = 0, ..., k$.

Theorem 4.16. *Let $\Omega \subset \mathbb{H}^d$ be a bounded open Lipschitz set of diameter δ. Suppose that Ω is connected and that*

$$\nu(\Omega) \leq \min\left\{\frac{1}{2}\omega_d \, sh(\delta)^{d-1}, \left(\frac{\alpha}{(1+\delta)e^{(d-1)\delta}}\right)^d\right\}, \tag{4.131}$$

where $\alpha(d)$ is the constant of proposition 4.15 and $\omega_d \, sh(r)^{d-1}$ the volume of a hyperbolic ball of radius r. Then,

$$\forall k \geq 1, \qquad \sigma_k(\Omega) \leq C(d)\left(\frac{k}{\delta}\right)^2, \tag{4.132}$$

for some constant $C(d)$ depending only on the dimension.

This theorem provides some sort of isodiametric control, only restricted to domains with exceedingly small measure relatively to the diameter[3]. Note that the power k^2 is even worse than that of the Euclidean isodiametric control of [2], but is consistent with the Weyl law (*id est* it doesn't contradict the Weyl law which provides a power $k^{1/(d-1)}$).

[3] *Facilius est camelum per foramen acus...*

Chapter 5

Discussion over the Intrinsic Diameter

Non venire non videreque. Hoc est optimus modus, ad non vincamur.[1]

Warning : the material contained in this chapter is the result of a small research work done during the internship, and is therefore liable to be heavily burdened with mistakes and errors of all kinds.

5.1 Motives

In all this chapter, we are exclusively concerned with Euclidean domains $\Omega \subset \mathbb{R}^d$.

The isodiametric control of the Steklov eigenvalues in [2] is done using the *extrinsic* diameter $\mathrm{diam}(\Omega)$ of the domain $\Omega \subset \mathbb{R}^d$,

$$\mathrm{diam}(\Omega) = \sup_{x,y\in\Omega} |x - y|. \tag{5.1}$$

One may ask if the isodiametric inequality is also true when using the *intrinsic* diameter $\mathrm{D}(\Omega)$ of Ω,

$$\mathrm{D}(\Omega) = \sup_{x,y\in\Omega} d(x, y) \tag{5.2}$$

where $d(x, y)$ is the geodesic distance between x and y defined as

$$d(x, y) = \inf_{\gamma} \mathrm{length}(\gamma). \tag{5.3}$$

In the infimum above γ ranges though all (continuous) piecewise C^1 curves linking x to y. This infimum is achieved by a curve $\gamma : [0, 1] \to \bar{\Omega}$ (in fact, possibly many) which are at least as regular as the boundary $\partial\Omega$ of the domain (see figure 5.1). Moreover, the supremum (5.2) is also achieved by a couple of points $(x, y) \in \partial\Omega \times \partial\Omega$ (in fact possibly many).

[1]Translated from R. Goscinny and A. Uderzo. *Asterix et les normands.*

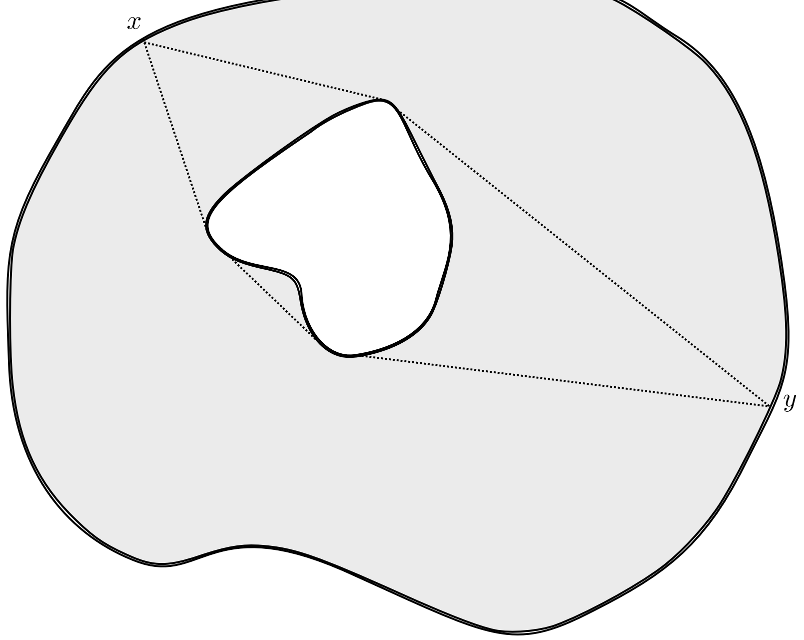

Figure 5.1 – There may be multiple geodesics linking two points.

Question : Suppose Ω connected. Is it true that, for some powers $\alpha(d), \beta(d) > 0$,

$$\sigma_k(\Omega) \leq C(d)\frac{k^\alpha}{\mathrm{D}(\Omega)^\beta} \ ? \tag{5.4}$$

This question is motivated by the fact that (5.4) is (almost trivially) always true in dimension 2.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be an open Lipschitz connected domain. Then*

$$\forall k \geq 1, \qquad \sigma_k(\Omega) \leq C(d)\frac{k}{D(\Omega)}. \tag{5.5}$$

Proof. Let $x, y \in \partial\Omega$ be extremal points for the geodesic distance. The points x and y necessarily are on the same connected component of $\partial\Omega$. Therefore, we can chose a path

$$\gamma : [0,1] \longrightarrow \partial\Omega \tag{5.6}$$

remaining on the boundary and linking x to y. Now, since γ covers only a part of the boundary $\partial\Omega$, we have

$$\mathrm{D}(\Omega) \leq \mathrm{length}(\gamma) \leq |\partial\Omega|. \tag{5.7}$$

We now use the isoperimetric control of the spectrum (theorem 2.5) which provides the result :

$$\sigma_k(\Omega) \leq C(d)\frac{k}{|\partial\Omega|} \leq \frac{k}{\mathrm{D}(\Omega)}. \tag{5.8}$$

□

Unfortunately, this proof is specific to dimension $d = 2$.

5.2 Geodesic Annuli

We might hope to adapt the method of [2] to obtain isodiamteric control of the spectrum for the intrinsic diameter. However, things start going wrong when constructing test functions.

Let $A = A(r, r+l)$ be a annulus for the geodesic distance of a domain $\Omega \subset \mathbb{R}^2$ and set (as in chapter 2) for some $t < \frac{1}{2}l$,

$$\phi(x) = \min\left\{1, \frac{1}{t}d(x, {}^cA)\right\}. \tag{5.9}$$

The Rayleigh quotient of ϕ also satisfies inequality (2.8) :

$$\mathcal{R}(\phi) \leq \frac{\int_\Omega |\nabla\phi|^2 \mathrm{d}x}{\oint_{\partial\Omega} \phi^2 \mathrm{d}S} \leq \frac{m(t)}{t^2 p(t)}, \tag{5.10}$$

where $m(t)$ and $p(t)$ are defined as before :

$$m(t) = |\Omega \cap A \diagdown A(r+t, r+l-t)|, \tag{5.11}$$

$$p(t) = |\partial\Omega \cap \bar{A}(r+t, r+l-t)|. \tag{5.12}$$

In order to proceed as in chapter 2, we must dispose of some relative isoperimetric inequality in the annulus $A_t = A(r+t, r+l-t)$. Let $M = |\Omega \cap A|$.

Question : is it true that for t small enough and some $c = c(d)$, $c(M - m(t))^{(d-1)/d} \leq p(t)$?

The answer to this question is negative. It is possible to construct domains which prove this inequality false. We thank Lucas Dahinden for the following example.

Take an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ and two points $x_0 \in \mathrm{Int}(\Omega)$ and $y \in \partial\Omega$ such that the line $[x_0, y]$ lies entirely in Ω. Let ϵ small enough. Then, around the line $[x_0, y]$, subtract from Ω a cylinder of width 2ϵ and add a cylinder of width ϵ prolonged by a long tube of extremity x (see figure 5.2). We call Ω_ϵ the resulting domain.

Now since x is an extremal point of Ω_ϵ, the annuli $A(r, r+l)$ (for the geodesic distance d) are such as described in figure 5.3 intersect the boundary $\partial\Omega$ with measure of order $l\epsilon^{d-2}$ but has a large measure $M - m(t)$. This makes it impossible to obtain inequalities of the desired form

$$\mathrm{Cte}(\Omega) \approx c(M - m(t))^{(d-1)/d} \leq p(t) \approx l\epsilon^{d-2} \to 0, \tag{5.13}$$

as long as $d \geq 3$.

This means that there is little hope of adapting the proof of [2] to the intrinsic diameter. However, the counter-example given by figure 5.3 contains a collapsing passage as described in [10] or in section 1.3.1, so the eigenvalues $\sigma_k(\Omega_\epsilon)$ tend to zero as $\epsilon \to 0^+$. Therefore, we may not deduce from our example

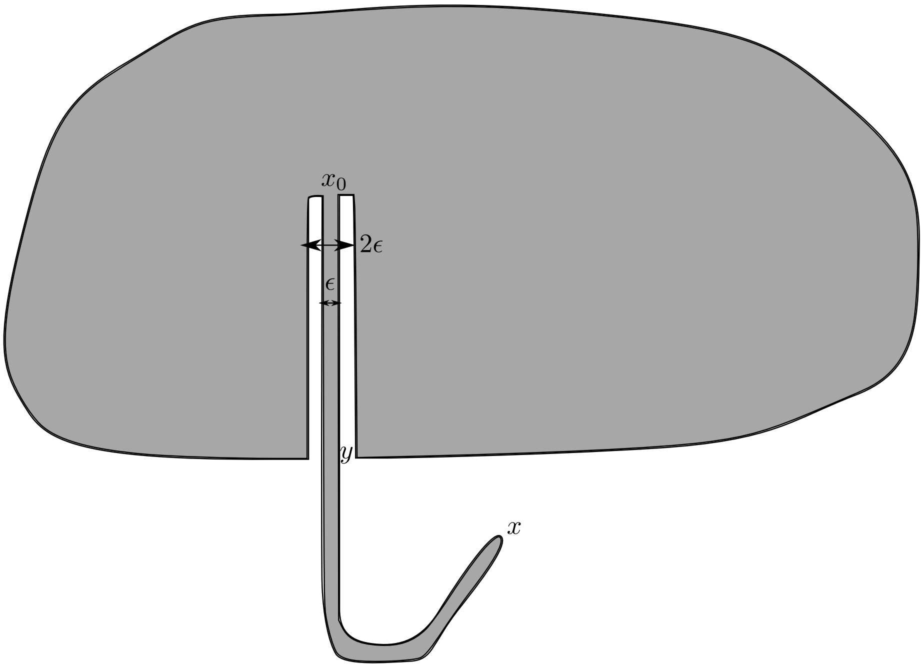

Figure 5.2 – *Dahindens Kartoffel* : a tube of width ϵ leads to x_0 in the interior of Ω and makes sure x is an extremal point of the resulting domain Ω_ϵ .

that the isodiametric inequality is false for the intrinsic diameter.

As the presence of a collapsing passage renders relative inequality impossible, we may wonder wether the converse is true : do domains which don't support relative inequalities $c(M - m(t))^{(d-1)/d} \leq p(t)$ necessarily possess thin collapsing passages ? If so, how thin are these passages ? Is it possible to use both these phenomena to prove some isodiametric control for the intrinsic diameter ?

5.3 Elongated Domains and Fourier Transforms

5.3.1 Radial Functions

This section contains some ideas of how to extract isodiametric inequalities for the first (nonzero) eigenvalue σ_1 on domains whose geometry is well known.

In this section, we shall consider *radial* test functions : functions depending only on the geodesic radius in Ω. If $x_0 \in \partial\Omega$ is an extremal point for the geodesic distance d, we consider functions of the form

$$f(x) = F(d(x_0, x)). \tag{5.14}$$

The advantage in doing this is that such functions can be easily integrates provided we have the knowledge of the *profile* of Ω : we call profile of Ω the function

$$\Theta : \begin{array}{c} [0, \mathrm{D}(\Omega)] \longrightarrow \mathbb{R}^+ \\ r \longmapsto |\partial B(r)| \end{array} , \tag{5.15}$$

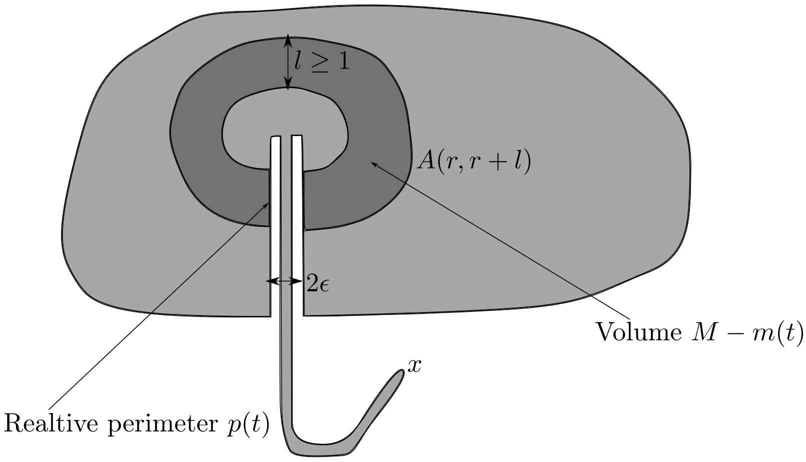

Figure 5.3 – The annulus $A(r, r+l)$ intersects the boundary $\partial\Omega$ with small measure.

where $|\partial B(r)|$ is the measure of the geodesic sphere $\partial B(r) = \{x \in \Omega | d(x_0, x) = r\}$. Then, the coarea formula provides

$$\int_\Omega f(x)\mathrm{d}x = \int_0^{\mathrm{D}(\Omega)} F(r)\Theta(r)\mathrm{d}r \tag{5.16}$$

which is a one dimensional integral. The numerator of the Rayleigh quotient $\mathcal{R}(f)$ is also easy to calculate since d is locally the Euclidean distance $|.|$ in $\mathrm{Int}(\Omega)$:

$$\int_\Omega |\nabla f(x)|^2\mathrm{d}x = \int_0^{\mathcal{D}(\Omega)} |F'(r)|^2\Theta(r)\mathrm{d}r. \tag{5.17}$$

However, the denominator $\oint_{\partial\Omega} f^2\mathrm{d}S$ is not given by the knowledge of the profile Θ. To get around this obstacle, we chose a function F that will make the denominator easy to handle, namely

$$F_z(r) = e^{izr}, \tag{5.18}$$

for some complex parameter $z \in \mathbb{C}$. Since this is a complex valued function, we must of course replace f^2 by $|f|^2$ in the Rayleigh quotient. Then, if $f_z(x) = e^{izd(x_0,x)}$, we have

$$\forall z \in \mathbb{C}, \qquad \mathcal{R}(f_z) = \frac{\int_\Omega |\nabla f_z|^2\mathrm{d}x}{\oint_{\partial\Omega} |f_z|^2\mathrm{d}S} = |z|^2\frac{|\Omega|}{|\partial\Omega|}. \tag{5.19}$$

Finally, for this to give some information about $\sigma_1(\Omega)$, f_z must be orthogonal to the space of constant functions (that is to be of mean value zero). We therefore seek the z such that

$$\int_0^{\mathrm{D}(\Omega)} \Theta(r)e^{izr}\mathrm{d}r = 0. \tag{5.20}$$

5.3.2 Fourier Transforms

Our problem is finding the zeros of the Fourier transform of the profile function :

$$\hat{\Theta}(z) = \int_0^{\mathrm{D}(\Omega)} \Theta(r) e^{izr} \mathrm{d}r. \tag{5.21}$$

Even if some information is known on the zeros of such functions ($\hat{\Theta}$ is an entire function), determining their location is difficult if nothing is known on the profile function.

Some work has been done on Fourier transforms of positive and increasing functions (section 3 in [7]), but, in the most general case, the problem of understanding the location of these zeros is largely unsolved.

We focus on the particular case where $\Theta(r)$ is close to the characteristic function of an interval $\mathbb{1}_{[0,\mathrm{D}(\Omega)]}$ in the L^1 topology, in which case the location of the zero z can be estimated with the knowledge of $\mathrm{D}(\Omega)$.

Theorem 5.2. *There are two absolute*[2] *constants $\epsilon, C > 0$ such that if the profile function satisfies*

$$\| D(\Omega)\Theta\left(t\, D(\Omega)\right) - 1 \|_{L^1(0 \le t \le 1)} \le \epsilon, \tag{5.22}$$

then we dispose of an upper bound of the first (nonzero) eigenvalue

$$\sigma_1(\Omega) \le C \frac{|\Omega|}{|\partial\Omega|\, D(\Omega)^2}. \tag{5.23}$$

Proof. **Step 1 :** we compute the Fourier transform of $\mathbb{1}_{[0,1]}$.

$$\forall a \in \mathbb{C}, \qquad \int_0^1 e^{iat} \mathrm{d}t = e^{ia/2} \frac{\sin(a/2)}{a/2}. \tag{5.24}$$

This function has a real zero $a = 2\pi$.

Step 2 : we prove the following approximation lemma :

Lemma 5.3. *Let $g : [0,1] \longrightarrow \mathbb{C}$ be an integrable function $g \in L^1[0,1]$. There are two absolute constants $\epsilon, \rho > 0$ such that if*

$$\|g - 1\|_{L^1[0,1]} \le \epsilon, \tag{5.25}$$

then the Fourier transform $\hat{g}$ of g,

$$\hat{g}(a) = \int_0^1 g(t) e^{iat}\, dt, \tag{5.26}$$

has a zero in the disk $D(2\pi, \rho)$. Moreover, by taking ϵ small enough, we can chose ρ as small as desired.

[2] A constant is called absolute if it depends on no parameter and could be explicitly and entirely computed if we so wished.

Proof of the lemma. Let

$$G(a) = e^{ia/2}\frac{\sin(a/2)}{a/2}. \tag{5.27}$$

Let $\rho > 0$ to be determined later. The argument principle gives the number of zeros of $\hat{g}$ near $a = 2\pi$ as a curve integral on a circle :

$$\frac{1}{2i\pi}\oint_{C(2\pi,\rho)} \frac{\hat{g}'(a)}{\hat{g}(a)}\mathrm{d}a \in \mathbb{Z}. \tag{5.28}$$

We wish to evaluate this integral by comparing $\hat{g}$ to G.

Both functions $\hat{g}$ and G are entire functions and are uniformly close on the circle $C(2\pi, \rho)$: for $|a - 2\pi| = \rho$,

$$|\hat{g}(a) - G(a)| \le \int_0^1 |1 - g(t)||e^{iat}|\mathrm{d}t \le \|g - 1\|_{L^1[0,1]}e^{2\pi+\rho}. \tag{5.29}$$

$$|\hat{g}'(a) - G'(a)| \le |a| \int_0^1 |1 - g(t)||e^{iat}|\mathrm{d}t \le \|g - 1\|_{L^1[0,1]}|a|e^{2\pi+\rho}. \tag{5.30}$$

The circle $C(2\pi, \rho)$ is compact so both approximations (5.29) and (5.30) yield, for ρ small enough,

$$\frac{1}{2i\pi}\oint_{C(2\pi,\rho)} \frac{\hat{g}'(a)}{\hat{g}(a)}\mathrm{d}a \underset{g\to 1 \text{ in } L^1}{\longrightarrow} \frac{1}{2i\pi}\oint_{C(2\pi,\rho)} \frac{G'(a)}{G(a)}\mathrm{d}a = 1, \tag{5.31}$$

and hence, since both of these integrals are integers, for $\|g - 1\|_{L^1}$ smaller than some $\epsilon > 0$, we have :

$$\frac{1}{2i\pi}\oint_{C(2\pi,\rho)} \frac{\hat{g}'(a)}{\hat{g}(a)}\mathrm{d}a = 1, \tag{5.32}$$

so that $\hat{g}$ has a zero in the disk $D(2\pi, \rho)$.

□

Step 3 : we apply lemma 5.3 to the profile function.

We rescale in order to have an integral on $[0, 1]$, as in the lemma :

$$\hat{\Theta}(z) = \int_0^{\mathrm{D}(\Omega)} \Theta(r)e^{izr}\mathrm{d}r = \mathrm{D}(\Omega)\int_0^1 \Theta(t\,\mathrm{D}(\Omega))e^{izt\,\mathrm{D}(\Omega)}\mathrm{d}t. \tag{5.33}$$

so that there are two absolute constants ϵ and ρ such that $\hat{\Theta}$ has a (unique) zero z_0 in the disk $D\left(\frac{2\pi}{\mathrm{D}(\Omega)}, \frac{\rho}{\mathrm{D}(\Omega)}\right)$ provided

$$\|\mathrm{D}(\Omega)\Theta\left(t\,\mathrm{D}(\Omega)\right) - 1\|_{L^1(0\le t\le 1)} \le \epsilon. \tag{5.34}$$

Step 4 : we use this to construct a test function and obtain the geometric inequality.

Let

$$f(x) = \exp(iz_0 d(x_0, x)). \tag{5.35}$$

The function f is of mean value zero so that, by taking ρ small enough,

$$\sigma_1(\Omega) \leq \mathcal{R}(f) = \frac{1}{|z_0|^2} \frac{|\Omega|}{|\partial\Omega|} \leq C \frac{|\Omega|}{|\partial\Omega|\, \mathrm{D}(\Omega)^2}. \tag{5.36}$$

□

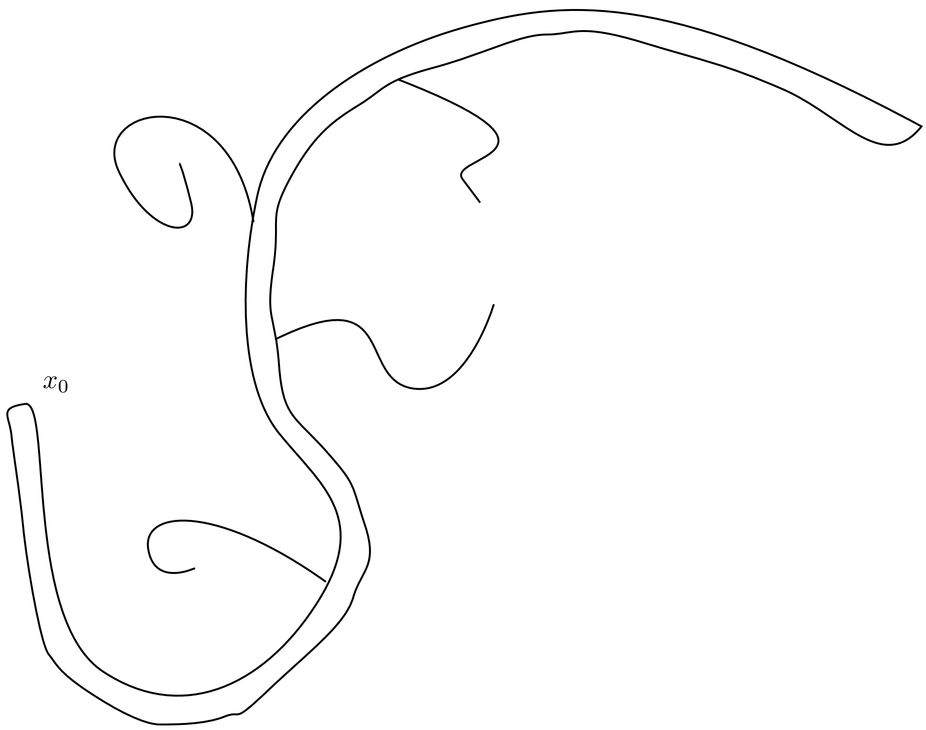

Figure 5.4 – Theorem 5.2 only applies to "elongated" domains, those whose profile is close to the characteristic function of an interval in the L^1 topology. This allows the domain to have parts of small measure without the geometric control to be affected.

The geometric control (5.23) relies on a very restrictive condition, which forces the domain to be more or less thin and tubular. But we know from [10] (or section 1.3.1) that such domains automatically have small eigenvalues. The only difference is that we allow Ω to have small deformations differentiating it from a tubular domain, as long as these are of small measure.

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