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Relative Trace Formulae in Analytic Number Theory

Solution to Set 2

The exercises in this set and the solutions form an overview of the very nice 1980 paper *Fourier coefficients of cusp forms and the Riemann zeta-function* by Henryk Iwaniec.

Exercise 2.1.a) The first estimate follows by using the Weil bound for the Kloosterman sums:

$$\begin{aligned}|B(c,N)| &\leq \sum_{N_1<n,m\leq N} |\overline{b_m} b_n S(n,m;c)| \\ &\ll c^{\frac{1}{2}+\epsilon} \sum_{N_1<n,m\leq N} |\overline{b_m} b_n|(n,m,c)^{\frac{1}{2}} \\ &\ll c^{\frac{1}{2}+\epsilon} \sum_{d|c} d^{\frac{1}{2}} \left(\sum_{\substack{\frac{N_1}{d}<n\leq \frac{N}{d}, \\ (n,c)=1}} |b_{dn}| \right)^2 \\ &\ll c^{\frac{1}{2}+\epsilon} N \sum_{d|c} d^{-\frac{1}{2}} \sum_{\substack{\frac{N_1}{d}<n\leq \frac{N}{d}, \\ (n,c)=1}} |b_{dn}|^2 \ll c^{\frac{1}{2}+\epsilon} N \sum_{N_1<n\leq N} |b_n|^2.\end{aligned}$$

To derive the second estimate we first note that since $c > N^{1-\epsilon}$ we have $A = \frac{N}{c}\theta < 3N^{\epsilon}$ and $A/2 < \frac{\sqrt{nm}}{c}\theta \leq A$. Let η be a smooth function with support in $[1/4, 4]$ so that $\eta|_{[1/2,2]} \equiv 1$. We obtain

$$B(c,N) = \sum_{N_1<n,m\leq N} \overline{b_m} b_n S(n,m;c) e(\theta \frac{\sqrt{nm}}{c}) \eta(\theta \frac{\sqrt{nm}}{cA}).$$

Put

$$R(s) = \int_0^{\infty} \eta(\frac{x}{A}) e(x) x^{s-1} dx.$$

This function is trivially bounded by $R(s) \ll_{\eta} A^{\Re(s)}$ and by partial integration one gets

$$R(s) \ll A^{\Re(s)} \Im(s)^{-2} \text{ for } |\Im(s)| > 4\pi A.$$

By Mellin inversion we get

$$\eta(x/A) e(x) = \frac{1}{2\pi i} \int_{(\sigma)} R(s) x^{-s} ds \text{ for } \sigma > 0.$$

The desired estimate now follows from the large sieve and the definition of the Kloosterman sum:

$$\begin{aligned}
B(c,N) &= \frac{1}{2\pi i}\int_{(\sigma)} R(s) \sum_{N_1<n,m\leq N} \overline{b_m} b_n \left(\frac{\sqrt{nm}}{c}\theta\right)^{-s} S(n,m;c)ds \\
&\ll c^{\sigma}\int_{(\sigma)} \sum_{\substack{h \bmod c,\\ (c,h)=1}} \Big| \sum_{N_1<m\leq N} \overline{b_m} m^{-\frac{s}{2}} e(m\frac{h}{c})\Big| \cdot \Big| \sum_{N_1<n\leq N} b_n n^{-\frac{s}{2}} e(m\frac{\overline{h}}{c})\Big| \cdot |R(s)|ds \\
&\ll c^{\sigma}\int_{(\sigma)} \left(\sum_{\substack{h \bmod c,\\ (c,h)=1}} \Big| \sum_{N_1<m\leq N} \overline{b_m} m^{-\frac{s}{2}} e(m\frac{h}{c})\Big|^2\right)^{\frac{1}{2}} \left(\sum_{\substack{h \bmod c,\\ (c,h)=1}} \Big| \sum_{N_1<n\leq N} b_n n^{-\frac{s}{2}} e(n\frac{h}{c})\Big|^2\right)^{\frac{1}{2}} \cdot |R(s)|ds \\
&\ll c^{\sigma}(c+N)\sum_{N_1<n\leq N} |b_n|^2 n^{\sigma} \int_{(\sigma)} |R(s)|ds \ll c^{\sigma} N^{-\sigma} A^{\sigma+1}(c+N) \sum_{N_1<n\leq N} |b_n|^2 \\
&\ll A(c+N)\sum_{N_1<n\leq N} |b_n|^2 \ll N^{\epsilon}(c+N)\sum_{N_1<n\leq N} |b_n|^2.
\end{aligned}$$

The finial estimate turns out to be the hardest, even though the ingredients are again elementary. First we write

$$\begin{aligned}
B(c,N) &= \sum_{N_1<n,m\leq N} \overline{b_m} b_n \sum_{d|(n,m,c)} dS(1,\frac{nm}{d^2};\frac{c}{d}) e(\theta\frac{\sqrt{nm}}{c}) \\
&= \sum_{d|c} d \underbrace{\sum_{\frac{N}{2d}<n,m\leq\frac{N}{d}} \overline{b_{dm}} b_{dn} S(1,mn;\frac{c}{d}) e(\theta\frac{\sqrt{mn}}{c/d})}_{=X(\frac{c}{d},\frac{N}{d})}
\end{aligned} \tag{1}$$

For notational sake we set $x_n = b_{dn}$, $q=\frac{c}{d}$, $M=\frac{N}{d}$ and $M_1 = \frac{N_1}{d}$. By Cauchy-Schwarz we have

$$|X(q,M)|^2 \leq \left(\sum_{M_1<n\leq M} |x_n|^2\right)\cdot \underbrace{\left(\sum_n \eta(\frac{n}{M})\Big| \sum_{M_1<m\leq M} \overline{x_m} S(1,mn;q) e(\theta\frac{\sqrt{mn}}{q})\Big|^2\right)}_{=\mathcal{S}(M)}.$$

We need to estimate the second factor. This is done by opening the square as well as the Kloosterman sum and moving the n-sum in. This leads to

$$\mathcal{S}(M) = \sum_{M_1<m_1,m_2\leq M} \overline{x_{m_1}} x_{m_2} \sum_{\substack{h_1,h_2 \bmod c,\\ (h_1h_2,c)=1}} e(\frac{\overline{h_1}-\overline{h_2}}{q}) \sum_n f(n),$$

where

$$f(n) = \eta(\frac{n}{M}) e(\frac{h_1m_1-h_2m_2}{q}n + \theta\frac{\sqrt{m_1}-\sqrt{m_2}}{q}\sqrt{n}) = \eta(\frac{n}{M}) e(an+b\sqrt{n}).$$

The n-sum is treated using Poisson summation:

$$\sum_n f(n) = \sum_n \widehat{f}(n) \text{ with } \widehat{f}(u) = \int_{\mathbb{R}} \eta(\frac{n}{M}) e((a-u)t + b\sqrt{t})dt.$$

Recall that $a \in \frac{1}{q}\mathbb{Z}$. In particular, if $u \neq a$, then $|a-u| \geq q^{-1}$. On the other hand, for $t \in \mathrm{supp}(\eta(\frac{\cdot}{M}))$, we can estimate

$$bt^{-\frac{1}{2}} \ll \frac{|m_1 - m_2|}{Mq} \ll q^{-1}.$$

Applying l-times partial integration yields

$$\widehat{f}(u) \ll_l (|u-a|M)^{-l} M.$$

With this at hand we can estimate

$$\sum_{n \neq a} \widehat{f}(n) \ll \left(\frac{q}{M}\right)^l M \ll N^{-\epsilon l} M.$$

In the last step we used $\frac{q}{M} = \frac{c}{N}$ and $c \leq N^{1-\epsilon}$. Now we can choose l as large as necessary (depending on ϵ) to obtain

$$\sum_n \widehat{f}(n) = \delta_{a \in \mathbb{Z}} \widehat{f}(a) + O(M^{-1}).$$

The condition $a \in \mathbb{Z}$ yields the congruence

$$h_1 m_1 \equiv h_2 m_2 \bmod q.$$

For this case we have to distinguish diagonal and off-diagonal contributions from the m_i-sums. Indeed for $m_1 = m_2$ we can not improve upon $\widehat{f}(a) \ll M$. However, if $m_1 \neq m_2$, then partial integration gives

$$\widehat{f}(a) = \int_{\mathbb{R}} \eta(\frac{t}{M}) e(b\sqrt{t})dt \ll \frac{\sqrt{M}}{|b|} \ll \frac{qM}{|m_1 - m_2|}.$$

Note that

$$\sum_{\substack{h_1, h_2 \bmod q, \\ (h_1 h_2, q) = 1, \\ h_1 m_1 \equiv h_2 m_2 \bmod q}} e(\frac{h_1 - h_2}{q}) \ll (m_1 - m_2, q).$$

Gathering these estimates we find

$$\begin{aligned} \mathcal{S}(M) &\ll qM \sum_{M_1 < m_1 \leq M} |x_{m_1}|^2 + qM \sum_{\substack{M_1 < m_1, m_2 \leq M, \\ m_1 \neq m_2}} \frac{(m_1 - m_2, q)}{|m_1 - m_2|} |x_{m_1} x_{m_2}| \\ &\ll qM \sum_{M_1 < m_1 \leq M} |x_{m_1}|^2. \end{aligned}$$

This gives

$$X(q,M) \ll \sqrt{qM} \sum_{M_1<m_1\leq M} |x_{m_1}|^2$$

and we finally obtain

$$B(c,N) = \sum_{d|c} dX(\frac{c}{d},\frac{N}{d}) \ll \sum_{d|c} \sqrt{cN} \sum_{\frac{N_1}{d}<n\leq\frac{N}{d}} |b_{dn}|^2 \ll c^{\frac{1}{2}} N^{\frac{1}{2}} \sum_{N_1<n\leq N} |b_n|^2.$$

This gives the last claim.

Exercise 2.1.b) We take η as in the previous section and define $\varphi(x) = \eta(\frac{x}{T})$. We obtain

$$\begin{aligned}\int_0^\infty \varphi(x)A(x,t)dx &= \int_0^\infty \eta(\frac{x}{T})\frac{\sinh(\pi x)}{\cosh(\pi x)^2+\sinh(\pi t)^2}dx \\ &\geq \int_{T/2}^T \frac{\sinh(\pi x)}{\cosh(\pi x)^2+\sinh(\pi t)^2}dx \\ &> \frac{1}{\cosh(\pi t)}\end{aligned}$$

for $t \in [T/2,T]$ as required. With this at hand we can estimate

$$\sum_{T/2\leq t_j\leq T} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N/2\leq n\leq N} a_n\rho_j(n)\right|^2 \leq \sum_{N/2\leq m,n\leq N} \overline{a_m}a_n \int_0^\infty \varphi(x) \sum_{j=1}^\infty A(x,t_j)\overline{\rho_j(m)}\rho_j(n)dx.$$

By positivity we can add the Eisenstein contribution and apply the forward Kuznetsov formula. This gives

$$\begin{aligned}\sum_{T/2\leq t_j\leq T} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N/2\leq n\leq N} a_n\rho_j(n)\right|^2 &\leq \frac{1}{2\pi^2}\int_0^\infty x\varphi(x)dx \sum_{N/2\leq n\leq N} |a_n|^2 \\ &+ \sum_{c=1}^\infty \frac{4\pi^2}{c^2} \sum_{N/2\leq n,m\leq N} \overline{a_m}a_n\sqrt{nm}S(n,m;c)\Phi(4\pi\frac{nm}{c}).\end{aligned}$$

Here

$$\Phi(x) = \int_0^\infty \frac{t\varphi(t)}{\cosh(\pi t)} \int_x^\infty (J_{2it}(u)+J_{-2it}(u))\frac{du}{u}dt$$

as required.

Exercise 2.1.c) First we simplify the integral defining Φ as follows

$$\begin{aligned}
\Phi(x) &= \int_0^\infty \frac{t\varphi(t)}{\cosh(\pi t)} \int_x^\infty (J_{2it}(u) + J_{-2it}(u)) \frac{du}{u} dt \\
&= \frac{4}{\pi}\Re \int_0^\infty \frac{t\varphi(t)}{\cosh(\pi t)} \int_x^\infty \int_0^\infty \sin(u\cosh(y) - i\pi t)\cosh(2ity)dy\frac{du}{u}dt \\
&= \frac{4}{\pi}\Re \int_0^\infty \frac{t\varphi(t)}{\cosh(\pi t)} \int_0^\infty \int_{x\cosh(y)}^\infty \sin(u - i\pi t)\frac{du}{u}\cosh(2ity)dydt \\
&= -\frac{2}{\pi}\Re \int_0^\infty \frac{\varphi(t)}{\cosh(\pi t)} \int_0^\infty i\sin(x\cosh(y) - i\pi t)\frac{\sinh(y)}{\cosh(y)}\sinh(2ity)dydt \\
&= -\frac{2}{\pi} \int_0^\infty \int_0^\infty \frac{\varphi(t)}{\cosh(\pi t)} \Re[i\sin(x\cosh(y) - i\pi t)\sinh(2ity)]dt\frac{\sinh(y)}{\cosh(y)}dy \\
&= \frac{2}{\pi} \int_0^\infty \int_0^\infty \varphi(t)\sin(2ty)dt\sin(x\cosh(y))\frac{\sinh(y)}{\cosh(y)}dy
\end{aligned}$$

In the fourth line we applied partial integration to the y-integral and afterwards some addition formula for trigonometric functions.
We put $\delta = N^\epsilon T^{-1}$. For $y > \delta$ partial integration one can estimate the inner integral by

$$\int_0^\infty \varphi(t)\sin(2ty)dt \ll_l T(yT)^l \ll KN^{-2}$$

when l is taken to be large enough. Thus we can insert a cutoff function ξ with support in $[0, 2\delta]$ and $\xi|_{[0,\delta]} \equiv 1$ into the y-integral with an acceptable error. This gives

$$\Phi(x) = \Delta(x)) + O(KN^{-2}) \text{ for } \Delta(x) = \frac{2}{\pi}\int_0^\infty \int_0^\infty \xi(y)\varphi(t)\sin(2ty)dt\sin(x\cosh(y))\frac{\sinh(y)}{\cosh(y)}dy.$$

We write

$$\Delta(x) = \int_0^\infty \int_0^\infty K(t,y)\sin(x\cosh(y))dtdy.$$

Note that

$$\|K\|_{L^1} = \int_0^\infty \int_0^\infty |K(t,y)|dtdy \ll \delta^2 T = N^{2\epsilon}T \leq 1.$$

For $x \gg T^2$ we apply partial integration in the y-integral to get

$$\int_0^\infty \xi(y)\sin(2ty)\sin(x\cosh(y))\frac{\sinh(y)}{\cosh(y)}dy = x^{-1}\int_0^\infty \frac{d}{dy}\left(\frac{\xi(y)\sin(2ty)}{\cosh(y)}\right)\cos(x\cosh(y))dy.$$

With this at hand we can write

$$\Delta(x) = \int_0^\infty \int_0^\infty L(t,y)x^{-1}\cos(x\cosh(y))dtdy,$$

for an appropriate kernel with the desired L^1-growth.

Exercise 2.1.d) We go back to the upper bound derived in part d) of the exercise. Since

$$\int_0^\infty t\phi(t) \ll T^2$$

the diagonal gives an acceptable contribution. We turn to the off diagonal and insert our approximation for $\Phi(x)$. This gives

$$\sum_{T/2\leq t_j\leq T} \frac{1}{\cosh(\pi t_j)} \left| \sum_{N/2\leq n\leq N} a_n\rho_j(n)\right|^2 \ll T^2 \sum_{N/2\leq n\leq N} |a_n|^2 + \underbrace{\sum_{c=1}^\infty \frac{4\pi^2}{c^2} \sum_{N/2\leq n,m\leq N} \overline{a_m}a_n\sqrt{nm}S(n,m;c)\Delta(4\pi\frac{nm}{c})}_{=\mathcal{T}(T,N)} + \mathcal{E}.$$

The error is treated trivial using the Weil bound:

$$\mathcal{E} \ll \sum_{c=1}^\infty \sum_{N/2\leq n,m\leq N} \frac{\sqrt{nm}(n,m,c)^{\frac{1}{2}}}{c^{\frac{3}{2}}}|a_m a_n| TN^{-2} \ll T \sum_{N/2\leq n\leq N} |a_n|^2.$$

It remains to estimate $\mathcal{T}(T,N)$. To so we split the c-sum in the four pieces $N^2 \leq c < \infty$, $N^{1-\epsilon} \leq c < N^2$, $NT^{-2} \leq c < N^{1-\epsilon}$ and $0 < c < NT^{-2}$. We call these intervals I_i for $i = 1,2,3,4$. Accordingly we get

$$\mathcal{T}(T,N) = \sum_{i=1}^4 \mathcal{T}_i(T,N).$$

Put $\theta = \theta_{\pm,y} = \pm 2\cosh(y)$ and $b_n = \sqrt{n}a_n$. Note that $0 < y < 2\delta$ so that we can assume $1 < |\theta| < 3$. We get

$$\mathcal{T}_i(T,N) = 2\pi^2 \sum_{\pm} \int_0^\infty \int_0^\infty L(t,y) \sum_{c\in I_i} \frac{1}{c^2} \sum_{N/2\leq n,m\leq N} \overline{b_m}b_n S(n,m;c)e(\theta_{\pm,y}\frac{\sqrt{nm}}{c})dydt.$$

We recognize the inner sums as $B(c,N)$, which we treated in a). Applying these results and treating everything else trivially concludes the proof.

Exercise 2.2.a) By considering the Taylor expansion

$$\varphi(t+x) = \varphi(x) + t\varphi'(x) + O((x+T)^{-2}t^2)$$

we get

$$\int_{-\infty}^\infty y^{it}\varphi(t)A(t,x)dt = \frac{y^{ix}}{\cosh(\pi x)} \int_{\mathbb{R}} y^{it}\varphi(t+x)\frac{dt}{\cosh(\pi t)} + O(e^{-2\pi x})$$
$$= y^{ix}\frac{\varphi(x)}{\cosh(\pi x)} \underbrace{\int_{\mathbb{R}} y^{it}\frac{dt}{\cosh(\pi t)}}_{=\alpha(y)} + y^{ix}\frac{\varphi'(x)}{\cosh(\pi x)} \underbrace{\int_{\mathbb{R}} y^{it}\frac{tdt}{\cosh(\pi t)}}_{=\beta(y)} + O((x+T)^{-2}e^{-\pi x}).$$

Note that α is real and satisfies

$$\alpha(y) = \alpha(1) + O(\log(y)) > \frac{1}{2}\alpha(1).$$

With this at hand we can write

$$\begin{aligned}\frac{\varphi(x)}{\cosh(\pi x)}\Big|\sum_l b_l l^{ix}\Big|^2 &= \frac{\varphi(x)}{\cosh(\pi x)}\sum_{l_1,l_2} b_{l_1}\overline{b_{l_2}}\left(\frac{l_1}{l_2}\right)^{ix} \\ &= \sum_{l_1,l_2}\frac{b_{l_1}\overline{b_{l_2}}}{\alpha(l_1/l_2)}\int_{\mathbb{R}}\left(\frac{l_1}{l_2}\right)^{it}\varphi(t)A(t,x)dt + O((x+T)^{-2}e^{-\pi x}L\|b_l\|_2^2).\end{aligned}$$

Note that β did not play a role here since it is purely imaginary. In particular, we can also open the n-sum in our original expression and transform it into

$$\pi\sum_{l_1,l_2}\frac{b_{l_1}\overline{b_{l_2}}}{\alpha(l_1/l_2)}\sum_{n,m}\overline{a_m}a_n\int_{\mathbb{R}}\left(\frac{l_1}{l_2}\right)^{it}\varphi(t)\sum_{j=1}^{\infty}A(t,t_j)\overline{\rho_j(m)}\rho_j(n)dt + \mathcal{E}_1.$$

The error $\mathcal{E}_1$ can be easily treated using the result from Exercise 2.1 and it turns out that $\mathcal{E}_1 \ll L\|b\|_2^2\|a\|_2^2$. By positivity we add the Eisenstein contribution and apply the forward Kuznetsov formula. The diagonal contribution gives directly $S_1(T,L,N)$, while $S_2(T,L,N)$ comes from the off-diagonal.

Exercise 2.2.b) The claimed estimate

$$\int_{\mathbb{R}} t\varphi(t)\left(\frac{l_1}{l_2}\right)^{it}dt \ll \min(T^2, \log(l_1/l_2)^{-2})$$

follows directly by applying partial integration twice. With this at hand we estimate $S_1(T,L;N)$ trivially and get

$$S_1(T,L,N) \ll T^2\|a_n\|_2^2\|b_l\|_2^2 + \sum_{|\frac{l_1}{l_2}|\gg 1}\log(l_1/l_2)^{-2}|b_{l_1}b_{l_2}|\|a_n\|_2^2 \ll (T^2+TL)\|a_n\|_2^2\|_2 b_l\|_2^2.$$

Exercise 2.2.c) To estimate the function $\Phi(x,y)$ we see that the same computation as in Exercise 2.1.c) shows that

$$\Phi(x,y) = \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} y^{it}\varphi(t)\sin(2tz)\sin(x\cosh(z))\frac{\sinh(z)}{\cosh(z)}dzdt.$$

For $z \geq |\log(y)|$ we partially integrating the t-integral twice to get

$$\int_0^{\infty} y^{it}\varphi(t)\sin(2tz)dt \ll T(zT)^{-2}.$$

(This can be seen for example by writing $y^{it}\sin(2tz) = \frac{i}{2}[e^{it(\log(y)-2z)} - e^{it(\log(y)+2z)}]$.) Inserting a suitable cutoff function ξ as before allows us to write

$$\Phi(x,y) = \frac{2}{\pi}\int_0^{\infty}\xi(z)\int_0^{\infty} y^{it}\varphi(t)\sin(2tz)dt\sin(x\cosh(z))\frac{\sinh(z)}{\cosh(z)}dz + O(T^{-1}).$$

The estimate is completed by applying partial integration twice in the z-integral (after taking the sine apart):

$$\int_0^\infty \xi(z) \frac{\sinh(z)}{\cosh(z)} e^{2itz \pm ix\cosh(z)} dz \ll t^{-2} \tag{2}$$

The claimed bound directly since our estimate for the z-integral implies

$$\Phi(x,y) \ll K^{-1} + \int_0^\infty t^{-2}\varphi(t)dt \ll K^{-1}.$$

With this estimate for $\Phi(x,y)$ at hand we can estimate $S_2(T,L,N)$. First note that $\frac{4\pi\sqrt{nm}}{c} \ll N \ll T$ and $\frac{l_1}{l_2} \in [\frac{1}{2}, 2]$. We start by using our bound for $\Phi(x,y)$ and the Weil bound to estimate

$$S_2(T,L,N) \ll T^{-1} \sum_{l_1,l_2} |b_{l_1} b_{l_2}| \sum_c \sum_{m,n} |a_n a_m| \frac{\sqrt{mn(n,m,c)}}{c^{\frac{3}{2}-\epsilon}} \ll \frac{LN^{2+\epsilon}}{T} \|a_n\|_2^2 \|b_l\|_2^2.$$

We are done since $N \ll T$.

Exercise 2.3.a) We first look at $r < N$. In this case we insert Basset's integral for the K-Bessel function to obtain

$$\begin{aligned}\tilde{f}(r) &= \int_0^\infty K_{2ir}(x) f(x) \frac{dx}{x} \\ &= \frac{3}{\sqrt{\pi}} \Gamma(\frac{1}{2} + 2ir) \int_0^\infty \int_0^\infty \left(\frac{2x}{w^2+x^2} \right)^{2ir} \frac{b(x)}{(x^2+w^2)^{\frac{3}{2}}} e\left(-\frac{3}{2\pi} \left(\frac{T^2x^2}{4\tau} \right)^{\frac{1}{3}} \right) \frac{dx}{x} w \sin(w) dw,\end{aligned}$$

where we use the shorthand $b(x) = a(\frac{4\pi\sqrt{ul}}{x}, l, u, \tau)$. Note that x is of size $N^{\frac{1}{2}}T^\epsilon$ due to the support of b. By partially integrating the x-integral sufficiently often we get

$$\tilde{f}(r) \ll N^{-6} e^{-\pi|r|} \log(T)$$

in this range. Note that the exponential comes from the Γ-factor after using Stirling's formula. We turn towards $|r| \geq N$. Here we use the representation of K_{2ir} in terms of the I-Bessel function and the Taylor series for the latter. Since $x^2 < |r|T^{-\epsilon}$ we can truncate the sum after finitely many terms (say $m_0(\epsilon)$ without introducing any significant error. Let $m \leq m_0(\epsilon)$. Then a typical term we need to treat is of the form

$$\begin{aligned}a_m &= \int_0^\infty b(x) x^{2m-1+2ir} e\left(-\frac{3}{2\pi} \left(\frac{T^2x^2}{4\tau} \right)^{\frac{1}{3}} \right) dx \\ &= \int_0^\infty b(x) x^{2m-1} \exp(2ir\log(x) - 3i \left(\frac{T^2x^2}{4\tau} \right)^{\frac{1}{3}}) dx.\end{aligned}$$

If r is not of size N, then we win again by partial integration. Indeed one gets for example

$$a_m \ll (N^3T^{-1})^{2m} |r|^{-6}$$

so that

$$\tilde{f}(r) \ll |r|^{-6}e^{-\pi|r|}.$$

In the remaining range, $r \asymp N$ we apply the stationary phase lemma as stated above. This gives the result desired result.

Exercise 2.3.b) According to equation (10) on the exercise sheet we have

$$\sum_{v=1}^{R} j(t_v)S_4(M,N,t_v) = \frac{T_0}{MN}\sum_{v=1}^{R}\sum_{u,l}\tau(l)\sum_{k}\frac{S(u,-l;k)}{k}\underbrace{a(k,l,u,t_v)e(-3\left(\frac{ulT^2}{2\pi k^2 t_v}\right)^{\frac{1}{3}})}_{=f(\frac{4\pi\sqrt{lu}}{k})} + O(RT_0).$$

Applying the backward Kuznetsov formula and inserting the asymptotic expansion from Exercise 2.3.a) we get

$$\begin{aligned}\sum_{k}\frac{S(u,-l;k)}{k}f(\frac{4\pi\sqrt{lu}}{k}) &= \int_{\mathbb{R}}(ul)^{ir}\sigma_{2ir}(u)\sigma_{2ir}(l)t_v^{ir}\frac{\theta(r)c(l,u,t_v,r)}{|\Gamma(\frac{1}{2}+ir)\zeta(1+2ir)|^2}dr \\ &\quad + 4\sum_{j=1}^{\infty}\rho_j(l)\rho_j(u)t_v^{it_j}\theta(t_j)c(l,u,t_v;t_j) \\ &\quad + \mathcal{E}_1 + \mathcal{E}.\end{aligned}$$

The errors $\mathcal{E}_1$ and $\mathcal{E}_2$ are handled by

$$\mathcal{E}_1 \ll \tau(u)\tau(l)\int_{\mathbb{R}}|\zeta(1+2ir)|^{-2}(|r|+N)^{-6}dr \ll N^{-1+\epsilon}$$

and

$$\mathcal{E}_2 \ll \sum_{j=1}^{\infty}\frac{|\rho_j(u)\rho_j(l)|}{(t_j+N)^6\cosh(\pi t_j)} \ll \sigma(u)\sigma(l)\sum_{j=1}^{\infty}\frac{|\rho_j(1)|^2}{(t_j+N)^6\cosh(\pi t_j)} \ll ulN^{-4+\epsilon} \ll N^{-1}T^{\epsilon}.$$

Note that here very crude estimates were sufficient and we used $l \asymp L$ and $u \asymp N$. Inserting this above directly yields the desired estimate. The genesis of the error term is as follows:

$$R\cdot\frac{T_0}{MN}\cdot L\cdot N\cdot N^{-1}T^{\epsilon} = RT_0T^{\epsilon}\cdot\frac{L}{MN} \ll RT_0T^{\epsilon}\cdot\frac{MN}{T} = \frac{T^{\frac{1}{2}}}{T_0} \ll RT_0T^{\epsilon},$$

where we used the constraints on the parameters L, M, N and the assumption concerning the size of T_0.

Exercise 2.3.c) Before we can apply our large sieve results from the previous exercises we have to separate the variables l, u and τ. This is done by writing

$$c(l,u,\tau,r) = \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\widehat{c}(x_1,x_2,x_3,r)e(x_1l+x_2u+x_3\tau)dx_1dx_2dx_3$$

via Fourier inversion. The Fourier transform $\widehat{c}$ is easily controlled by partial integration:

$$\widehat{b}(x_1,x_2,x_3,r) \ll \beta(x_1,x_2,x_3) = (x_1^2+1)^{-1}(x_2^2+1)^{-1}(x_3^2+1)^{-1}.$$

With this at hand we can bound

$$\sum_{v=1}^{R} j(t_v) S_4(M,N,t_v) \ll RT_0 T^{\epsilon} + \mathcal{S}_{\text{cusp}} + \mathcal{S}_{\text{Eis}},$$

where

$$\mathcal{S}_{\text{cusp}} = \frac{T_0}{MN} \sum_{t_j \asymp N} \frac{1}{t_j \cosh(\pi t_j)} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Big| \sum_{v=1}^{R} t_v^{ir} e(x_3 t_v) \Big| \Big| \sum_{u \asymp N} \rho_j(u) e(x_2 u) \Big| \Big| \sum_{l \asymp L} \rho_j(l) \tau(l) e(x_1 l) \Big| d\beta$$

and

$$\mathcal{S}_{\text{Eis}} = \frac{T_0 \log(N)^2}{MN^2} \int_{r \asymp N} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Big| \sum_{v=1}^{R} t_v^{ir} e(x_3 t_v) \Big| \Big| \sum_{u \asymp N} \sigma_{2ir}(u) u^{ir} e(x_2 u) \Big| \Big| \sum_{l \asymp L} \sigma_{2ir}(l) \tau(l) l^{ir} e(x_1 l) \Big| dr d\beta$$

We first treat the Eisenstein contribution. First we apply Cauchy-Schwarz and bound the u-sum trivially. This gives

$$\begin{aligned}
\mathcal{S}_{\text{Eis}} &\ll \frac{T_0 N^{\epsilon}}{MN} \int_{\mathbb{R}^3} \Bigg(\underbrace{\int_{\asymp N} \Big| \sum_{v=1}^{R} t_v^{ir} e(x_3 t_v) \Big|^2 dr}_{\ll (N + \frac{T}{T_0}) R} \Bigg)^{\frac{1}{2}} \Bigg(\underbrace{\int_{\asymp N} \Big| \sum_{l \asymp L} \sigma_{2ir}(l) \tau(l) l^{ir} e(x_1 l) \Big|^2 dr}_{(N\sqrt{L} + L) L^{1+\epsilon}} \Bigg)^{\frac{1}{2}} d\beta \\
&\ll \frac{T_0 T^{\epsilon}}{MN} \cdot (N + \frac{T}{T_0})^{\frac{1}{2}} R^{\frac{1}{2}} \cdot (N\sqrt{L} + L)^{\frac{1}{2}} L^{\frac{1}{2}} \ll R^{\frac{1}{2}} T_0^{-\frac{1}{2}} T^{1+\epsilon}.
\end{aligned}$$

The sum $\mathcal{S}_{\text{cusp}}$ is treated by using Cauchy-Schwarz on the t_j-sum to get precisely the averages treated in Exercise 2.1 and 2.2. Applying these yields

$$\begin{aligned}
\mathcal{S}_{\text{cusp}} &= \frac{T_0}{MN^2} \int_{\mathbb{R}^3} \Bigg(\sum_{t_j \asymp N} \frac{1}{\cosh(\pi t_j)} \Big| \sum_{l \asymp L} \rho_j(l) \tau(l) e(x_1 l) \Big|^2 \Bigg)^{\frac{1}{2}} \\
&\quad \cdot \Bigg(\sum_{t_j \asymp N} \frac{1}{\cosh(\pi t_j)} \Big| \sum_{v=1}^{R} t_v^{ir} e(x_3 t_v) \Big|^2 \Big| \sum_{u \asymp N} \rho_j(u) e(x_2 u) \Big|^2 \Bigg)^{\frac{1}{2}} d\beta \\
&\ll \frac{T_0 T^{\epsilon}}{MN^2} (N^2 + L)^{\frac{1}{2}} (N + R)^{\frac{1}{2}} L^{\frac{1}{2}} N R^{\frac{1}{2}} \ll R^{\frac{1}{2}} T_0^{-\frac{1}{2}} T^{1+\epsilon}.
\end{aligned}$$

This completes the proof.