Kleine AG: Around the Mumford–Tate Conjecture

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Lecture 1: Setup and statement

Let A be an abelian variety over a field k. Recall the construction of the ℓ–adic Galois representation $\mathrm{V}_\ell A$ for ℓ different from the characteristic of k, and of the associated rational Hodge structure $\mathrm{V}_0 A$ for $k = \mathbb{C}$, as well as the comparison isomorphism between these objects (for $k = \mathbb{C}$). Define the Mumford–Tate group associated with a complex abelian variety in terms of the complex representation $\mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathrm{GL}(\mathrm{V}_0 A \otimes \mathbb{C})$, and in terms of Tannakian formalism ([Del82]). State the Mumford–Tate conjecture for abelian varieties over fields that are finitely generated over $\mathbb{Q}$. If time permits, give a short status report on the conjecture (see Introduction of [Vas08]).

Lecture 2: Elementary cases

Let A be a polarised abelian variety over a number field $k \subset \mathbb{C}$, let $\mathfrak{l} \subseteq \mathfrak{gl}(\mathrm{V}_\ell A)$ and $\mathfrak{h} \subseteq \mathfrak{gl}(\mathrm{V}_0 A)$ be the Lie algebras of the image of the absolute Galois group of k in $\mathrm{GL}(\mathrm{V}_\ell A)$ and of the Mumford–Tate group of A respectively. Using Faltings' theorem on endomorphisms of abelian varieties, show that the Lie algebras $\mathfrak{l}$ and $\mathfrak{h}$ are both reductive, and that they are both contained in $\mathfrak{gsp}$ with respect to the Weil pairing. Also, show that $\mathfrak{l}$ and $\mathfrak{h}$ have the same commutator algebra. Prove the Mumford–Tate conjecture for elliptic curves (see [Ser68] IV.2, but use Faltings).

Lecture 3: Absolute Hodge cycles

Define the notion of an absolute Hodge cycle on an abelian variety A over a finitely generated field k of characteristic zero. Show how absolute Hodge cycles behave in families ([Del82] Theorem 2.15). Show that if every Hodge cycle on A is absolute, then the inclusion

$$\{\text{Image of Galois group in } \mathrm{GL}(\mathrm{V}_\ell A)\} \ \subseteq\ \{\mathbb{Q}_\ell\text{–points of Mumford–Tate group}\}$$

holds. State Deligne's theorem on absolute Hodge cycles ([Del82] Theorem 2.11). If time premits, show that two conjectures out of the Hodge–, the Tate– and the Mumford–Tate conjecture imply the third one.

Lecture 4: Abelian varieties of CM–type

Recall the definition and basic properties of abelian varieties of CM–type (for example from [Mil06]). Show that on abelian varieties of CM–type, the Hodge– and the Tate–conjecture on algebraic cycles are equivalent following Pohlmann [Poh68] (mind that Faltings' results were not available, so there are some simplifications). Deduce the Mumford–Tate conjecture for abelian varieties of CM–type.

Lecture 5: Deligne's theorem on absolute Hodge cycles

Deduce Deligne's theorem on absolute Hodge cycles for abelian varieties of CM–type from Lecture 4. Complete the proof of Deligne's theorem by showing that every abelian variety can be put into a smooth family of abelian varieties which contain abelian varieties of CM–type ([Del82] §6).

Literatur

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