# KLEINE AG: CRYSTALLINE COHOMOLOGY IN CHARACTERISTIC 0

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Crystalline cohomology arose in the search for a "good $p$-adic cohomology theory". In Grothendieck's words ([Gro68, p. 315]):

> Such a cohomology theory should associate, to each scheme $X$ of finite type over a perfect field $k$ of characteristic $p > 0$, cohomology groups which are modules over an integral domain, whose quotient field is of *characteristic* 0, and which satisfy all the desirable formal properties (functoriality, finite dimensionality, Poincaré duality, Künneth formula, invariance under base change, ...). This cohomology should also, most importantly, explain torsion phenomena, and in particular $p$-torsion.

Over the field of complex numbers, it is known that algebraic de Rham cohomology, i.e. the hypercohomology of the de Rham complex $\Omega^\bullet_{X/\mathbb{C}}$, agrees with the analytic singular cohomology $H^*(X^{\mathrm{an}}, \mathbb{C})$ and, hence, is a theory with the "desirable formal properties" mentioned above. If $k$ is of characteristic $p > 0$, algebraic de Rham cohomology is no longer that well behaved, essentially because $d(f^p) = 0$ for every function $f$, which has the consequence, for example, that the de Rham cohomology groups are "too big".

The rough idea of crystalline cohomology then is, for a variety $X$ over a perfect field $k$ of positive characteristic $p$, to take a lifting of $X$ to a base of characteristic 0 (e.g. the ring of Witt vectors of $k$) and to set $H^*_{\mathrm{Cris}}$ equal to the algebraic de Rham cohomology of the lifting. Unfortunately, not every $X$ can be lifted to characteristic 0, so the definition of crystalline cohomology needs to be formulated in a way which is independent of liftings. This program has been implemented by Grothendieck, Berthelot and others; it has turned out that crystalline cohomology theory does satisfies the properties mentioned in the beginning in the case where $X$ is smooth and proper.

In this Kleine AG we follow parts of [Gro68],where Grothendieck defines crystalline cohomology and sketches a program later carried out by Pierre Berthelot, [Ber74]. We, however, merely scratch the surface: If $S$ is a scheme, we define the *infinitesimal cohomology* of an $S$-scheme, which agrees with its crystalline cohomology if $S$ is defined over $\mathbb{Q}$. We then show that, if $X$ is smooth over $S$, this theory computes the algebraic de Rham cohomology of $X/S$. To define crystalline cohomology in positive characteristic, one would have to add divided power structures to the mix, which we will not touch upon.

We fix a locally noetherian scheme $S$ and a smooth morphism $f : X \to S$. For simplicity, we assume that $f$ is separated.

**Talk 1: Sites, Topoi, Cohomology:** In this talk we recall some general facts on sites and topoi. Due to time restraints we have to gloss over many details; in particular, we ignore set-theoretical issues.

- Define a site as in [SP, 8.6, 8.37], that is, as a category equipped with a pretopology.
- Mention that originally, in [SGA4], sites were defined differently, but that a site according to our definition gives rise to a site in the original sense.
- Define presheaves and sheaves on a site, e.g. as in [SP, 8.37]. Define the topos of a site as the category of sheaves of sets on that site.
- Explain the Yoneda embedding of a site $C$ into the associated topos $\tilde{C}$: The Yoneda map $U \mapsto h_U$ gives a fully faithful functor from $C$ into the category of presheaves of sets $\hat{C}$ of $C$, and composing with sheafification yields a morphism $\epsilon_C : C \to \tilde{C}$.
- Given a site $C$, we put the following topology on its associated topos $\tilde{C}$: Covering families are families $(\mathscr{F}_i \to \mathscr{F})_i$ such that $\coprod_i \mathscr{F}_i \to \mathscr{F}$ is surjective as a map of sheaves. State that a covering $(U_i \to U)_i$ in $C$ gives rise to a covering $(\epsilon_C(U_i) \to \epsilon_C(U))_i$ with respect to the above topology. Mention without proof that this topology on $\tilde{C}$ is the so-called *canonical topology* (see e.g. [SP, 8.37]), which is the finest topology (in a suitable sense) such that every representable presheaf is a sheaf. This is based on the fact that topoi, just like the

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category of sets, admit fibered products. The proofs for these statements can be found in [SGA4, II.2.5,II.4.3.2] and [SP, 8.12.5].

- State that if the topology of $C$ is coarser than the canonical topology (i.e. if every representable presheaf is a sheaf), then the functor $\epsilon_C$ is fully faithful; see e.g. [SP, 8.12] or [SGA4, II.4.40].
- Define the global section functor $\Gamma(T,\cdot) = \Gamma_T(\cdot)$ on a topos $T$ to be $\mathrm{Hom}_T(e_T, -)$, where $e_T$ is the final object in $T$. After restricting $\Gamma_T(\cdot)$ to abelian groups in $T$, we may consider its right derived functor $(H^*(T,\cdot), \partial)$. Similarly, for any object $X$ in $T$ we define $\Gamma(X,\cdot) = \mathrm{Hom}_T(X,\cdot)$ and $H^*(X,\cdot)$.
- "Localization/Restriction" of topoi and cohomology, c.f. [BO78, Prop. 5.23, 5.24, 5.25]: If $T$ is a topos and $Z$ an object of $T$, then the category $T|_Z$ of morphisms $X \to Z$ in $T$ is a topos, again. For any $X \in T$, we have $H^*(Z, X) \cong H^*(T|_Z, X|_Z)$, where we write $X|_Z$ to denote object $pr_2 : X \times Z \to Z$ of $T|_Z$.

**Talk 2: The Stratifying Topos, Stratifications and Crystals:** From now on, we follow [Gro68].

- Define the infinitesimal site $\mathrm{Inf}(X/S)$ as in [Gro68, 4.1]; let $(X/S)_{\mathrm{Inf}}$ denote the associated so-called infinitesimal topos.
- Show that an object $F$ of the infinitesimal topos can be described as family of Zariski-sheaves $F_{(U \hookrightarrow T)}$ together with certain morphisms (c.f. [Gro68, 4.1]). For more details, see also [BO78, Prop. 5.1], where for our purposes every mention of divided power structures should be ignored, and every occurence of Cris should be replaced by Inf.
- Show that any representable presheaf on $\mathrm{Inf}(X/S)$ is a sheaf, so that the Yoneda map $\epsilon_{\mathrm{Inf}(X/S)} : \mathrm{Inf}(X/S) \to (X/S)_{\mathrm{Inf}}$ is fully faithful.
- Using the concrete description of objects of $(X/S)_{\mathrm{Inf}}$, define the structural sheaf $\mathcal{O}_{(X/S)_{\mathrm{Inf}}}$ which provides $(X/S)_{\mathrm{Inf}}$ with the structure of a ringed topos.
- Recall the definition of smoothness via the infinitesimal lifting criterion, and note that for $X \to S$ smooth, any object $U \hookrightarrow T$ of the infinitesimal site $T$-locally admits a retraction $T \to X$. Hence, in the smooth case, the infinitesimal site agrees with the *stratifying site* defined by Grothendieck in [Gro68, 4.2]. As we restrict ourselves to the smooth case, we will henceforth use both sites synonymously.
- State the main theorem [Gro68, Thm. 4.1], whose proof is the goal of this Kleine AG. (As a reminder, quickly state the definition of algebraic de Rham cohomology as the hypercohomology of the relative de Rham complex.)
- Define stratifications of $\mathcal{O}_X$-modules, following either [Gro68, Appendix] or [BO78, Def. 2.10]. Mention the interpretation of a stratification as an "infinitesimal descent datum".
- Show that a stratified $\mathcal{O}_X$-module $F$ on $X$ defines what Grothendieck calls a "special sheaf" in [Gro68, 4.2], and which nowadays is called a crystal in $\mathcal{O}_{(X/S)_{\mathrm{Strat}}}$-modules. As an example, show that $\mathcal{O}_{(X/S)_{\mathrm{Strat}}}$ is a crystal in $\mathcal{O}_{(X/S)_{\mathrm{Strat}}}$-modules.
- Prove or sketch that, conversely, a crystal in $(X/S)_{\mathrm{Inf}} = (X/S)_{\mathrm{Strat}}$ gives rise to a stratified sheaf $F$. Note that this correspondence depends crucially on the $S$-smoothness of $X$. For a more detailed proof (which shows more than we actually need) you can also consult [BO78, Prop. 2.11].
- You don't have to adress the relationship between stratifications, connections and $\mathcal{D}_{X/S}$-modules; this will be the subject of a later talk.

**Talk 3: Cohomology of crystals on $(X/S)_{\mathrm{Strat}}$:** In this talk, we explain the construction of [Gro68, 5.1], which to a given crystal of $\mathcal{O}_{(X/S)_{\mathrm{Strat}}}$-modules $F$ associates a complex $\mathscr{F}^\bullet$ in $X_{\mathrm{Zar}}$ whose hypercohomology computes the cohomology of $F$.

- Show that in general $\mathrm{Inf}(X/S) = \mathrm{Strat}(X/S)$ does not have a final object. Let $\tilde{X}$ denote the object of $(X/S)_{\mathrm{Strat}}$ that is represented by $\mathrm{id}_X$, and let $e$ be the final object of $(X/S)_{\mathrm{Strat}}$. Show that the unique morphism $\tilde{X} \to e$ is a covering (in the topology introduced in Talk 1) by using the definition of (formal) smoothness. See also [BO78, 5.28] where, as usual, every mention of divided powers should be ignored and where Cris should be replaced by Strat or Inf. (In the notation of *loc. cit.* we can take $X = Y$ smooth; then the "PD-envelope $D_{X,\gamma}(Y)$" is just $X$.)
- Follow Grothendieck by writing down the associated Čech-to-derived-functor spectral sequence, and use it to show that for affine $X$ we have canonical isomorphisms

$$H^q((X/S)_{\mathrm{Strat}}, F) \cong H^q(F(X^\bullet/X)) = H^q(\varprojlim_i F(\Delta^\bullet_{X/S}(i)) \ .$$

The "Mittag-Leffler style arguments" from [Gro68, top of p. 337] may be skipped. The natural isomorphisms $H^q((U \hookrightarrow T), F) \cong H^q(T_{\mathrm{Zar}}, F_{(U\hookrightarrow T)})$ at [Gro68, bottom of p. 336] exist because the restriction functor $(X/S)_{\mathrm{Strat}}/u \to T_{\mathrm{Zar}}$ admits an exact left adjoint; for more details see [BO78, 5.26].

- In the situation where $X$ is not necessarily affine, define the complex $\mathscr{F}^\bullet$. It is not necessary to use simplicial language, simply define the differentials of the complex in terms of alternating sums. Finally, prove that there are canonical isomorphisms

$$H^q((X/S)_{\mathrm{Strat}}, F) \cong \mathbb{H}^q(X_{\mathrm{Zar}}, \mathscr{F}^\bullet).$$

**Talk 4: Differential Operators and Linearization:** In this talk we define differential operators, and their linearization.

- Define differential operators, e.g. following [BO78, Ch. 2]. State that for $\mathcal{O}_X$-modules $F$ and $G$, the differential operators $F \to G$ form a sheaf $\mathscr{D}\!\mathit{iff}_{X/S}(F,G)$.
- Describe the local structure of $\mathscr{D}_{X/S} := \mathscr{D}\!\mathit{iff}_{X/S}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathcal{P}^n_{X/S}$ (notation of [BO78]) in the situation where $X$ is a smooth scheme over $\mathbb{Q}$, cf. [BO78, Prop. 2.2 and Rem. 2.7]. Conclude that in this situation, $\mathscr{D}_{X/S}$-module structures on some $\mathcal{O}_X$-module $F$ correspond to flat connections on $F$ (where a flat connection on $F$ is a morphism of sheaves of Lie algebras $\mathscr{D}\!\mathit{er}_{f^{-1}(\mathcal{O}_S)}(\mathcal{O}_X, \mathcal{O}_X) \to \mathscr{E}\!\mathit{nd}_{f^{-1}(\mathcal{O}_S)}(F)$, where $\mathscr{D}\!\mathit{er}_{f^{-1}(\mathcal{O}_S)}(\mathcal{O}_X, \mathcal{O}_X)$ is the sheaf of $f^{-1}(\mathcal{O}_S)$-derivations on $\mathcal{O}_X$).
- Define the linearization functor $Q^0$ as follows, cf. [Gro68, 6.2]: Let $\mathcal{P}^\nu_{X/S}(i)$ denote the structure sheaf of the $i$-th infinitesimal neighborhood of the diagonal in the $(\nu+1)$-fold fiber product of $X$ over $S$, and write $\mathcal{P}^\nu_{X/S}$ for $\varprojlim_i \mathcal{P}^\nu_{X/S}(i)$. Let $M$ be an $\mathcal{O}_X$-module; we define $Q^\nu(M) := \varprojlim_i (\mathcal{P}^{\nu+1}(i)_{X/S} \otimes_{\mathcal{O}_X} M)$ (considered as an $\mathcal{O}_X$-module via the left $\mathcal{O}_X$-structures of $\mathcal{P}^\nu_{X/S}(i)$, while the tensor product is defined using the right $\mathcal{O}_X$-structure). Sketch that $Q^0(M)$ carries an $S$-stratification (compare [BO78, 2.14], where $L$ is used to denote $Q^0$). For our purposes, we only need to study $Q^\nu(\Omega^k_{X/S})$, which is $\mathcal{P}^{\nu+1}_{X/S} \otimes \Omega^k_{X/S}$ because $\Omega^k_{X/S}$ is coherent.
- Show that $Q^0$ defines a functor from the category of $\mathcal{O}_X$-modules with morphisms given by differential operators to the category of stratified $\mathcal{O}_X$-modules with horizontal $\mathcal{O}_X$-morphisms: If $D : E \to F$ is a differential operator of order $\le k$, then its linearization $\mathcal{P}^1_{X/S}(k) \otimes_{\mathcal{O}_X} E \to F$ together with the canonical $\delta^{n-k,k} : \mathcal{P}^1_{X/S}(n) \to \mathcal{P}^1_{X/S}(n-k) \otimes_{\mathcal{O}_X} \mathcal{P}^1_{X/S}(k)$ induces for every $n$ an $\mathcal{O}_X$-linear morphism $\mathcal{P}^1_{X/S}(n) \otimes_{\mathcal{O}_X} E \to \mathcal{P}^1_{X/S}(n-k) \otimes_{\mathcal{O}_X} F$, see [BO78, 2.14].

**Talk 5: Cohomology of Linearizations and Proof of the Main Theorem:** In this talk we prove the main theorem relating crystalline cohomology and algebraic de Rham cohomology.

- Restate the main theorem and summarize the strategy of its proof, e.g. as in [Gro68, 6.1].
- Let $M^\bullet$ be a complex of differential operators on $X$ bounded from below. If you wish you may take $M^\bullet = \Omega^\bullet_{X/S}$. Show (e.g. by checking termwise) that[1] $Q^\bullet(M^\bullet)$ is a resolution of $M^\bullet$ and that, hence, there is a canonical isomorphism

$$\mathbb{H}^p(X_{\mathrm{Zar}}, M^\bullet) \cong \mathbb{H}^p(X_{\mathrm{Zar}}, \mathrm{Tot}(Q^\bullet(M^\bullet))).$$

To prove that $Q^\bullet(M^q)$ is a resolution of $M^q$, it is important to note that $Q^\nu = \mathcal{P}^{\nu+1}$ – that is, $Q^\nu \to Q^{\nu+1}$ is defined by $\sum_{i=0}^{\nu+1}(-1)^i d_i^\nu$, where $d_i^\nu : \mathcal{P}^{\nu+1} \to \mathcal{P}^{\nu+2}$ are the canonical maps. There are $\nu+2$ such maps, but the last one is not used! Compare e.g. [Ber74, Lemme V.2.2.1].

- We have seen that $Q^0(M^\nu)$ carries a stratification for every $\nu$; hence it defines a crystal in $(X/S)_{\mathrm{Strat}}$, and it follows that we obtain a complex of crystals $Q^0(M^\bullet)_{\mathrm{Strat}}$ on $\mathrm{Strat}(X/S)$. By the calculations from Talk 3, we get, for every $p$ and $q$, a canonical isomorphism

$$H^p((X/S)_{\mathrm{Strat}}, Q^0(M^q)_{\mathrm{Strat}}) \cong \mathbb{H}^p(X_{\mathrm{Zar}}, C^\bullet(Q^0(M^q)_{\mathrm{Strat}})), \tag{1}$$

where Grothendieck now uses the notation $C^\bullet(-)$, for the Zariski complex associated to a crystal in Talk 3. Show that this implies that we have canonical isomorphisms

$$\mathbb{H}^p((X/S)_{\mathrm{Strat}}, Q^0(M^\bullet)_{\mathrm{Strat}}) \cong \mathbb{H}^p(X_{\mathrm{Zar}}, \mathrm{Tot}(C^\bullet(Q^0(M^\bullet)_{\mathrm{Strat}}))).$$

[1] Grothendendieck writes $\varprojlim Q^\bullet(M^\bullet)$ where we write $Q^\bullet(M^\bullet)$, because he considers $Q^\nu(M)^q = (\mathcal{P}^{\nu+1}_{X/S}(n) \otimes_{\mathcal{O}_X} M^q)_n$ as pro-object, while we identify it with its limit.

- Note that $C^\bullet(Q^0(M^\bullet)_{\mathrm{Strat}}) \cong Q^\bullet(M^\bullet)$ (see end of [Gro68, 6.3]), and show that this implies the existence of canonical isomorphisms
$$\mathbb{H}^p(X_{\mathrm{Zar}}, M^\bullet) \cong \mathbb{H}^p((X/S)_{\mathrm{Strat}}, Q^0(M^\bullet)_{\mathrm{Strat}}).$$
Deduce the existence of a spectral sequence
$$E_2^{p,q} = H^p((X/S)_{\mathrm{Strat}}, \mathcal{H}^q(Q^0(M^\bullet)_{\mathrm{Strat}})) \Longrightarrow \mathbb{H}^{p+q}(X_{\mathrm{Zar}}, M^\bullet)\ . \tag{2}$$
- For our purposes, we may take $F = \mathcal{O}_X$ in [Gro68, 6.5]. We then have a complex of differential operators $\Omega^\bullet_{X/S}$, so by what we have sowh above, we have canonical isomorphisms
$$\mathbb{H}^p(X_{\mathrm{Zar}}, \Omega^\bullet_{X/S}) \cong \mathbb{H}^p((X/S)_{\mathrm{Strat}}, Q^0(\Omega^\bullet_{X/S})_{\mathrm{Strat}})\ . \tag{3}$$
To finish the proof, we must naturally identify the right hand side with the stratified cohomology of the structural sheaf. To do so, we need a formal Poincaré Lemma which crucially depends on the assumption that our base $S$ is defined over $\mathbb{Q}$ (and that $X$ is smooth):

**Lemma** (Poincaré Lemma)**.** *If $S$ is a $\mathbb{Q}$-scheme, then the morphism*
$$\mathcal{O}_{(X/S)_{\mathrm{Strat}}} \to Q^0(\Omega^\bullet_{X/S})_{\mathrm{Strat}}$$
*of complexes arising from $\mathcal{O}_X \to \mathcal{P}^1_{X/S} = Q^0(\mathcal{O}_X)$, $x \mapsto x \otimes 1$ is a quasi-isomorphism.*

For the proof, follow [BO78, 6.11,6.12]; a certain amount of care, however, is needed to handle the differences between our situation and the crystalline case. Here's a brief outline of the argument: First note that the map in the statement of the Lemma really defines a morphism of complexes, i.e. that the composition $\mathcal{O}_X \to Q^0(\mathcal{O}_X) \to Q^0(\Omega^1_{X/S})$ is 0. For this, note the following local description of the complex $Q^0(\Omega_{X/S})$: If $x_1, \ldots, x_n$ are local coordinates for $X/S$ and if $\xi_i$ denotes the image of $1 \otimes x_i - x_i \otimes 1$ in $\mathcal{P}^1_{X/S}$, then $\mathcal{P}^1_{X/S} = \mathcal{O}_X[\![\xi_1, \ldots \xi_n]\!]$ and $Q^0(\Omega^p_{X/S}) = \bigoplus \mathcal{O}_X[\![\xi_1, \ldots, \xi_n]\!] dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, where the differential $Q^0(d)$ is given by
$$\xi_1^{i_1} \cdot \ldots \xi_n^{i_n} \omega \mapsto \sum_{j=1}^n i_j \xi_1^{i_1} \cdot \ldots \cdot \xi_j^{i_j - 1} \cdot \ldots \xi_n^{i_n} dx_j \wedge \omega + \xi_1^{i_1} \cdot \ldots \cdot \xi_n^{i_n} d\omega\ .$$
To prove the Lemma, we may work locally on $\mathrm{Strat}(X/S)$, so we may consider $(U \hookrightarrow T) \in \mathrm{Strat}(X/S)$, with $T$ small enough such that there exists a lifiting $h : T \to X$; then $h^* Q^0(\Omega^\bullet_{X/S}) = (Q^0(\Omega^\bullet_{X/S})_{\mathrm{Strat}})_{(U,T)}$. Now let $x_1, \ldots, x_n$ be local coordinates on $X$, then wee need to check that $(\bigoplus \mathcal{O}_T[\![\xi_1, \ldots, \xi_n]\!] dx_{i_1} \wedge \ldots \wedge dx_{i_r}, d)_r$ is a resolution of $\mathcal{O}_T$, which is true and which follows from an explicit calculation because we are in characteristic 0. See also [Har75, proof of Prop. 7.1].
- The Poincare Lemma now implies that (3) induces a canonical isomorphism
$$H^p((X/S)_{\mathrm{Strat}}, \mathcal{O}_{(X/S)_{\mathrm{Strat}}}) \to \mathbb{H}^p((X/S)_{\mathrm{Strat}}, Q^0(\Omega^\bullet_{X/S})_{\mathrm{Strat}}) \cong \mathbb{H}^p(X_{\mathrm{Zar}}, \Omega^\bullet_{X/S}),$$
which finishes the proof of the theorem.

## References

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